Intelligent Systems, Control and Automation: Science and Engineering

Honglei Xu
Xiangyu Wang *Editors*

Optimization and Control Methods in Industrial Engineering and Construction

Springer

Intelligent Systems, Control and Automation: Science and Engineering

Volume 72

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Optimization and Control Methods in Industrial Engineering and Construction

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ISSN 2213-8986 ISSN 2213-8994 (electronic)
ISBN 978-94-017-8043-8 ISBN 978-94-017-8044-5 (eBook)
DOI 10.1007/978-94-017-8044-5
Springer Dordrecht Heidelberg New York London

Library of Congress Control Number: 2013956328

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Preface

> As it was stated in the Nine Chapters on the Mathematical Art (Jiu Zhang SuanShu) "Mathematics problems are able to vary to be extremely infinite, fine or unmeasurable. In spite of the much complexity, the approaches can always be discovered, not as difficultly as supposed, which involve no more than measurement, reasoning and calculation to learn common laws".
>
> —Hui Liu, The Nine Chapters on the Mathematical Art, no later than 100 BC

The field of optimization is vast with applications appearing in almost every area of science and engineering. Generally speaking, optimization is to do with minimizing or maximizing an objective function (e.g. cost, energy, profit) subject to various types of constraints that arise due to engineering requirements or physical specifications. The optimization techniques for solving optimization problems are particularly important in the aspects of engineering and science applications. There are many efficient optimization techniques available in the literature, while many new techniques continue to be developed so as to meet the needs of solving various new practical problems in areas such as industrial engineering and construction, which are motivated by the need of satisfying more stringent requirements on energy saving, environment protection, and green manufacturing and construction. The natural formulations of the corresponding optimization problems have become much more complicated. The purpose of this edited book is to gather papers which address interesting optimization and control methods and new applications of optimization methods in industrial engineering and construction. Topics include optimization and control theory, statistical measurement, monitoring, fault detection, process control, construction design and production management. This edited book could be used as a reference book for researchers and postgraduate students in science and engineering.

The book is composed of three parts. The first three chapters are devoted to the development of new optimization methods. From "Optimum Confidence Interval Analysis in Two-factor Mixed Model with a Concomitant Variable for Gauge Study" to "Economic Scheduling of CCHP Systems Considering the Tradable Green Certificates", the focus is on the new applications of optimization and control methods in industrial engineering. For the rest of the chapters, different optimization problems in construction projects are being addressed.

In “Robustness of Convergence Proofs in Numerical Methods in Unconstrained Optimization”, the robustness of convergence proofs in numerical methods of unconstrained optimization is presented. It is developed based on an important principle in dynamic control system theory, where control policies are preferred to be of feedback form, rather than in an open loop manner. In “Robust Optimal Control of Continuous Linear Quadratic System Subject to Disturbances”, the robust optimal control of linear quadratic system is considered. It is formulated as a minimax optimal control problem which admits a unique solution. A control parameterization scheme is developed to transform the infinite dimensional optimal control problem to one with finite dimension. It is further shown that the transformed finite-dimensional optimal control problem can be solved through semi-definite programming. In “A Linearly-Growing Conversion from the Set Splitting Problem to the Directed Hamiltonian Cycle Problem”, a linearly growing conversion from the set splitting problem to the directed Hamiltonian cycle problem is discussed. A constructive procedure for such a conversion is given, and it is shown that the input size of the converted instance is a linear function of the input size of the original instance.

In “Optimum Confidence Interval Analysis in Optimum Confidence Interval Analysis in Two-Factor Mixed Model with a Concomitant Variable for Gauge Study”, the efforts on optimum confidence interval analysis in two-factor mixed model for gauge study are studied. The analysis of variance is performed in the model and variabilities in the model are represented as a linear combination of variance components. Optimum confidence intervals are constructed using a modified large sample approach and a generalized inference approach is proposed to determine the variability such as repeatability, reproducibility, parts, gauge and the ratio of variability of parts to the variability of gauge. In “Optimization of Engineering Survey Monitoring Networks”, the focus is on various ways of engineering survey monitoring networks, such that those used for tracking volcanic and large-scale ground movements may be optimized to improve the precision. These include the traditional method of fixing control points, the Lagrange method, free net adjustment, the g-inverse method and the singular value decomposition (SVD) approach using the pseudo-inverse. In “Distributed Fault Detection Using Consensus of Markov Chains”, a fault detection procedure appropriate for use in a variety of industrial engineering contexts is proposed, where consensus among a group of agents about the state of a system is employed. Markov chains are used to model subsystem behaviours, and consensus is reached by way of an iterative method based on estimates of a mixture of the transition matrices of these chains. In “Engineering Optimization Approaches of Nonferrous Metallurgical Processes”, an intelligent sequential operating method based on genetic programming is developed for solving nonferrous metallurgical processes, where optimization is being carried out while avoiding violent variation by operating the parameters in the ordered sequence. Real practical industrial data are used for carrying out the verification. In “Development of Neural Network Based Traffic Flow Predictors Using Pre-processed Data”, a simple but effective training method by incorporating the mechanisms of back-propagation algorithm and the exponential smoothing method is proposed to pre-process traffic flow data before training purposes. The pre-processing approach intends to aid the back-propagation algorithm to develop more accurate neural networks, as the pre-processed traffic flow data are more smooth and continuous than the original unprocessed traffic flow data. This approach is evaluated based on some sets of traffic flow data captured on a section of the freeway in Western Australia. Experimental results indicate that the neural networks developed based on this pre-processed data outperform those that are developed based on either original data or data which are pre-processed by the other pre-processing approaches. In "Economic Scheduling of CCHP Systems Considering the Tradable Green Certificates", tradable green certificate mechanism is introduced for the operation of CCHP system, and the impacts of tradable green certificate on the scheduling of CCHP system are studied. Then the economic dispatch model for multi-energy complementary system considering the TGC is proposed to maximize renewable energy utilization. This is a non-convex scheduling optimization problem. A global descent method is applied, which can continuously update the local optimal solutions by global descent functions. Finally, one modified IEEE 14-bus system is used to verify the performance of the proposed model and the optimization solver.

