Logic, Epistemology, and the Unity of Science 32

Michèle Friend

Pluralism in Mathematics: A New Position in Philosophy of Mathematics

Springer

Pluralism in Mathematics: A New Position in Philosophy of Mathematics

LOGIC, EPISTEMOLOGY, AND THE UNITY OF SCIENCE

VOLUME 32

Logic, Epistemology, and the Unity of Science aims to reconsider the question of the unity of science in light of recent developments in logic. At present, no single logical, semantical or methodological framework dominates the philosophy of science. However, the editors of this series believe that formal techniques like, for example, independence friendly logic, dialogical logics, multimodal logics, game theoretic semantics and linear logics, have the potential to cast new light on basic issues in the discussion of the unity of science.

This series provides a venue where philosophers and logicians can apply specific technical insights to fundamental philosophical problems. While the series is open to a wide variety of perspectives, including the study and analysis of argumentation and the critical discussion of the relationship between logic and the philosophy of science, the aim is to provide an integrated picture of the scientific enterprise in all its diversity.

For further volumes:
http://www.springer.com/series/6936

Michèle Friend

Pluralism in Mathematics: A New Position in Philosophy of Mathematics

Springer

Michèle Friend
The George Washington University
Washington, DC, USA

ISBN 978-94-007-7057-7 ISBN 978-94-007-7058-4 (eBook)
DOI 10.1007/978-94-007-7058-4
Springer Dordrecht Heidelberg New York London

Library of Congress Control Number: 2013953900

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

I dedicate this book to my teachers:
human and equine.

Preface

I came to write about this position as a result of my professional experience. I attended and participated in a lot of conferences, mostly in Europe. They were philosophy, mathematics and logic conferences. I observed that, for the most part, mathematicians and logicians did not behave as though they adhered to a philosophy of mathematics. In particular, with some exceptions, they did not seem to show adherence to one foundation of mathematics in a philosophical way. Some were working on some issues in a foundation, and were wedded to it as a result of their invested time and energy, not so much (again with some exceptions) for philosophical reasons. Yet, my studies in the philosophy of mathematics would have me believe that it is imperative that one have a philosophical outlook or position, and one should work within it. I was puzzled.

Oversimplifying: the philosophers seemed to be convinced that mathematics is *one thing* and that to show this one just pointed to the foundation of mathematics, and this was a particular theory in mathematics. The philosophers seemed to be completely ignoring the fact that there are several rival foundations, and none has a completely privileged position, except maybe Zermelo-Fraenkel set theory – but even that could not support the philosophical claims, since there were all sorts of equi-consistency proofs around. There would be no point in making such proofs if the other 'rival' foundations were for nought. Mathematicians and logicians in their presentations and in casual speech were quite willing to take seriously other theories that conflicted with the ones they were working in. In fact that is one of the reasons they go to conferences: to find out what is going on in other fields, to see how results in one area of mathematics share features with their own. They would quite happily talk of rival foundations in the same breath, and not be casting one away. Instead, they embraced the lot.

I was convinced that if one wanted to give a philosophy of all of today's working mathematics, one had to give a philosophy that was not foundational. I was going to call the position Meinongian structuralism, but Bill Griffiths convinced me that the name was too baroque. It later occurred to me that 'pluralism' would work as a name. Once I fastened on 'pluralism', I noticed the word used by a few philosophers of mathematics such as Shapiro and Maddy. In contrast to the philosophers, mathematicians for the most part behave in a pluralist way. I conclude that pluralism is ‘in the air’. But if we look at how the word is used, we find it is used in so many different ways as to be almost useless! It occurred to me that it would be a useful service to develop a philosophical account of pluralism as a philosophy, as opposed to ‘pluralism’ being used to gesture towards a vague and ambiguous attitude of tolerance.

I confess to feeling I am a bit of a philosophical charlatan, since I hardly think I am doing anything original, again, since the idea is already very much in the air. At other times I think I am a charlatan on the grounds that the position is so obvious, as to be platitudinous. It seems to hardly qualify as a position at all, since it is just an articulation of the prevailing attitude of practicing mathematicians. But, then I quickly realise that this is not at all the case. Once developed in its entirety, I discovered how radical the position is. It is deeply radical. As such, if my arguments are persuasive, then the book will either convert readers, or act as a strong warning to treat the word ‘pluralism’ with care, use it sparingly, or only in the negative. One person’s *modus ponens* is another’s *modus tollens*.

