Mathematical and Analytical Techniques with Applications to Engineering

Petre P. Teodorescu

Treatise on Classical Elasticity

Theory and Related Problems

Springer

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Treatise on Classical Elasticity

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Petre P. Teodorescu
Faculty of Mathematics
University of Bucharest
Bucharest
Romania

ISSN 1559-7458 ISSN 1559-7466 (electronic)
ISBN 978-94-007-2615-4 ISBN 978-94-007-2616-1 (eBook)
DOI 10.1007/978-94-007-2616-1
Springer Dordrecht Heidelberg New York London

Library of Congress Control Number: 2012952139

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

The mechanics of deformable solids brings its contribution both to the knowledge of the phenomena of the surrounding physical reality, from a theoretical point of view, and to the determination of the state of strain and stress in various elements of construction, from practical considerations. The deformable solids have a particularly complex character; a mathematical modelling of them is not simple and often leads to inextricable difficulties of computation. One of the most simple mathematical models and, at the same time, the most used one, is the model of the elastic body—especially the linear elastic model; despite its simplicity, even this model of real body may lead to great difficulties of calculation.

In general, the engineering constructions have been based, during the centuries, on empirical methods; beginning with the seventeenth century, one has obtained a lot of results, which form what now is called “Strength of Materials”, where simplifying supplementary hypotheses have been introduced. As a matter of fact, this denomination is not a proper one, because it corresponds only to a mechanical phenomenon modelled by the so-called “strength theory”; we maintained this denomination, being still currently used. The theory of elasticity, chapter of the mechanics of deformable solids with a theoretical character, succeeds to express better the physical phenomenon, giving results closer to the reality, in certain limits; it became a science only at the middle of ninth century, being in continuous development even today.

The practical importance of a book on the theory of elasticity, which is—at the same time—an introduction to the mechanics of deformable solids, consists in putting in evidence points of view and scientific methods of computation in a domain in which simplified methods or with a non-accurate limit of validity are still used. The actual technical progress and the necessity to use a minimum of materials in various constructions ask for a better determination of the state of strain and stress which takes place in a civil or mechanical construction; the engineering design may be thus improved.

The first eight chapters deal with the construction of the mathematical model of a deformable solid, giving special attention to the linear elastic bodies; the formulation of the fundamental problems is followed by their solution in displacements end stresses. The importance of the concentrated loads is put into evidence, as well in the case of Cosserat-type bodies. Another group of four chapters contains static and dynamic spatial problems, treated systematically by the same method of potential functions.

The following two chapters deal with some special problems: particular cases, treated in the same systematical manner and the case of anisotropic and non-homogeneous bodies.

The last two chapters contain introductions to thermoelasticity and linear viscoelasticity. Special accent is put on the solving methodology as well as on the mathematical tool used: vectors, tensors and notions of the field theory. Continuous and discontinuous phenomena and various mechanical quantities are presented in unitary form by means of the theory of distributions. Some appendices give the book an autonomy with respect to other works, a special mathematical knowledge being not necessary.

Concerning the first six chapters, I must mention the kind co-operation of Professor Vasile Ille, Technical University of Cluj-Napoca, who unfortunately has passed away. I am grateful to Mariana Gheorghiţă for her valuable help in the presentation of this book. The excellent cooperation of the team of Springer, Dordrecht, is gratefully acknowledged.

The book covers a wide number of problems (classical or new ones) as one can see from its contents. It used the known literature, as well as the original results of the author and his more than 50 years' experience as Professor of Mechanics and Elasticity at the University of Bucharest. It is addressed to a large circle of readers: mathematicians (especially those involved in applied mathematics), physicists (particularly those interested in mechanics and its connections), engineers of various specialities (civil, mechanical engineers, etc., who are scientific researchers or designers), students in various domains etc.

Bucharest, Romania, January 2013 P. P. Teodorescu

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Chapter 1
Introduction

The mathematical model of mechanics of deformable bodies is shortly presented in this chapter; stress is put on the notion of mechanical system. Using these introductory notions, one can pass to a mathematical study of these mechanical systems.

1.1 Aim of Mechanics of Deformable Solids

The scope of mechanics of deformable solids is to determine the state of strain and stress of a solid body subjected to the action of external loads in static or dynamic equilibrium.