The remainder of the book relates to construction engineering optimization, more or less. Many types of optimization problems arise in construction engineering, such as sizing optimization, shape optimization, topology optimization, production optimization, contract dispatching and project management. Considering the differences in production conditions in the manufacturing industry, these problems are worth studying and complex for seeking valuable laws in optimization. First, the construction is rooted in place and conducted as on-site manufacturing. Second, every construction project is unique and a one-of-a-kind production, managed by a temporary organization, and consists of several companies. Third, highly interdependent activities have to be conducted in limited space, with multiple components, a lack of standardization and with many trades and subcontractors represented on-site. In "Optimizations in Project Scheduling: A State-of-Art Survey", a state-of-art survey of project management and scheduling is presented. This survey focuses on the new optimization formulations and new solution algorithms developed in the recent years. In "Lean and Agile Construction Project Management: As a Way of Reducing Environmental Footprint of the Construction Industry", a way of reducing the environmental footprint of the construction industry is proposed with the concept of lean and agile construction project management. It focuses on the construction project management with respect to the agility and leanness perspective and provides an in-depth analysis of the whole project life cycle phases based on lean and agile principles. Considering managing construction projects in Hong Kong, dynamic implications of industrial improvement strategies are analysed in "Managing Construction Projects in Hong Kong: Analysis of Dynamic Implications of Industrial Improvement Strategies". Based on a series of face-to-face interviews with experienced practitioners and a focus group exercise, this chapter presents the mapping of various interacting and fluctuating behaviours patterns during the site installation stage of building services in construction projects, with the aid of a generic system dynamics model, and draws interesting conclusions about the relationships among factors in construction project management. In "Dynamic Project Management: An Application of System Dynamics in Construction Engineering and Management", system dynamics (SD) are taken into consideration for construction engineering and project management. It is expected to serve as a useful guideline for the application of SD in construction and to contribute to expanding the current body of knowledge in construction simulation. Since production control is an essential part of any complex and constrained construction project, a lean framework for production control in complex and constrained construction projects (PC^4P) is discussed in "A Lean Framework for Production Control in Complex and Constrained Construction Projects (PC^4P)", which is based on an open system-theory mindset and consists of components, connections and inputs. In "Optimization in the Development of Target Contracts", by formulating the sharing problem in optimization terms, specific quantitative results will be obtained for all the various combinations of the main variables that exist in the contractual arrangements and project delivery. Such variables include the risk attitudes of the parties (risk-neutral, risk-averse), single or multiple outcomes (cost, duration, quality), single or multiple agents (contractors, consultants), and cooperative or non-cooperative behaviour. This chapter will be particularly of interest to academics and practitioners in the discipline of the design of target contracts and project delivery. It provides an understanding of optimal sharing arrangements within projects, broader than currently available.

We take this opportunity to express our immense gratitude to Prof. Kok Lay Teo for his guidance and encouragement all the time. We would also like to acknowledge financial support from Curtin University and the Natural National Science Foundation of China (11171079). In addition, we wish to thank Nathalie Jacobs and Cynthia Feenstra from Springer for their kind cooperation and professional support. Our special thanks go to Dr. Xiaofang Chen for his technical support during this book's editing process. Finally, we would like to convey our appreciation to all contributors, authors and reviewers who made this book possible.

Contents

Robustness of Convergence Proofs in Numerical Methods in Unconstrained Optimization

B. S. Goh, W. J. Leong and K. L. Teo

Abstract Numerical methods to solve unconstrained optimization problems may be viewed as control systems. An important principle in dynamic control system theory is that control policies should be prescribed in a feedback manner rather than in an open loop manner. This is to ensure that the outcomes are not sensitive to small errors in the state variables. A standard proof in numerical methods in unconstrained optimization like the Zoutendijk method is, from the control theory point of view, an open loop type of analysis as it studies what happens along a total trajectory for various initial state variables. In this chapter, an example is constructed to show that the eventual outcome and convergence to a global minimum point or otherwise can be very sensitive to initial values of the state variable. Convergence of a numerical method in unconstrained optimization can also be established by using the Lyapunov function theorem. The Lyapunov function convergence theorem provides feedback type analysis and thus the outcomes are robust to small numerical errors in the initial states. It requires that the level sets of the objective function are properly nested everywhere in order to have global convergence. This means the level sets of the objective function must be topologically equivalent to concentric spherical surfaces.

1 Introduction

An iterative method to compute the minimum point in an unconstrained optimization problem can be viewed as a control system. Thus to achieve robust solutions it is desirable to have feedback solution rather than open loop control policies [1].

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H. Xu and X. Wang (eds.), *Optimization and Control Methods in Industrial Engineering and Construction*, Intelligent Systems, Control and Automation: Science and Engineering 72,
DOI: 10.1007/978-94-017-8044-5_1, © Springer Science+Business Media Dordrecht 2014

A typical proof of a numerical method in optimization examines what happens along the total path of a trajectory for all admissible initial values. Thus, it is an open loop type of analysis. On the other hand, a proof of convergence of a numerical method by Lyapunov theorem in an unconstrained optimization problem examines what happens to changes in the value of the objective function relative to the level sets of the function in a typical iteration and it is re-started with numerical errors of the state variable. This is an example of feedback type control analysis and thus it is robust to numerical errors in the computation of the current position.

We shall draw on an example due to Barbashin and Krasovskii [1–3], and use Lyapunov function theory to illustrate the differences between open loop and closed loop convergence analysis of a numerical method in unconstrained optimization. It will also be demonstrated that open loop type of convergence along each trajectory for all possible initial conditions may not guarantee convergence to a global minimum point. It only establishes convergence to stationary points. What is needed is the concept of properly nested level sets of the objective function which is a key requirement for global convergence in a proof by using Lyapunov function theorem. Globally, an objective function has properly nested level sets if all the level sets are topologically equivalent to concentric spherical surfaces.

For convenience, brief reviews of Lyapunov function theorem for the global convergence of an iterative system and the Zoutendijk theorem for the convergence of a line search method in optimization will be given.

2 Convergence Proof by Using Lyapunov Function Theorem in Optimization

The traditional statement of the Lyapunov function theorem [1, 4–7] for a system of iterative equations is as follows: Let x^* be the optimal solution in an optimization problem. It is the equilibrium point of a system of iterative equations. Let L and C be positive constants. The vector iterative equation is,

$$x(k+1) = F[x(k)], \quad x \in R^n, \quad k = 0, 1, 2, \dots , \tag{1}$$

where $F(x)$ is a vector of continuous functions which does not explicitly contain the time variable k. It is said to be a time independent system. Thus, this analysis is not immediately applicable to time varying iterative systems like Quasi-Newton iterations in optimization. Some changes of this analysis can be made and they would then be applicable to time dependent systems.

We seek a continuous and nonnegative scalar function, $V(x)$, such that,

$$\Delta V[x(k)] = V[F(x(k))] - V[x(k)] < 0, \quad k = 0, 1, 2, \dots \tag{2}$$

for all $x(k) \in \{x|0 \leq V(x) \leq L\} = \Omega(x, x^*, L)$ and where $x \neq x^*$. At the equilibrium point, $V(x^*) = 0$ and trivially, $\Delta V(x^*) = 0$. By definition, a *sublevel set* of the function $V(x)$is defined by $\Omega(x, x^*, L) = \{x|0 \leq V(x) \leq L\}$. Here, if L is a large positive constant, it defines a large sublevel set of $V(x)$. On the other hand, a *level set* of the function $V(x)$ is the set given by $\Gamma = \{x|V(x) = C\}$. If condition (2) is satisfied, the function $\Delta V(x)$ is said to be *negative definite* in the sublevel region, $\Omega(x, x^*, L)$. It is important to differentiate between a level set and a sublevel set in a convergence analysis.