Washington, DC, USA Michèle Friend

Acknowledgments

First, I should like to thank my co-conspirators. Andrea Pedeferri is co-author of two of the chapters of this book. It is through discussions with him, and collaboration in writing, that the chapters really took shape. Joe Mourad acted as my main 'mathematical consultant'. Second, I should like to thank the editorial board of Springer, and the managing team who realised the book. Third, I should like to thank my two academic hosts during my sabbatical year in Europe. The Arché group working on the foundations of logical consequence at the University of St. Andrews received me as a visiting fellow. István Németi and Hajnal Andréka of the Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, received me as a guest.

Close colleagues who gave gentle nudges, corrections and encouragement include: Andrew Aberdein, Norma B. Goethe, Tatyana Romanovskaya, Göran Sundholm and Tadeusz Zawidzki. In the next circle of influence, we find: Nimrod Bar-Am, Diderik Batens, Roy Dyckhoff, Valentina Harizanov, Benedikt Löwe, Colin McLarty, Graeme Priest, Stephen Read, Stewart Shapiro and Neil Tennant. Not all of these will want to be associated with the contents of the book, by the way. I leave the reader to guess who would. The following might not even think of themselves as having had an influence, partly because of their charming sense of modesty. Nevertheless, our discussions and their kindness have had an effect on at least a few sentences in the book: Hajnal Andéka, Paul Churchill, José Felix Costa, Bill Griffith, Nicholas Joll, Brendan Larvor, Bob Lockie, João Marcos, Michael Morreau, István Németi, Olga Opombo, Mark Powell, Guiseppe Primeiro, Georges Rey, Dirk Schlimm, Stewart Shapiro, Michael Sigrist, Gergeli Székely, Sara Uckleman, Petr Vopěnka, Alan Weir and Crispin Wright.

Pluralism in mathematics was neither invented by me, nor was it invented yesterday. It has been in the air for a while, although it has never been articulated as a fully developed position in the philosophy of mathematics. I would have been unable to formulate the position without long deliberation, so less directly, I have been influenced by: Sherry Ackerman, Otávio Bueno, Luiz-Carlos Peirreira, Peter Caws, Peter Clark, David Davies, Bill Demopoulos, Mic Detlefsen, Albert Dragalin, Bob Hale, Michael Hallett, Joachim Lambeck, Michael Makkai, Mathieu Marion, Marcin Mostowski, Alex Oliver, Jeff Paris, Rebecca Pates, Michael Potter, Jehuda Rav, Peter Simons, Jairo daSilva, Hartry Slater, Peter Sullivan, Vítěslav Švejdar and Michael Wright. Again, some of these might deeply regret being mentioned here! They might regret that they did not have *more* influence in guiding me *away* from the pluralist path.

Academics who are not specialists in the philosophy of logic or mathematics but who also were helpful include all of my colleagues from George Washington University. It is a remarkably collegial department, and they have all been morally supportive, and more concretely they helped to reduce my administrative duties. They include: Jeff Brand, Peter Caws, Paul Churchill, David DeGrazia, Lloyd Eby, Bill Griffith, Josh Miller, Tom Morris, Andrea Pedeferri, Tatyana Romanovskaya, Eric Saidel, Athenassis Samaras, Geert Van Cleemput, Michael Sigrist, (Sigrist, Miller and I shared many a lovely beery evening), Chris Venner, Gail Weiss and Tadeusz Zawidzki.

On a less academic level, I must acknowledge the unbridled support of my parents: Tony and Henriette Friend. My friends tolerated my unsociable behaviour while at home writing and correcting the manuscript. I wrote most of the first draft of the book while on a sabbatical year in Europe. Separately I should like to thank my hosts during that year, in chronological order: my host in England, Graeme Hall; my hostess in France, Odile Geoffroy-Gibson; my host in Scotland, Roy Dyckhoff; and my host in Hungary, György Pándi.

Contents

List of Figures

Chapter 1
Introduction

Abstract The introduction is meant as a guide to reading the book. I briefly describe the parts and individual chapters of the book. I also outline some conventions adopted in the book.