In what follows we make some general considerations and put in evidence the basic computational hypotheses, which specify the mathematical model of the considered solids.

1.1.1 General Considerations

To specify the position of mechanics of deformable solids with respect to various technical disciplines, especially strength of materials, one must discuss its mathematical model. To do this, it is necessary to put in evidence the state of strain and stress of a solid body.

P. P. Teodorescu, *Treatise on Classical Elasticity*, Mathematical and Analytical Techniques with Applications to Engineering, DOI: 10.1007/978-94-007-2616-1_1,

1.1.1.1 State of Strain and Stress

Under the action of *external loads* (either concentrated or distributed forces, concentrated or distributed moments, other concentrated loads (applied on the external surface of the body or within it), volume forces or moments, inertial forces, loads due to a thermal field, loads due to an electromagnetic field, loads due to radioactive radiations, deformations induced by various means, imposed displacements etc.), the particles (infinitesimal elements) constituting a solid body change (eventually in time) the positions (with respect to a fixed frame of reference) they had before the action of these forces. If, by a translation and a rotation, we can impose on the particles of the body undergoing the action of the external forces to regain the positions they had before the application of these loads, then we have to deal with a *rigid body motion*. Otherwise, the body is subjected to a *deformation*. The totality of the deformations (which we shall define later) undergone by a particle of the body constitutes *the state of deformation at a point* (the centre of mass of the particle). The totality of the states of deformation corresponding to all the points of the solid body constitutes *the state of deformation of the body*.

At the same time, the notion of *displacement* may be emphasized. The totality of the displacements corresponding to a point of the solid body constitutes *the state of displacement at a point*. The totality of the states of displacement corresponding to all the points of the solid body constitutes *the state of displacement of the body*. Often, by the state of deformation of the body its state of displacement is equally intended, the final scope being to determine the components of the displacement vector. In general, we use the notion of *state of deformation* (*state of strain*).

The bodies undergoing only rigid body displacements are called *rigid solids*; the other solid bodies are called *deformable solids*.

Owing to the deformations, the balance (static or dynamic) of the forces linking the various particles of the body is broken and additional internal forces arise; the totality of these internal forces (called *efforts*, if one deals with an arbitrary section in the body, or *stresses*, in case of efforts which act upon a unit area), corresponding to a particle, constitutes *the state of stress at a point* (the centre of mass of the above considered particle). The totality of the states of stress corresponding to all the points (particles) of the solid body constitutes *the state of stress of the body*.

The state of deformation (or the state of displacement) and the state of stress of a body form *the state of strain and stress* of it.

1.1.1.2 Mechanics of Deformable Solids and Strength of Materials

The purpose of *mechanics of deformable solids* is to determine the *state of strain and stress* of a solid body subjected to the action of external loads that are in static or dynamic equilibrium. One has to deal with a theoretical discipline, the results of which are of great importance for many technical disciplines.

In general, the *strength of materials* deals with the same problems as the mechanics of deformable solids. But this discipline involves a *technical solving method* that for each problem or group of problems formulates separate hypotheses, reflecting more or less the physical reality in isolated cases. On the basis of these *working hypotheses*, one obtains the equations which govern the phenomenon and endeavour thus to simplify the computation as much as possible.

By contrast, the mechanics of deformable solids is a chapter of mechanics, fundamental science of nature, which is based on a *mathematical theory*, built on *unitary hypotheses*; it is applicable whatever the form of the body, the kind of the material or the manner of action of the external loads may be. Each problem is specified by its own *limiting conditions* (*boundary conditions* and *initial conditions* (for the movable bodies)), since the fundamental hypotheses always remain the same. The computational methods applied are general methods leading to an accurate solution of the problem (more properly, as accurate as necessary and possible).

In the strength of materials, the construction elements are simplified, by reducing them to the *medium fiber* or to the *middle surface*, so as to diminish the number of variables involved in the computation. On the other hand, *additional simplifying hypotheses* concerning the distribution of the deformations on a cross section of the body, e.g., the hypothesis of plane cross section [96], as it has been given by Jacob Bernoulli [4] for the *straight bars* (equally applied later in the case of curved bars). If to this hypothesis is added a linear relation between strain and stress (Hooke's law) and the mechanical equilibrium is written for a portion of the bar, then we shall find the known formulae that give the so-called *simple stresses* for a straight bar. Assuming that the strains are small with respect to unity and can be thus disregarded, it follows that the principle of superposition of effects can be applied, hence the results obtained can be extended to the case of *combined stresses*. Thus, computation methods valid for ordinary beams are obtained.