In an unconstrained optimization problem with objective function $f(x)$, the following function is a natural Lyapunov function

$$V(x) = f(x) - f(x^*). \tag{3}$$

Clearly, $V(x)$ is a merit function in optimization theory, with an additional requirement that it has a zero value at the optimal point, x^*. Furthermore, all the level sets of the function $V(x)$ must be *properly nested* in a sublevel set or global region, which means that they are topologically equivalent to concentric spherical surfaces. The function $V(x)$ with the required properties is called a *Lyapunov function*. The condition that the level sets of a function are properly nested can be verified easily for a function of two variables. This is done by plotting samples of the level sets of the function and by invoking the assumption that the function is continuous.

Suppose that $f(x)$ is the objective function for an unconstrained optimization problem with higher dimension. Then a sufficient condition to ensure that the level sets of a Lyapunov function are properly nested globally is that there exists a positive constants, γ, such that

$$V(x) - V(x^*) = f(x) - f(x^*) \geq \gamma||x - x^*||, \tag{4}$$

for all $x \in R^n$, where $||.||$ is a norm. If (4) is satisfied globally, the Lyapunov function is also said to be *radially unbounded*. In (4), a *fixed point* at the point x^* is used. On the other hand, a Lipschitz type condition in place of (4) for use in convergence analysis in numerical methods, would require that for all x and y in a finite region,

$$||\nabla f(x) - \nabla f(y) \leq \gamma||x - y||. \tag{5}$$

Note that the inequality signs in (4) and (5) are in opposite directions. Furthermore, (4) is a condition on the objective function rather than its gradient function in (4).

Theorem 2.1 *The equilibrium, x^*, of the iterative equation* (1) *is globally convergent if*

(i) there exists a continuous nonnegative function $V(x)$ with $V(x^*) = 0$, such that the function change $\Delta V(x)$ in (2) *is negative definite globally and*
(ii) all the level sets of $V(x)$, are properly nested.

Proof Suppose as $k \to \infty$ the function $V[x(k)] \to K_\infty \neq 0$

We maximize the function

$$W(x) = \Delta V[x(k)] = V[F(x(k))] - V[x(k)] \quad (6)$$

for all $x(k) \in \Omega(x, K_\infty, V(x(0))) = \{x | K_\infty \leq V(x) \leq V(x(0))\}$. This set, which is bounded by the two level sets of the function, $V(x)$, is a closed and bounded (i.e., compact) set because of the assumption that the level sets of $V(x)$ are properly nested. Thus, by Weierstrass's theorem for continuous functions, the maximum of $W(x) = \Delta V(x) = V[F(x)] - V(x)$ in $\Omega(x, K_\infty, V(x(0)))$ exists and it is attained in this compact set. Let the maximum value of $W(x) = -\theta$. Furthermore, θ is a nonzero positive parameter as $\Delta V(x)$ is negative definite and by assumption, $K_\infty \neq 0$. We have

$$V[x(N)] = \sum_{0}^{N-1} \Delta V[x(k)] + V[x(0)] \leq -N\theta + V[x(0)]. \quad (7)$$

This implies that $V[x(N)] \to -\infty$ as $N \to \infty$. This is impossible as $V(x)$ is nonnegative for all values of N. Hence we must have $K_\infty = 0$.This shows that the equilibrium is globally asymptotically convergent.

Corollary 2.1 *Suppose that the two conditions in Theorem 2.1 are satisfied only in a finite sublevel region, $\Omega(x, x^*, L)$. Then the convergence is valid in the finite region.*

To apply Theorem 2.1 to a numerical method in an unconstrained optimization problem, minimize $f(x)$, a natural choice of the Lyapunov function is,

$$V(x) = f(x) - f(x^*).$$

This implies that,

$$\Delta V(x) = \Delta[f(x) - f(x^*)] = \Delta f(x). \quad (8)$$

It is an important practical result, because $\Delta V(x)$ can be calculated in each step of an iterative method for an optimization problem even though the Lyapunov function $V(x)$ is not explicitly defined. *This property provides an important way to ensure that the Lyapunov function theorem is satisfied in a specific problem when a numerical method is used.*

On careful examination of (7), it is observed that the reduction of value of the Lyapunov function is finite and negative in a typical iteration. When the Lyapunov function theorem is applied to a numerical method for finding a solution of a specific problem, $\Delta V(x) = \Delta f(x)$ can be computed at each step. If numerical errors of an algorithm cause it to be positive in a particular iteration, $\Delta V(x) = \Delta f(x)$ would require re-computation until it is negative or stop—indicating failure of the numerical method.

3 Zoutendijk Convergence Analysis of a Numerical Method in Optimization

The Zoutendijk theorem is a set of prototype conditions which are used to establish the convergence of a numerical method for computing the minimum point of an optimization problem. It examines what happens along the *total* trajectory for a given initial state. A numerical method in unconstrained optimization may be viewed as a control system where the position is called the state vector and the steplengths and directions are control variables. For a control system, a feedback control policy is preferred over an open loop control policy [1, 8]. This is because an open loop control can be very sensitive to errors in the initial or current values of the state variables. This sensitivity of outcomes to numerical errors in the initial or current state variables will be explicitly and clearly demonstrated in an example.

For convenience, we briefly describe the application of Zoutendijk theorem to establish convergence of a line search method in unconstrained optimization of the objective function $f(x)$. Assume a line search method generates the iterative equation,

$$x(k+1) = x(k) + \alpha(k)p[x(k)], \quad x \in R^n, \quad k = 0, 1, 2, \ldots . \tag{9}$$

The key conditions required are: (i) The objective function is bounded below; and (ii) the gradient vector of the objective function satisfies the Lipschitz condition in an open subset $\Omega(x, x_0)$ of the sublevel set $\{x|f(x) \leq f(x_0)\}$. This means that for any pair of points x and y in $\Omega(x, x_0)$, there exists a positive constant γ such that,

$$||\nabla f(y) - \nabla f(x)|| \leq \gamma ||y - x||. \tag{10}$$

Furthermore, the steplength in the iterative equation (9) is chosen to satisfy the Wolfe's conditions, namely,

$$f[x(k) + \alpha(k)p(k)] \leq f[x(k)] + c_1\alpha(k)\nabla f[x(k)]^T p(k), \tag{11}$$

$$\nabla f[x(k) + \alpha(k)p(k)]^T p(k) \geq c_2 \nabla f[x(k)]^T p(k). \tag{12}$$

The positive constants $c_1$andc_2 are such that $0 < c_1 < c_2 < 1$.

Under these conditions, the Zoutendijk theorem states that,

$$\sum \cos^2\theta(k)||\nabla f[x(k)]||^2 < \infty. \tag{13}$$

Here, $\theta(k)$ is the angle between the search direction $p(k)$ and the steepest descent direction, $-\nabla f[x(k)]$. Thus, if there exists a positive constant σ such that

$$\cos(\theta(k)) \geq \sigma > 0, \tag{14}$$

then it can be deduced that

$$\lim ||\nabla f[x(k)]|| = 0 \tag{15}$$

as $k \to \infty$. This means that the trajectory generated from an arbitrary initial point x_0 would converge to a stationary point.