1.1 Introduction

There are four parts to this book. The first is motivational. I give motivations for adopting pluralism from four separate starting points: realism, Maddy's naturalism, Shapiro's structuralism and formalism. For reading this part of the book, I suggest reading the first chapter on realism in order to gain some orientation concerning pluralism, and as an introduction to some vocabulary which is used idiosyncratically. There is a glossary for further reference, or to use as a reminder.

The three other chapters of the first part are self-contained, and are directed towards philosophers with certain inclinations. That is, if one has naturalist inclinations, one should read the naturalism chapter. If one has structuralist inclinations, one should read the structuralism chapter and if one has formalist inclinations, one should read the formalism chapter. If the reader is none of the above, then she can read these chapters only to become acquainted with some motivations for adopting pluralism. This part of the book is not exhaustive in discussing all possible motivations for pluralism. Not only are there only a few non-pluralist positions discussed, but even within the motivational chapters on naturalism and structuralism I target one philosopher's philosophy in this area, not all of the well received versions. The philosophers in question are Maddy and Shapiro, respectively. Motivating pluralism from other starting points is part of the greater pluralist programme. Similarly, comparing and contrasting other positions with pluralism is part of the greater programme. I return to it in one section of Chap. 14.

Pluralism is not just one position in the philosophy of mathematics, it is a family of positions. This is one of the reasons I call it a 'programme'. This book gives a starting push to the programme. Different members of the family are distinguished

M. Friend, *Pluralism in Mathematics: A New Position in Philosophy of Mathematics*, Logic, Epistemology, and the Unity of Science 32, DOI 10.1007/978-94-007-7058-4_1,

along the dimensions of: degree (of pluralism), underlying logic and sort of pluralism. Examples of sorts are: foundational, methodological, epistemological and alethic. The pluralist not only distinguishes himself from other positions in the philosophy of mathematics, he is inspired by other positions. In particular, the pluralist retains lessons from the realist, the naturalist, the structuralist, the formalist and the constructivist. The last source of inspiration will be put to work in the fourth part of the book.

The second part of the book concerns the details of how to cope with the inevitable conflicts and contradictions which surface when entertaining very different philosophical positions and mathematical theories under one theory. This part concerns reasoning in the light of contradiction and conflict. I first present pluralism as a philosophical position in its own right. I make reference to a paraconsistent formal system as a guide to reasoning about conflicting ideas *without* necessarily having to decide that one idea is correct and the other is not, or that one position 'wins' over another. Sometimes one does win, but in more sophisticated arguments, there will not be a clearly correct position. Since I am presenting a philosophical position, I can only make *reference to* a formal system of logic, as opposed to *using* a formal system. This is because the pluralist philosopher is not comparing propositions or well-formed formulas and reasoning from these to theorems or conclusions. This is why I write about using a formal logic metaphorically in Chap. 7. The logic is used to set parameters and to sanction and guide the reasoning about whole theories. For this reason, the notions of rigour of argument and the idea of communication become very important to the pluralist. These are discussed in Chaps. 8, 9, and 14. We shall see in these chapters a tension and a struggle with meaning, ontology and truth. These are traded for the more practice reflecting: communication, rigour and protocol. The struggle is the struggle of the pluralist. It is the cost of taking on board the task of explaining what mathematics is about without compromising on the real subtleties in operation in mathematics.

In the third part of the book I work with the paradoxes of tolerance and the idea of transcending one's own position. The paradoxes of tolerance surface when we ask the questions: 'does it makes sense to be tolerant towards those who are intolerant of our own tolerant position?' and 'are there not some things the pluralist is intolerant towards?' In Chap. 10, I discuss the paradoxes, and explain that the pluralist is not a *global* pluralist, but a *maximal* pluralist. He would be a global pluralist if he were tolerant of everything. He is a maximal pluralist if he is as tolerant as possible without his position becoming self-defeating. The maximal pluralist is intolerant towards dogmatism, and particular moves made by, say, realists, naturalist and structuralists against other positions, and pluralism.

In Chap. 11 we visit the more subtle question of whether the pluralist is pluralist towards himself. Another way to ask this is to ask if the pluralist is dogmatic in the ways identified in the previous chapter. To answer this question, we first explore Meyer's collapsing lemma. I use this to a very modest end, to indicate that the paraconsistent logician wedded to LP (a particular paraconsistent logic) will have to be pluralist about interpretations of his logic. The result generalises to anyone who is both pluralist and fixes on a particular logic to underpin his pluralism.