Let us also consider the hypothesis of *the straight line element*, due to Kirchhoff [38] for *thin plates* and to A.-E.-H. Love [43] for *thin shells*, which leads to the usual computational methods for these construction elements.

In case in which the state of strain and stress is to be determined for wall beams, thick (or moderately thick) plates, blocks or in case of stress concentrations etc., more accurate computational methods should be applied. In this case, it is no longer permitted to consider the equilibrium of a finite element (section), but that of an infinitesimal element cut out of the body. A *local study* in the neighbourhood of every point of the body is completed by a *global study*, referring to the whole body. The computational method (exact, approximate and even experimental) is general and is only restricted by difficulties which are liable to arise in practical applications.

The methods of the mechanics of deformable solids allow one to verify the limits of applicability of the results obtained by the methods of the strength of materials. Chiefly, it permits one to solve problems that cannot be tackled by more elementary methods, such as all *three-dimensional problems*.

Hereafter, we shall apply the methods of the mechanics of deformable solids.

1.1.2 Models in the Mechanics of Deformable Solids

To make a study of a deformable solid from the mechanical point of view, it is necessary, firstly, to set up a mathematical model of it.

Usually, by *model* one means an object or a device artificially created by man, resembling to a certain extent to another one, the latter one being an object of scientific research or of practical interest. The scientific notion of model refers to a certain technique to know the reality representing the phenomenon under study by an artificially-built system. The most general property of a model is therefore its capacity to reflect and to reproduce properties and phenomena of the objective world, as well as their necessary order and their structure.

1.1.2.1 Technical Models

From the very start, the models can be divided into two large classes: *technical* (or material) *models* and *ideal* (or imagined) *models*; this division is made according to the manner of construction the models and according to the means by which the objects under study may be reproduced.

The technical models are created by man, but they exist objectively, independently of his awareness, being materialized in metal, wood, electromagnetic fields etc. Their purpose is to reproduce for a cognitive goal the object under study, so as to put in evidence its structure or some of its properties. The model can or cannot retain the physical nature of the object under study or its geometrical likeness. If the similitude is maintained, but the model differs in its physical nature, we have to deal with analogic systems. For instance, electrical models can reproduce processes analogue to those taking place in the mechanics of deformable solids, which are qualitatively different, but are described by similar equations. These models, as well as other of the same kind, are classed as *mathematical models*.

One can construct such models, e.g., to study the torsion of a cylindrical straight bar of arbitrary simply or multiply connected cross section. If the bar is homogeneous, isotropic and linearly elastic (subjected to infinitesimal deformations), then the phenomenon is governed by a Poisson type equation in B. de Saint-Venant's [12] theory. L. Prandtl showed that the same partial differential equation is valid in the case of a membrane supported on a given contour and undergoing a constant internal pressure; if the contour is similar to the frontier of the plane domain corresponding to the cross section of the straight bar, then we have a correspondence of the boundary conditions, hence the classical *membrane analogy* (or of the soap-bubble film) is obtained. Other analogies for these problems are used too, i.e.: electrical modelling, optical interference modelling, hydrodynamical modelling etc.

It is interesting to remark that the above analogies can moreover be extended to more complex phenomena. Thus, in case of a multiply connected domain (especially that of a tubular bar with intermediate thin walls), it is possible to use a membrane supported along internal contours. As well, in case of a perfect plastic straight bar, the analogy with small sand heaps of A. Nadai is used. In the case of elastic plastic bodies, the analogy to be used is obtained be combining the Prandtl analogy with the Nadai one.