It is important to note that condition (13) or (15) is a property of the total trajectory from an arbitrary initial point x_0. Thus, there is no way to predicts what will happen if there are numerical errors in the initial state vector or a current vector as the iterative method progresses. In control system terminology, this may be viewed as an open loop control policy which is sensitive to numerical errors in the state variable during the computation of successive iterations. We shall demonstrate this by a specific example in the next section. Furthermore, the convergence of the iterative method is only to a stationary point which may not be even a local minimum point. This will be shown in an example.

4 Analysis of a Counterexample Without Properly Nested Level Sets

We shall adapt a counterexample due to Barbashin and Krasovskii [1–3] in Lyapunov theory for a system of ordinary differential equations to a system of iterative equation equations. For a system of ordinary differential equations, without the property that all the level sets are properly nested, an objective function can be monotonic decreasing, but the trajectories may not converge to the global minimum point.

From this counterexample, it is observed that if all the level sets of the objective function are not properly nested, then the solutions can be very sensitive to errors in the values of initial variables and hence they are not robust against numerical errors.

Example 4.1 Consider an unconstrained optimization with its objective function,

$$V(x) = f(x) = x_1^2/(1 + x_1^2) + x_2^2. \tag{16}$$

Its global minimum is at the origin. For convenience, let

$$w = (1 + x_1^2). \tag{17}$$

It does not have properly nested level sets for states in the set defined by

$$\{x|V(x) \geq b > 1\},$$

where b is a constant.

Assume that the iterative equations to compute the minimum point are given by

$$x_1(k + 1) = x_1(k) - \alpha(k)[6x_1(k)/w^2(k) + 2x_2(k)], \tag{18}$$

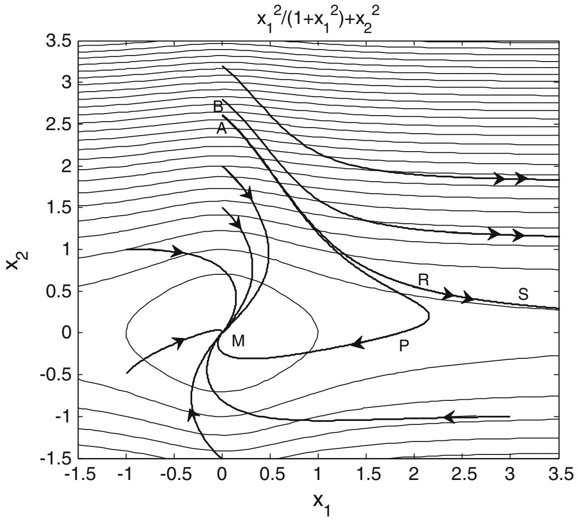

Fig. 1 Objective function has properly nested level sets only in a sublevel set $\{x|f(x) \le c < 1\}$. Lyapunov theorem guarantees convergence only is such sets. Zoutendijk theorem applies globally as function is monotonic decreasing everywhere. But trajectories with initial condition (0,a) with $a \ge 2.61$ converge to $(\infty, 0)$. BRS and APM show sensitivities to initial values

$$x_2(k+1) = x_2(k) - \alpha(k)[2(x_1(k) + x_2(k))/w^2(k)], \tag{19}$$

with the steplength, $\alpha(k) = 0.01$

With sufficiently small steplengths, it follows from Taylor's approximation that

$$\begin{aligned} \Delta V(x) &= V[x(k+1)] - V[x(k)] \\ &= \nabla V[x(k)]^T \Delta x(k) \\ &= -12\alpha x_1^2/w^4 - 4\alpha x_2^2/w^2 < 0. \end{aligned} \tag{20}$$

Thus, the objective function is monotonic decreasing globally except at the origin.

Apply the Zoutendijk theorem to this example, we deduce that

$$\lim \cos^2\theta ||\nabla f(x)||^2 \to 0, \tag{21}$$

with iterations from any point, globally. But the outcomes could be the global minimum point at the origin or a stationary point at $(\infty, 0)$ or $(-\infty, 0)$.

By Lyapunov function theorem, the level sets of the objective function are only properly nested in a sublevel set, $\{x|f(x) = V(x) \le c < 1\}$, where c is a constant. Thus, by Lyapunov function theorem, we are ensured that all trajectories with initial points in this sublevel set will converge to the global minimum at the origin (0,0), as depicted in Fig. 1.

Zoutendijk theorem can be applied to all initial points but for initial points, such as (0,a) with $a \geq 2.61$, the trajectories converge to $(\infty, 0)$, rather than the global minimum point. A more important issue is that as under an open loop control policy, the trajectories can be sensitive with respect to numerical errors in the initial state vector. This is illustrated by the trajectories APM from (0,2.6) and BRS from (0,2.61) in Fig. 1. Here, a small change in initial conditions leads to entirely different outcomes. Thus, Zoutendijk theorem in a proof of convergence only provides conditions for a trajectory from a typical initial point to converge to a stationary point. More importantly, the trajectories can be very sensitive to the choice of the values of the initial state variables. This phenomena is a well known weakness of an open loop policy in control systems.

5 Conclusion

Numerical methods in unconstrained optimization can be viewed as control systems. It is well know that a feedback control policy is much preferred over an open control policy in control systems. Proofs of convergence of a numerical method, such as those based on Zoutendijk theorem, are in the context of open loop control policies. They examine what happens along the total path of a trajectory for different initial values. Thus the outcome could be sensitive to numerical errors of the initial values or the current state. Furthermore, Zoutendijk theorem ensures only convergence to stationary points.

On the other hand, the Lyapunov theorem proof of the convergence of a numerical method in unconstrained optimization is a feedback type of analysis. It requires that in a typical iteration the decrease in the objective function must be finite and negative. If numerical errors caused the failure of this monotonic decrease condition of the objective function, then it requires new iterations by line search or otherwise to recompute a new iteration which causes a decrease in the objective function. Thus the Lyapunov function approach has feedback properties.

Furthermore, the Lyapunov function requires that the objective function has properly nested level sets globally or in a finite sublevel set which defines an estimate of its region of convergence. With the properly nested level sets property, convergence to a minimum point is guaranteed and not just to a stationary point.

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Robust Optimal Control of Continuous Linear Quadratic System Subject to Disturbances

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Abstract In this chapter, the robust optimal control of linear quadratic system is considered. This problem is first formulated as a minimax optimal control problem. We prove that it admits a solution. Based on this result, we show that this infinite-dimensional minimax optimal control problem can be approximated by a sequence of finite-dimensional minimax optimal parameter selection problems. Furthermore, these finite-dimensional minimax optimal parameter selection problems can be transformed into semi-definite programming problems or standard minimization problems. A numerical example is presented to illustrate the developed method.