In contrast, if we are logical pluralists, then we have another reason to be pluralist about pluralism. Using other logics will give a different flavour to pluralism. The pluralist is pluralist towards himself just in virtue of admitting alternative logical formal systems to underpin pluralism. Again, *qua* programme, here we see that we can make different versions of pluralism by adopting different underlying formal logical systems.

The fourth part of the book puts the pluralist to work. I indicate some sample pluralist exercises. The first concerns the notion of proof in mathematics. The pluralist analyses the notion of proof as it is used by the working mathematician and draws conclusions about the role of proof in mathematics. In Chap. 13, I undertake a different sort of exercise. This concerns pluralism about conflicting *philosophies* of logic. In Chap. 14, I launch three sorts of project, one is to take a feather from Maddy's hat, and identify an aspiration of some mathematicians, articulate and define the aspiration and put it to work to partly resolve a technical problem, and to deepen our understanding concerning the problem. The second project is to explore the notion of rational reconstruction, to see what they can teach us. The third discusses the issue of working in a trivial setting. This part of the book demonstrates how pluralism is programmatic. There is a lot of work for the pluralist philosopher of mathematics. For the reader who is interested in reading as little as possible, while still forming a view of pluralism, I suggest reading Chaps. 2, 6, and 11.

1.2 A Note on Conventions

Definitions for technical terms are usually given at their first mention, but not invariably, for example in this introduction I have used many such words without giving a definition. Technical terms are given a definition in the glossary. The index should provide further guidance.

'The pluralist' is used to name a character who takes on some sort of pluralist philosophy of mathematics. The definite article is used in the same way as when we say 'the logician' and are referring, not to an individual (person) but to a species, or type of person. More technically, 'the pluralist' is not a first-order singular term, but a second-order singular term. Pluralism is a family of positions. As such, the different pluralisms have many features in common, and can all avail themselves of most of the same arguments against other positions.

I use 'he' throughout for the pluralist. This is because I am a 'she' and I do not want to show prejudice. Other philosophical or mathematical characters might be given the preposition 'he', 'she' or 'it'. I use 'it' for the more obscure, remote or extreme positions, which are just philosophical constructs. In these cases it is possible that no one ever did or ever will hold the position. It is supposed that if someone were to hold the position, such a person would not hold it for long. It is more a position to be temporarily entertained than seriously defended. An example is the trivialist. An example is the trivialist position. A trivialist is an 'it'.

Foreign words, and phrases which I wish to emphasise, are italicised. There should not be very much confusion resulting from italics playing two roles. ‘Or’ is taken as inclusive throughout the text.

There are two chapters which were co-written with Pedeferri. Therefore, in both these chapters I use the first person plural. In other chapters, I use the first person singular. After the acknowledgments, preface and introduction, names of philosophers or mathematicians are only ever written using the family name.

My punctuation might also raise eyebrows. I part company with Fowler and Gowers, and put a comma after ‘for’ when it is used in the sense close to that of ‘since’. I part company with the conventions of the grammar check on my computer, and do not always precede ‘which’ with a comma. Single quotation marks, or inverted commas, are use to show that a term is a technical term. They are also sometimes used to show mild irony. For the most part, they could be replaced by the words: ‘as it were’ or ‘so called’, but this would be more tedious than using the elegant single inverted comma.

Finally, I should add a word about the index. The index is lengthy, and has some odd entries. The purpose of the index is twofold. One purpose is for a reader interested in, say, finding out what I have to say about realism, and nothing else. But the other use is when a reader wants to re-read, say, an example, and remembers that it concerned an unusual phrase such as: ‘the inconsistency of UN declarations’. For this second reason, odd entries, such as, ‘UN declaration’ are in the index.

Part I
Motivations for the Pluralist Position; Considerations from Familiar Positions

Chapter 2
The Journey from Realism to Pluralism

Abstract In this chapter I take the reader on a journey from a naïve realist position through to the beginnings of pluralism. Some simplifying assumptions are made, but this is done in order to *introduce* some of the concepts we find in pluralism, not to defeat all realist positions. In particular, in order to set the stage, the naïve realist will take Zermelo Fraenkel set theory to be the foundation for mathematics in a philosophically robust sense of capturing the essence, ontology and absolute truth of mathematics. The reader is given several reasons to abandon the naïve realist conception and to consider a more pluralist conception. The main aspect of pluralism discussed here is pluralism in foundations. 'Pluralism in foundations' is an oxymoron, and therefore, is unstable. Some other aspects of pluralism are then introduced: pluralism in perspective, pluralism in methodology and pluralism in measure of success.