1 Introduction

A fundamental problem of theoretical and practical interest, that lies at the heart of control theory, is the design of controllers that yield acceptable performance for a family of plants under various types of inputs and disturbances [1]. This problem is often referred to as a robust optimal control problem. Normally, there are two kinds of criteria to achieve robust controller design. One is based on a statistical description, i.e., the criterion of the expectations of the cost and the constraints is adopted [17]. For the other one, the worst-case performance criterion is adopted [2–4, 9–12, 15, 18].

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H. Xu and X. Wang (eds.), *Optimization and Control Methods in Industrial Engineering and Construction*, Intelligent Systems, Control and Automation: Science and Engineering 72,
DOI: 10.1007/978-94-017-8044-5_2, © Springer Science+Business Media Dordrecht 2014

The dynamical systems can be classified into two kinds—discrete dynamical system and continuous dynamical system. For the robust optimal control of linear discrete dynamical system with quadratic cost function, there are many results available [2–5, 9–12, 15, 18]. If disturbances lie in an ellipsoid, then it is shown in [3] that such an optimal control problem without constraints is equivalent to a semi-definite programming (SDP) problem. If the optimal control problem is subject to constraints on the state and control, it can be relaxed (see [3]) as a second-order cone programming (SOCP). If disturbances lie in a polyhedral, then such a robust optimal control problem becomes computationally highly demanding, (see [2, 12]). For other results on such robust optimal control problems, see, for example, [2, 3, 10–12, 18]. For robust optimal control governed by continuous dynamical system, a computational scheme is developed in [16]. By introducing a linear operator and resorting to its norm, the original minimax optimal control problem can be transformed into a standard optimal control problem. This method depends crucially on the special form of the cost function. If the cost function is with the terminal cost, then this method does not work.

In this chapter, we consider a class of robust optimal control problems governed by continuous dynamical systems subject to constraints on the admissible controls and the disturbances. The cost function involves not only a quadratic integral cost, but also a terminal cost expressed in the form of quadratic terminal state. Furthermore, we will use piecewise functions, rather than orthornormal basis as in [16], to approximate admissible control functions. We first show that this robust optimal control problem admits a solution. Based on this result, we show that this infinite-dimensional minimax optimal control problem can be approximated by a sequence of finite-dimensional minimax optimal parameter selection problems. Then, we show that these minimax optimal parameter selection problems can be transformed into SDPs. We also show that these minimax optimal parameter selection problems can also be transformed into standard minimization problems. Thus, gradient-based optimization methods can be applied. To illustrate our developed method, a numerical example is presented.

2 Problem Formulation

Consider the continuous linear dynamical system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + C(t)w(t), t \in [0, T], \\ x(0) &= x^0, \end{aligned} \tag{1}$$

where T is the given terminal time, $x(t) \in \mathbb{R}^n$ is the state at time t, x^0 is a given initial state, $u(t) \in \mathbb{R}^m$ is the input at time t, $w(t) \in \mathbb{R}^r$ is the uncertainty at time t, and A, B and C are matrices with appropriate dimension.

Let

$$\mathscr{W} = \left\{ w \in L^2\left([0,T], \mathbb{R}^r\right) : \|w\|_{L^2}^2 = \int_0^T (w(t))^T w(t)\, dt \le \zeta^2 \right\}, \tag{2}$$

and

$$\mathscr{U} = \left\{ u \in L^2\left([0,T], \mathbb{R}^m\right) : \|u\|_{L^2}^2 = \int_0^T (u(t))^T u(t)\, dt \le \eta^2 \right\}. \tag{3}$$

A function u is said to be an admissible control if $u \in \mathscr{U}$. Note that $\mathscr{W}$ and $\mathscr{U}$ are weakly closed in $L^2([0,T], \mathbb{R}^r)$ and $L^2([0,T], \mathbb{R}^m)$, respectively. For brevity, they are simply referred to as weakly closed.

Now our robust optimal control problem can be stated as follows.

Problem (P). Choose $(u^*, w^*) \in \mathscr{U} \times \mathscr{W}$ such that

$$J\left(u^*, w^*\right) = \min_{u\in\mathscr{U}} \max_{w\in\mathscr{W}} J(u,w) = (x(T))^T P x(T) + \int_0^T (x(t))^T Q(t) x(t) + (u(t))^T R(t) u(t)\, dt, \tag{4}$$

where P, $Q(t)$ and $R(t)$ are all positive definite matrices with appropriate dimensions.

To proceed, we assume that the matrices $A(t)$, $B(t)$, $C(t)$, $Q(t)$ and $R(t)$ are continuous matrix-valued functions defined on $[0,T]$.

3 Existence Theorem

Note that for each given $t \in [0,T]$, P, $Q(t)$ and $R(t)$ are all positive definite matrices. Let $F(t,\tau)$ be the $n \times n$ state transition matrix that satisfies

$$\begin{aligned} \dot{F}(t,\tau) &= A(t) F(t,\tau), \\ F(\tau,\tau) &= I, \end{aligned} \tag{5}$$

where I is the identity matrix. Then, for each given u and w, the solution of (1) can be expressed as

$$x(t|u,w) = F(t,0)x_0 + \int_0^t F(t,\tau) B(\tau) u(\tau)\, d\tau + \int_0^t F(t,\tau) C(\tau) w(\tau)\, d\tau. \tag{6}$$

Since P and $Q(t)$ are positive definite matrices for each given $t \in [0,T]$, $J(u,w)$ is strictly convex with respect to x. From (6), it follows that x is linear with respect to w. Thus, $J(u,w)$ is strictly convex with respect to w. For each given $u \in \mathscr{U}$, since $\mathscr{W}$ is a weakly sequentially compact subset of $L^2([0,T], \mathbb{R}^r)$, there exists a $w(u)$

such that

$$J\left(u,w\left(u\right)\right)=\max_{w\in\mathscr{W}}J\left(u,w\right).$$

Let

$$\mathscr{G}\left(u\right)=\int_0^T\left(u\left(t\right)\right)^TR\left(t\right)u\left(t\right)dt.$$

Note that for each given $t\in[0,T]$, $R(t)$ is positive definite, it is easily to verify that $\mathscr{G}(u)$ is a strictly convex function with respect to u. Now we have the following lemmas.

Lemma 1. *If $u_n\rightharpoonup u$ as $n\to\infty$, ($u_n\rightharpoonup u$ means that u_n converges to u weakly in $L^2([0,T],\mathbb{R}^m)$). Then,*

$$u\in\mathscr{U}\ and\ \mathscr{G}\left(u\right)\leq\varliminf_{n\to\infty}\mathscr{G}\left(u_n\right).\tag{7}$$

If $u_n\to u$ as $n\to\infty$, ($u_n\to u$ means that u_n converges to u in the norm of $L^2([0,T],\mathbb{R}^m)$), where $\{u_n\}\subset\mathscr{U}$, then

$$u\in\mathscr{U}\ and\ \lim_{n\to\infty}\mathscr{G}\left(u_n\right)=\mathscr{G}\left(u\right).\tag{8}$$

Proof. Suppose that $u_n\rightharpoonup u$. Clearly, $u\in\mathscr{U}$, as $\mathscr{U}$ is a weakly closed set in $L^2([0,T],\mathbb{R}^m)$. By the convexity of $\mathscr{G}(u)$, we have

$$\mathscr{G}\left(u_n\right)\geq\mathscr{G}\left(u\right)+\left\langle D\mathscr{G}\left(u\right),u_n-u\right\rangle=\mathscr{G}\left(u\right)+2\int_0^T\left(u_n\left(t\right)-u\left(t\right)\right)^TR\left(t\right)u\left(t\right)dt.\tag{9}$$

Note that $\{u_n\}\subset\mathscr{U}$ and $R(\cdot)$ is continuous on $[0,T]$, we can show that

$$\int_0^T\left(u_n\left(t\right)\right)^TR\left(t\right)u_n\left(t\right)dt$$

is bounded uniformly with respect to n. Thus, $\varliminf_{n\to\infty}\mathscr{G}(u_n)$ exists. Since $R(\cdot)$ is continuous on $[0,T]$ and $u\in L^2([0,T],\mathbb{R}^m)$, it follows that $R(\cdot)u(\cdot)\in L^2\left([0,T],\mathbb{R}^{n\times m}\right)$. Thus,

$$\lim_{n\to\infty}\int_0^T\left(u_n\left(t\right)\right)^TR\left(t\right)u\left(t\right)dt=\int_0^T\left(u\left(t\right)\right)^TR\left(t\right)u\left(t\right)dt\tag{10}$$

as $u_n\rightharpoonup u$. Therefore, (7) holds.

Suppose that $u_n\to u$, *i.e.*,

$$\|u_n-u\|_{L^2}\to 0.\tag{11}$$

Clearly, $u \in \mathscr{U}$. Since $\{u_n\} \subset \mathscr{U}$ and $R(\cdot)$ is continuous on $[0, T]$, there exists a constant $\varkappa$ such that

$$\|R(\cdot)\, u_n(\cdot)\|_{L^2} \leq \varkappa \text{ for all } n = 1, 2, \ldots,$$

and

$$\|R(\cdot)\, u(\cdot)\|_{L^2} \leq \varkappa.$$

Thus,

$$\begin{aligned} |\mathscr{G}(u_n) - \mathscr{G}(u)| &\leq \left| \int_0^T (u_n(t) - u(t))^T R(t)\, u_n(t)\, dt \right| + \\ &\quad \left| \int_0^T (u_n(t) - u(t))^T R(t)\, u(t)\, dt \right| \\ &\leq \varkappa \|u_n - u\|_{L^2} + \left| \int_0^T (u_n(t) - u(t))^T R(t)\, u(t)\, dt \right| \qquad (12) \\ &\leq 2\varkappa \|u_n - u\|_{L^2}. \end{aligned}$$

Since $u_n \to u$, it follows that $\lim_{n\to\infty} \mathscr{G}(u_n) = \mathscr{G}(u)$. This completes the proof.

Define

$$\mathscr{F}(u, w) = (x(T|u, w))^T P x(T|u, w) + \int_0^T (x(t|u, w))^T Q(t)\, x(t|u, w)\, dt,$$

We have the following lemma.

Lemma 2. *Suppose that $u_n \rightharpoonup u$ and $w_n \rightharpoonup w$ as $n \to \infty$, where $\{u_n\} \subset \mathscr{U}$ and $\{w_n\} \subset \mathscr{W}$. Then,*

$$\lim_{n\to\infty} \mathscr{F}(u_n, w_n) = \mathscr{F}(u, w), \qquad (13)$$

where $u \in \mathscr{U}$ and $w \in \mathscr{W}$.

Proof. Since $\mathscr{U}$ and $\mathscr{W}$ are weakly closed, $u \in \mathscr{U}$ and $w \in \mathscr{W}$. By the continuity of $A(t)$, $F(t, \cdot)$ is continuous on $[0, t]$ for each $t \in [0, T]$ Note that

$$\begin{aligned} &|x(t|u_n, w_n) - x(t|u, w)| \\ &= \left| \int_0^t F(t, \tau)\, B(\tau)\, (u_n(\tau) - u(\tau))\, d\tau + \int_0^t F(t, \tau)\, C(\tau)\, (w_n(\tau) - w(\tau))\, d\tau \right| \\ &\leq \left| \int_0^t F(t, \tau)\, B(\tau)\, (u_n(\tau) - u(\tau))\, d\tau \right| + \left| \int_0^t F(t, \tau)\, C(\tau)\, (w_n(\tau) - w(\tau))\, d\tau \right| \\ &= \left| \int_0^T \tilde{F}(t, \tau)\, B(\tau)\, (u_n(\tau) - u(\tau))\, d\tau \right| + \left| \int_0^T \tilde{F}(t, \tau)\, C(\tau)\, (w_n(\tau) - w(\tau))\, d\tau \right|, \end{aligned}$$

where

$$\tilde{F}(t,\tau) = \begin{cases} F(t,\tau), & if \tau \le t, \\ 0_{n\times n} & \text{else.} \end{cases}$$

Clearly, $\tilde{F}(t,\tau) B(\tau)$ and $\tilde{F}(t,\tau) C(\tau)$ are continuous on $[0,T]$ except at the point $\tau = t$ and hence $\tilde{F}(t,\tau) B(\tau) \in L^2\left([0,T], \mathbb{R}^{n\times m}\right)$ and $\tilde{F}(t,\tau) C(\tau) \in L^2\left([0,T], \mathbb{R}^{n\times r}\right)$. Thus, for each $t \in [0,T]$, we have

$$\lim_{n\to\infty} x_n(t|u_n, w_n) = x(t|u, w). \tag{14}$$

On the other hand,

$$\begin{aligned}
|x(t|u_n, w_n)| &= \bigg| F(t,0) x_0 + \int_0^t F(t,\tau) B(\tau) u_n(\tau) d\tau + \\
&\quad \int_0^t F(t,\tau) C(\tau) w_n(\tau) d\tau \bigg| \\
&\le |F(t,0) x_0| + \left| \int_0^t F(t,\tau) B(\tau) u_n(\tau) d\tau \right| + \\
&\quad \left| \int_0^t F(t,\tau) C(\tau) w_n(\tau) d\tau \right| \\
&\le |F(t,0) x_0| + \left[\sum_{i=1}^m \left(\int_0^t \left((F(t,\tau) B(\tau))_i \right)^2 d\tau \right) \right]^{1/2} \\
&\quad \left[\sum_{i=1}^m \int_0^T \left(u_{n,i}(\tau) \right)^2 d\tau \right]^{1/2} \\
&\quad + \left[\sum_{i=1}^r \int_0^t \left((F(t,\tau) C(\tau))_i \right)^2 d\tau \right]^{1/2} \\
&\quad \left[\sum_{i=1}^r \int_0^T \left(w_{n,i}(\tau) \right)^2 d\tau \right]^{1/2},
\end{aligned}$$

where $(F(t,\tau) B(\tau))_i$ is the i-th element of $F(t,\tau) B(\tau)$. By the continuity of $\int_0^t \left((F(t,\tau) B(\tau))_i \right)^2 d\tau$, $\int_0^t \left((F(t,\tau) C(\tau))_i \right)^2 d\tau$ and $F(t,0) x_0$, there exists a ρ such that

$$\begin{aligned}
\rho = \max_{i=1,\ldots,m;\, j=1,\ldots,r;\, t\in[0,T]} \bigg\{ &\int_0^t \left((F(t,\tau) B(\tau))_i \right)^2 d\tau, \\
&\int_0^t \left((F(t,\tau) C(\tau))_j \right)^2 d\tau, |F(t,0) x_0| \bigg\}.
\end{aligned}$$