2.1 Introduction: ZF Monism

Since this is the beginning of the of book, I should issue a warning. Especially in this chapter, I tell some lies. Or, rather, I begin with oversimplifications. This will be alarming for the more sophisticated readers. However, rest assured that as we proceed through the book, most of these oversimplifications will be re-expressed, refined, honed and made more explicit. The reason for the oversimplifications is that since this is a new position in the philosophy of mathematics, I prefer to start with some very naïve ideas.

In explaining a philosophical position, it is sometimes useful to start from a quite different, but easily recognised position, even if we think no one occupies it.[1] Realism is a familiar position in the philosophy of mathematics. However, since

[1]As we shall see in the subsequent chapters, I shall take the reader through journeys with other starting points: naturalism, structuralism and formalism. Balaguer (1998) works through different versions of realism, and teaches us to use the word carefully. It is quite possible that no one

M. Friend, *Pluralism in Mathematics: A New Position in Philosophy of Mathematics*, Logic, Epistemology, and the Unity of Science 32, DOI 10.1007/978-94-007-7058-4_2,

'realism' is such a broad term with so many connotations and aspects, I shall fix the term and restrict 'realism' to a 'monist foundationalist' position, where Zermelo-Fraenkel set theory (henceforth: 'ZF') is The Foundation.[2]

Explaining and defining the terms just used: ZF is an axiomatic theory. Zermelo developed most of the axioms and Fraenkel added the axiom of replacement (Potter 2004, 296). The theory is very general. In ZF we study sets of objects, combinations of sets, the comparison of sets with each other, and the creation of one set from another, or of a new set from several others: for example, by taking their intersection or union.

Definition[3] *The Foundation* is an axiomatically presented mathematical theory to which all or most of successful existing mathematics can be reduced. It can be used normatively to exclude from *bona fide* mathematics any *purported* mathematics which cannot be reduced to the axiomatic theory.

Definition *Successful existing mathematics* is the body of mathematical theories and results about those theories that are currently judged by the mathematical community to be 'good mathematics' (as indicated by publication, reference in discussion, use in classrooms and study groups, airing at conferences and so on). This will include past mathematics not presently under mathematical investigation, but, for all that, not dismissed as bad mathematics.

What counts as successful existing mathematics is revisable. We might find out that what we thought was a good mathematical theory turns out not to be. Thus, 'successful existing mathematics' is a vague term, but the imprecision of the boundaries of application of the term need not concern us in the present context.

Definition The *monist foundationalist* believes that there is a unique correct, or true, foundation for mathematics, and uses The Foundation normatively to determine what is to count as successful existing mathematics.

What might motivate someone to adopt monist foundationalism? In the late nineteenth century and the early twentieth century, we developed various set theories. They were not all fully axiomatised at first. Cantor's set theory was not presented as a fully axiomatised theory at all. Despite they're not being presented as fully axiomatised, we discovered that set theories were very powerful. By 'powerful' we mean that a great deal of mathematics can be reduced to set theory. That is, we can translate, say, arithmetic, into the language of, say, ZF, avail ourselves of the axioms and inference rules of the proof theory of ZF, contribute some definitions in the language of ZF, and obtain, through proof, a number of theorems or 'results'.

---

presently holds the position I give here. It is, admittedly, a caricature. That does not matter for present purposes, since (1) the point is to start from a familiar position, not an occupied and carefully defended position, and (2) this chapter is not meant as a knock-down argument against realism in all its forms. Rather, we begin with a naïve and familiar view in order to introduce pluralism.

[2] This is quite different from a 'full-blooded realism' (Balaguer 1998, 5).

[3] The definitions in this chapter are to be read as working definitions. As such, in a more exacting context, they might require further refinement. Definitions are repeated in the glossary.