It follows that

$$|x(t|u_n, w_n)| \leq \rho + \rho^{1/2}\left(\|u_n\|_{L^2} + \|w_n\|_{L^2}\right) \leq \rho + \rho^{1/2}(\zeta + \eta), \ \forall t \in [0, T].$$

Since $Q(t)$ is continuous on $[0, T]$ and is positive definite for each $t \in [0, T]$, we have, for any $t \in [0, T]$,

$$0 \leq (x(t|u_n, w_n))^T Q(t) x(t|u_n, w_n) \leq \max_{i,j=1,\ldots,n; t\in[0,T]} \left|Q_{i,j}(t)\right| \left(\rho + \rho^{1/2}(\zeta + \eta)\right)^2.$$

Therefore, by Lebesgue Dominated Convergence Theorem (Theorem 2.6.4 in [14]), it holds that

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T (x(t|u_n, w_n))^T Q(t) x(t|u_n, w_n)\, dt \\
&= \int_0^T \lim_{n\to\infty} (x(t|u_n, w_n))^T Q(t) x(t|u_n, w_n)\, dt \\
&= \int_0^T (x(t|u, w))^T Q(t) x(t|u, w)\, dt.
\end{aligned} \tag{15}$$

By virtue of (14) with $t = T$ and (15), we obtain

$$\lim_{n\to\infty} \mathscr{F}(u_n, w_n) = \mathscr{F}(u, w).$$

This completes the proof.

From Lemma 1 and Lemma 2, we have the following lemma.

Lemma 3. *If $u_n \rightharpoonup u$ and $w_n \rightharpoonup w$, where $\{u_n, w_n\} \subset \mathscr{U} \times \mathscr{W}$, then,*

$$(u, w) \in \mathscr{U} \times \mathscr{W} \text{ and } J(u, w) \leq \underline{\lim}_{n\to\infty} J(u_n, w_n).$$

If $u_n \to u$ and $w_n \rightharpoonup w$, where $\{u_n, w_n\} \subset \mathscr{U} \times \mathscr{W}$, then,

$$(u, w) \in \mathscr{U} \times \mathscr{W} \text{ and } J(u, w) = \lim_{n\to\infty} J(u_n, w_n).$$

Now we have the following main theorem in this section.

Theorem 1. *Consider Problem (P). Then, there exists a $(u^*, w^*) \in \mathscr{U} \times \mathscr{W}$ such that*

$$J\left(u^*, w^*\right) = \min_{u\in\mathscr{U}} \max_{w\in\mathscr{W}} J(u, w). \tag{16}$$

Proof. Note that $L^2([0,T], \mathbb{R}^r)$ is reflexive and $\mathscr{U}$ is a compact and convex set. It follows that $\mathscr{U}$ is weakly sequentially compact. To prove (16), it suffices, by Proposition 38.12 in [19], to prove that

$$J(u) = \max_{w \in \mathscr{W}} J(u, w)$$

is weakly sequentially lower semi-continuous. That is to say, we only need to prove

$$J(u) \leq \underline{\lim}_{n\to\infty} J(u_n) \text{ when } u_n \rightharpoonup u. \tag{17}$$

Suppose that $u_n \rightharpoonup u$. From Lemma 3, we know that

$$J(u, w) \leq \underline{\lim}_{n\to\infty} J(u_n, w), \text{ for any } w \in \mathscr{W}.$$

Clearly,

$$\max_{w \in \mathscr{W}} J(u_n, w) \geq J(u_n, w).$$

It follows that

$$J(u, w) \leq \underline{\lim}_{n\to\infty} J(u_n, w) \leq \underline{\lim}_{n\to\infty} \max_{w \in \mathscr{W}} J(u_n, w), \text{ for any } w \in \mathscr{W}.$$

Thus,

$$J(u) = \max_{w \in \mathscr{W}} J(u,w) \leq \underline{\lim}_{n\to\infty} \max_{w \in \mathscr{W}} J(u_n, w) = \underline{\lim}_{n\to\infty} J(u_n, w(u_n)) = \underline{\lim}_{n\to\infty} J(u_n).$$

This completes the proof.

4 Problem Approximation

Consider a monotonically non-decreasing sequence $\{S^p\}_{p=1}^{\infty}$ of finite subsets of $[0, T]$. For each p, let $n_p + 1$ points of S^p be denoted by $t_0^p, t_1^p, \ldots, t_{n_p}^p$. These points are chosen such that

$$t_0^p = 0, t_{n_p}^p = T, \text{ and } t_{k-1}^p < t_k^p, k = 1, 2, \ldots, n_p.$$

Thus, associated with each S^P there is the obvious partition $\mathscr{I}^p$ of $[0, T]$ defined by

$$\mathscr{I}^p = \{I_k^p : k = 1, \ldots, n_p\},$$

where $I_k^p = [t_{k-1}^p, t_k^p)$.

We choose S^P such that $\lim_{p\to\infty} S^p$ is dense in $[0, T]$, that is

$$\lim_{p\to\infty} \max_{k=1,\ldots,n_p} \left|I_k^p\right| = 0,$$

where $\left|I_k^p\right| = t_k^p - t_{k-1}^p$, the length of the *kth* interval.

Let

$$u^p(t) = \sum_{k=1}^{n_p} \sigma^{p,k} \chi_{I_k^p}(t), \tag{18}$$

$$w^p(t) = \sum_{k=1}^{n_p} \theta^{p,k} \chi_{I_k^p}(t), \tag{19}$$

and

$$\sigma^p = [(\sigma^{p,1})^T, \ldots, (\sigma^{p,n_p})^T]^T \text{ and } \theta^p = [(\theta^{p,1})^T, \ldots, (\theta^{p,n_p})^T]^T,$$

where

$$\sigma^{p,k} = \left[\sigma_1^{p,k}, \ldots, \sigma_m^{p,k}\right]^T, \text{ and } \theta^{p,k} = \left[\theta_1^{p,k}, \ldots, \theta_r^{p,k}\right]^T,$$

χ_I denotes the indicator function of I defined by

$$\chi_I(t) = \begin{cases} 1, & t \in I, \\ 0, & \text{elsewhere.} \end{cases}$$

Define

$$\Pi^p = \left\{\sigma^p \in \mathbb{R}^{mn_p} : \left(\sigma^p\right)^T U^P \sigma^p \le \eta^2\right\}, \tag{20}$$

$$\Xi^p = \left\{\theta^p \in \mathbb{R}^{rn_p} : \left(\theta^p\right)^T W^P \theta^p \le \zeta^2\right\}, \tag{21}$$

$$\mathscr{U}^p = \left\{u^p(t) = \sum_{k=1}^{n_p} \sigma^{p,k} \chi_{I_k^p}(t) : \sigma^p \in \Pi^p\right\},$$

and

$$\mathscr{W}^p = \left\{w^p(t) = \sum_{k=1}^{n_p} \theta^{p,k} \chi_{I_k^p}(t) : \theta^p \in \Xi^p\right\},$$

where

$$U^P = diag\left(\left|I_1^p\right| I_{m\times m}, \left|I_2^p\right| I_{m\times m}, \ldots, \left|I_{n_p}^p\right| I_{m\times m}\right),$$

and

$$W^{P} = diag\left(\left|I_1^p\right| I_{r\times r}, \left|I_2^p\right| I_{r\times r}, \ldots, \left|I_{n_p}^p\right| I_{r\times r}\right).$$

It is clear that $\mathscr{U}^p \subseteq \mathscr{U}$ and $\mathscr{W}^p \subseteq \mathscr{W}$. Furthermore, we have the following lemma.

Lemma 4. *For any $u \in \mathscr{U}$ and $w \in \mathscr{W}$, let*

$$u^p(t) = \sum_{j=1}^{n_p} \sigma^{p,j} \chi_{I_j^p}(t) \tag{22}$$

and

$$w^p(t) = \sum_{j=1}^{n_p} \theta^{p,j} \chi_{I_j^p}(t), \tag{23}$$

where

$$\sigma^{p,j} = \frac{1}{\left|I_j^p\right|} \int_{I_j^p} u(t)dt$$

and

$$\theta^{p,j} = \frac{1}{\left|I_j^p\right|} \int_{I_j^p} w(t)dt.$$

Then, $u^p \in \mathscr{U}^p$ and $w^p \in \mathscr{W}^p$. Furthermore,

$$u^p \to u \text{ and } w^p \to w. \tag{24}$$

Proof. Note that

$$\begin{aligned}
\int_0^T \left(u^p(t)\right)^T u^p(t)dt &= \int_0^T \left(\sum_{j=1}^{n_p} \sigma^{p,j} \chi_{I_j^p}(t)\right)^T \left(\sum_{j=1}^{n_p} \sigma^{p,j} \chi_{I_j^p}(t)\right) dt \\
&= \sum_{j=1}^{n_p} \int_{I_j^p} \left(\sigma^{p,j}\right)^T \sigma^{p,j} dt = \sum_{j=1}^{n_p} \frac{1}{\left|I_j^p\right|} \int_{I_j^p} u^T(t)dt \int_{I_j^p} u(t)dt \\
&\leq \sum_{j=1}^{n_p} \frac{1}{\left|I_j^p\right|} \left|I_j^p\right| \int_{I_j^p} u^T(t)u(t)dt = \int_0^T u^T(t)u(t)dt.
\end{aligned} \tag{25}$$

Thus, $u^p \in \mathscr{U}^p$. In a similar way, we can show that $w^p \in \mathscr{W}^p$. From Lemma 6.4.1 of [14], we have

$$u^p(t) \to u(t), \text{ for almost all } t \in [0, T],$$

and

$$w^p(t) \to w(t), \text{ for almost all} t \in [0, T].$$

Note that $\{u^p\} \times \{w^p\} \subset \mathscr{U} \times \mathscr{W}$ and $u \times w \in \mathscr{U} \times \mathscr{W}$. We have $\|u^p\|_{L^2}^2 \le \eta^2$ and $\|w^p\|_{L^2}^2 \le \zeta^2$ for all $p = 1, \ldots$, while $\|u\|_{L^2}^2 \le \eta^2$ and $\|w\|_{L^2}^2 \le \zeta^2$. Since T is a finite number, the conclusion of the lemma follows readily.

With $u \in \mathscr{U}^p$ and $w \in \mathscr{W}^p$, the dynamical system (1) becomes

$$\dot{x}(t) = A(t)x(t) + B(t)\sum_{k=1}^{n_p}\sigma^{p,k}\chi_{I_k^p}(t) + C(t)\sum_{k=1}^{n_p}\theta^{p,k}\chi_{I_k^p}(t),$$
$$x(0) = x^0, \tag{26}$$

and $J(u, w)$ becomes

$$\tilde{J}\left(\sigma^p, \theta^p\right) = (x(T))^T Px(T) + \int_0^T \Bigg\{(x(t))^T Q(t)x(t) + \left(\sum_{k=1}^{n_p}\sigma^{p,k}\chi_{I_k^p}(t)\right)^T R(t)\left(\sum_{k=1}^{n_p}\sigma^{p,k}\chi_{I_k^p}(t)\right)\Bigg\}dt.$$

Now we define the following minimax optimal parameter selection problem.
Problem (P_p) : For the given dynamical system (26), choose $(\sigma^{p,*}, \theta^{p,*}) \in \Pi^p \times \Xi^p$ such that

$$\tilde{J}\left(\sigma^{p,*}, \theta^{p,*}\right) = \min_{\sigma^p \in \Pi^p} \max_{\theta^p \in \Xi^p} \tilde{J}\left(\sigma^p, \theta^p\right).$$

Remark 1. Following a similar argument given for the proof of Theorem 1, we can show that for Problem (P_p), there exists a $(\sigma^{p,*}, \theta^{p,*}) \in \Pi^p \times \Xi^p$ such that

$$J\left(\sigma^{p,*}, \theta^{p,*}\right) = \min_{\sigma^p \in \Pi^p} \max_{\theta^p \in \Xi^p} \tilde{J}\left(\sigma^p, \theta^p\right). \tag{27}$$

Theorem 2. *Suppose that (u^*, w^*) and $(\sigma^{p,*}, \theta^{p,*})$ are the optimal solutions of Problem (P) and Problem (P_p), respectively. That is,*

$$J\left(u^*, w^*\right) = \min_{u \in \mathscr{U}} \max_{w \in \mathscr{W}} J(u, w) \text{ and } \tilde{J}\left(\sigma^{p,*}, \theta^{p,*}\right) = \min_{\sigma^p \in \Pi^p} \max_{\theta^p \in \Xi^p} \tilde{J}\left(\sigma^p, \theta^p\right).$$

Then,

$$\lim_{p \to \infty} \tilde{J}\left(\sigma^{p,*}, \theta^{p,*}\right) = J\left(u^*, w^*\right). \tag{28}$$

Proof. Suppose that (28) is not true. Then, there exists an $\varepsilon_0 > 0$ and a sub-sequence $\{\sigma^{p_k,*}, \theta^{p_k,*}\}$ such that