We can then translate back into the language used in the original arithmetic. If we compare the results we obtained in the original arithmetic to those we obtain in the ZF version of arithmetic, then we can prove that we can in principle reproduce all of the results of the original arithmetic in ZF. That is, there is no theorem of arithmetic, which does not have an analogue in ZF. Therefore, in principle, there is a complete reduction of arithmetic into ZF. The *power* of set theory consists in the fact that not only arithmetic, but also analysis and geometry, and therefore most of working mathematics can all be reduced to set theory. Because of the power of ZF, it can be presented as a candidate for founding mathematics.

ZF was not the only set theory developed. There were (and still are) rival theories of sets, and there arose problems with some of the theories with the discovery of paradoxes. Even apart from the paradoxes, other conceptual puzzles surfaced such as how to conceive of very large totalities, which many of us now think of as proper classes. These were both philosophical and technical problems. The paradoxes and puzzles produced a crisis in mathematics (Giaquinto 2002) and hailed the foundational and axiomatic movements. It was thought that mathematics needed a 'secure' foundation, since it was clear that some mathematical activity was deeply flawed. Philosophers, or professional mathematicians assuming a philosophical role, (henceforth: philosophers)[4] contributed in culling some set theories, such as the so-called naïve theories. The culling did not eliminate all but one set theory. So we had no clear unique founding set theory, we had several. Nevertheless, we can say that presently we have honed in on one. Under the received view today, we can say that ZF is the 'orthodoxy' of mathematics (Maddy 1997, 22). 'Orthodoxy' can be taken to mean 'the most accepted theory' or reference point for mathematics. Or, it can mean that ZFC (ZF with the axiom of choice) 'codifies current mathematical practice' (Hrbacek et al. 2009, 2). How might a philosopher interpret such phrases? There are conceptually distinct roles that ZF can play as 'orthodoxy'. Let us start at one extreme. The position: monist foundationalism reads 'orthodoxy' to mean that ZF sets the parameters for what is to count as mathematics. The reason for starting with this is not plausibility, but familiarity and conceptual simplicity. Philosophers are all familiar with some (less extreme) version of realism. At this extreme end of realism, The Foundation plays the following four roles.

1. All of what is counted as 'mathematics' has to be reducible to, or can be faithfully[5] translated into, The Foundation. The Foundation gives the scope of

[4]Obviously, being a professional mathematician does not preclude one from having philosophical thoughts or from writing quite philosophically about mathematics. The distinction here is not professional but conceptual, in the sense of philosophical and mathematical problems or puzzles requiring different sorts of solution.

[5]By 'faithfully' I mean that the language being translated into, here the language of ZF, has the expressive power to capture the nuances of the original concepts as expressed in the original language. A test for loyalty of a definition, say, would be that analogues of all of the same theorems can be derived when the definition is expressed in ZF as can be derived using the definition of the original language, all other definitions, theorems, lemmas and proof techniques remaining equal. In contrast, a reduction would be unfaithful if fewer or more (non-equivalent) theorems could be derived. Ancient Latin can only make an unfaithful translation of a modern computer manual.

the correct use of the word ‘mathematics’, or, we might say, the foundational theory determines the extension of the term ‘mathematics’. We might call this ‘the semantic determining role of The Foundation’.

Another, but related, role is that

2. the foundational theory tells us what the basic ontology of mathematics is: what it is that mathematicians ultimately study. In the case of ZF, it is sets, and not, for example, lines, planes, numbers or cuts. We might call this ‘the ontological role of The Foundation’.

As a corollary,

3. what counts as correct, or legitimate methodology is also determined by ZF. This is ‘the methodological role’ played by The Foundation. We give some axioms, elaborate definitions and then prove theorems within ZF.

We might even

4. confer an epistemic role to ZF, by saying that to really understand and know mathematics, we have to study set theory. The rest of mathematics, written in other languages, is a pale imitation, and studying mathematics, not presented as a part of set theory, might even mislead us into thinking that we know an area of mathematics when we do not. Call this ‘the epistemological role of The Foundation’. All these roles meant to have normative force over the practice of mathematics.

To hold that ZF plays all four roles is quite extreme, but this is where we shall begin our journey. The monist foundationalist who confers all four roles on the foundational theory is also the most extreme opponent to pluralism. So the journey from monist foundationalism to pluralism is long. In the course of the journey, we shall meet considerations that trouble the extreme position. Since considerations are not full arguments, each elicits different legitimate reactions. There will be better arguments in the chapter on structuralism. Thus, before we see the considerations we should add a note about how to think of them.

Upon thinking about the considerations, a reader might be prompted to muster arguments *against* the pluralist, or she might modify, or even change, her position. Thus, the considerations, can be thought of as: (a) points of re-entrenchment for the convinced monist foundationalist, (b) calls for conservative modification of the monist foundationalist view or (c) points of rejection or doubt towards monist foundationalism. The last leads us closer to the pluralist position. Since not every reader will be willing to follow me for the whole journey, we can think of the journey as an exercise in mapping out the philosophical territory and discovering where one stands *ab initio* (Fig. 2.1).

The monist foundationalist holds an extreme position because it has a vision of reformation.

Definition *The reformation* is a movement to constrain successful existing mathematics by The Foundation.

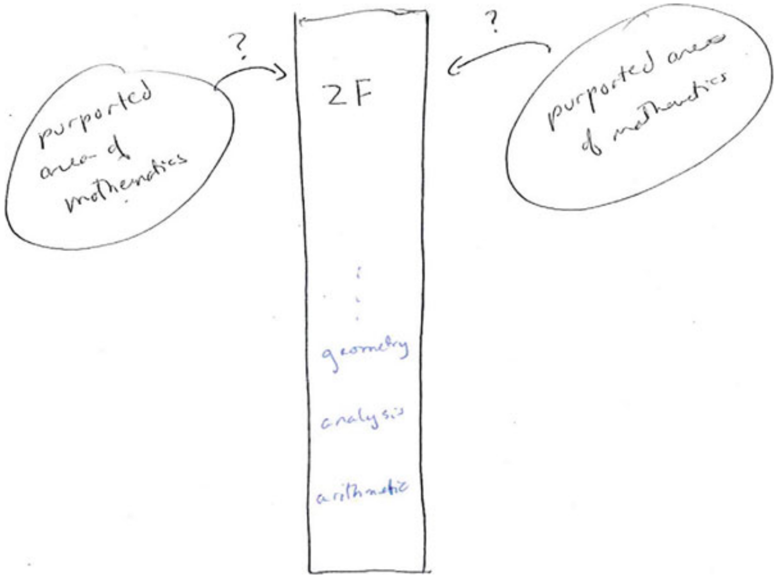

Fig. 2.1 ZF set theory as a foundation

If ZF is the orthodoxy of mathematics in a strong sense and is a good candidate for The Foundation, then we might think that we have an equal trade-off between good mathematics, as practiced, and set theory. If we have an equal trade-off, then we can do mathematics in the language of set theory or we can do mathematics in the original language developed for that theory, and the two are equivalent. In other words, we could set up two communities of mathematicians. One would continue to work in the languages of our plethora of mathematical theories: Euclidean geometry, topology, calculus, algebra and so on, but not set theory directly (in the sense of using only the language of set theory). The other would simply work in set theory. According to our reducing results concerning The Foundation, in the long run, the plethora community would produce a number of results; and the set theory community would produce analogues of all of the results of the plethora community. The order of producing the results would be different – because notation is differently suggestive, but 'at the end of the day' equivalent results would be generated. If this is an accurate prediction of what would happen, then this confirms (in a somewhat circular manner) that we have an equal trade-off.

However, the prediction would probably not be met. This is because the set theoretic community will produce some results not produced by the plethora community. The latter would concern results unique to set theory. It would seem that the set theory community is, therefore, *better off*, at least in the long run.

We now have good reason to initiate the reformation. It would be a wonderful feat to reform mathematical practice by stipulating that we *only* do mathematical work in the language of set theory, and we confine ourselves to the axioms of set theory and are allowed to introduce definitions only in the language of set theory. This would clear up misunderstandings, cut down on the time spent learning new symbols and vocabulary, and cut out all of the work which we do showing that two theorems in different areas of mathematics are equivalent in some respects since this would be clear and explicit if our work was all done within set theory in the first place. Under the reformation, mathematics would become an explicitly unified discipline.

Unfortunately, the suggested reformation would incur considerable loss. It is not at all clear that the set theory community simply reproduces all of the results of the plethora community, plus some more. Sometimes we use one area of mathematics to inform us about another area. We translate from the first area into the second, make a proof in the second and re-translate back to the original area. The reason for taking this circuitous route is that the execution of the proof, and what to look for, are much more obvious in the second area than in the first. For example, Arrighi and Dowek (2010) turn to quantum computing to make some sense of the notion of computable function in a space of infinite dimensions. Note that, strictly speaking, they could have generated the same results in the original classical theory.[6] However, it would not have been obvious, and it would not have been at all evident in the classical logic framework. Lobachevsky, whose work we shall discuss later in the book, turns to hyperbolic geometry to make sense of the notion of an indefinite integral in Euclidean geometry. So there is a heuristic advantage, and maybe even an epistemic advantage to working in different frameworks or theories. Moreover, it would be a mistake to think that these are isolated cases. Therefore, this is one reason to be cautious about the reformation.

However, the reformer would be quite right to retort that this is not a serious objection. The difference between the plethora community and the set theory community influences the *order* in which results are discovered, not in the body of results themselves. Furthermore, strictly speaking, and as we noted earlier, the set theory community will produce some results not produced in the other community. Moreover, their work will be more efficient, since they are not doubling up on results, and then have to prove the equivalence of theorems.

Notice that this retort is strongly underpinned by the sort of realism that emphasises monism in the sense of pre-supposing that there is a unique body of truths, or theorems, to be discovered. If we do not hold this pre-supposition, then the retort carries no weight. But it is a difficult presupposition to give up altogether, as witnessed by the very fact that philosophers are at pains to give arguments as to why non-uniqueness should not worry us. For example, see Balaguer (1998). Let us introduce some more vocabulary, and re-express the two antagonistic positions in that new vocabulary.

[6]This point was made in the oral presentation of the material, (Computability in Europe 2010) but is not obvious in the written version.

Definition *Realism* in mathematics has two conceptually distinct versions: realism in ontology and realism in truth-value (Wright 1986, 9).

Definition *Realism in ontology* is the position that the ontology of the subject we are realists about is independent of our investigations or knowledge.[7]

Some pluralists are anti-realist in ontology.

Definition *Anti-realists* are all those who are not realists. Following Wright (1986, 2), there are two sorts. Anti-realists can assume just a negative view *vis-à-vis* the realist and be sceptics. On the positive side they can be idealists, who believe that it is our ideas that shape the world around us, and determine our ontology. The idealist anti-realist is someone who epistemically constrains truth (rejects verification-transcendent truths).[8]

Definition A *realist in truth-value* of the sentences of a theory holds that the truth-value of sentences of a theory is independent of our ability to judge or establish or discover them.

Some pluralists are anti-realist in truth-value.

Note that one person can be a realist about one area of discourse and an anti-realist about another; for example, it is common to be a realist about physical objects, but an anti-realist about humour. Such split positions are fairly common. It is less common to be a global realist or anti-realist. Returning to mathematics, the anti-realist thinks of the mathematician as a sort of creator. In contrast, the realist thinks of the mathematician as a discoverer, who then enters the discoveries in a well-organised form in, what is suggestively called 'The Book of Proofs'.

Definition *The Book of Proofs* is a unique book that records all of the proofs of mathematics made in the foundational theory in normal form.[9]

Under this conception, mathematics consists in the results of the completed Book of Proofs. The process of doing mathematics is subservient to the discovery of those results.

The alternative anti-realist view emphasises the epistemology over the ontology, and thinks of mathematics as a process that leads to results. The importance of results lies in their continuing the process of establishing knowledge – 'results' are not ends. They are steps in a process.

---

[7]Ontology is usually presupposed to be consistent. There are no impossible objects, there are no pairs of objects whose existence precludes each other. Of course, paraconsistent ontologies are a different matter (no pun intended). For our gross sketch, we need not consider this added complexity.

[8]In the last chapter of the book, I am more explicit and subtle than this. It will turn out that the pluralist is, in some respects, a type of sceptic, and he is neutral on the realist, idealist axis of debate; but this added subtlety will be introduced in due course.

[9]The idea of The Book of Proofs has a history. In the original conception, all perfect proofs were entered. There was no guarantee that there would only be one proof for each theorem, since there was no presupposition that there was only one founding theory for mathematics. However, if we assume monism in foundations, then The Book of Proofs will only have one proof per theorem.