A slightly different *mathematical analogy* is that between *the classical plane problem of the theory of elasticity* and *the bending problem of a thin plate*. In this case, the model (a thin plate subjected to a bending in the absence of loads normal to its middle plane) has the same physical nature as the object under study (thin plate loaded in its middle plane, therefore undergoing a plane state of stress). By bending the thin plate, which is thus undergoing certain displacements and slopes of the middle surface, deformed along the contour, one obtains an image of the G.-B. Airy stress function of the plane problem, for which the values as well as those of its normal derivative along the boundary are given. This analogy, due to K. Wieghardt [99], can be extended to multiply connected domains or to nonhomogeneous and anisotropic bodies. Let us also remark that the contour values of the Airy function and of its normal derivative are obtained as the bending moment and the axial force in a fictitious bar along the boundary of the considered plane domain, subjected to the action of given external loads, the contour being travelled through counterclockwise, from an arbitrarily chosen point; by so doing, an interesting analogy appears, which is particularly useful in computation. Recently, this analogy has been partially extended to the space problem. It is also to be mentioned that, by a judicious choice of the similitude, the model can be made of a material with more accentuated bending properties, which is very important from a practical standpoint.

Another type of technical model, used in the mechanics of deformable solids, is that which corresponds to the intuitive notion of model. Various construction elements are performed partially or wholly at a reduced scale; this allows to obtain results concerning the maximal deformations and stresses liable to occur. These models can be built up of the same material as the object to be studied or of other materials, which may raise rather difficult similitude problems.

Technical models of a slightly different type are used in photoelastic studies; in this case, although the geometrical likeness is preserved, the physical nature of the model is different from that of the object (a thin plate, made of an optically sensible material, which acquires birefringence properties under the action of external loads imposed in its middle plane). In this manner, an additional phenomenon appears that enables one to specify the various properties related to the object which is studied.

1.1.2.2 Character of the Ideal Models

Generally, the ideal models are not materialized and—sometimes—they neither can be. From the viewpoint of their form, they can be of two types.

The models of first order are built up of intuitive elements having a certain likeness to the corresponding elements of the real modelled phenomenon; we observe that this similitude must not be limited only to space relationships but can be extended also to other aspects of the model and of the object (e.g., the character of the motion). The intuitiveness of these models is put into evidence on one hand by the fact that the models themselves, formed by elements sensorial perceptible (planes, levers, tubes, fluids, vortices etc.) and on the other hand by the fact that they are intuitive images of the objects themselves. Sometimes, these models are fixed in the form of schemata.

The models of second order are systems of signs the elements of which are special signs; the logical relations between them form—at the same time—a system and are expressed by special signs. In this case, there is no likeness between the elements of the models and the corresponding elements of the object. These models have not intuitiveness with respect to geometrical likeness or physical analogy; they have, by their physical nature, nothing in common with the nature of the modelled objects. The models of second order reflect the reality on a gnosiological plane, by virtue of their isomorphism with the reality; a one to one correspondence is thus assumed between each element and each relation of the model. These models reproduce the objects under study in a simplified form, constituting thus (as, in fact, all models do) a certain idealization of the reality.

The types of ideal models mentioned above can be considered as *limit cases*. Indeed, there are ideal models combining common features of both model types which have been described; they contain systems of notions and axioms which characterize quantitatively and qualitatively the phenomena of nature, for instance representing mathematical models. Such models are particularly important and their systematic use has permitted—among other things—the large development of the mechanics of deformable solids in the last time.

The basic dialectic contradiction of modelling (the model serves to the knowledge of the object just because it is not identical with the latter one) is useful, e.g., to put into evidence the properties of continuous deformable solids. In fact, a model contains the more information concerning the object, when it is more like this one. Nevertheless, the physical reality is rather intricate; the contradiction is solved by using a sequence of models which are more and more complete, where each one brings new contributions to the knowledge of real deformable solids. We shall endeavour to emphasize the very process of continuously improving methods of the mechanics of deformable solids, a process constituting—as a matter of fact—*the main tenor* of the development of this branch of mechanics.

In general, after a certain ideal model is adopted, it is absolutely necessary to compare the results obtained by theoretical reasoning with physical reality. If these results are not satisfactory (sometime this can occur between some limits, which can be sufficiently narrow), then it is necessary to make corrections or to improve the chosen model. In fact, this is the manner in which the mechanics of deformable solids was developed, the word "model" being more and more used by researchers dealing with this branch of mechanics.

1.1.2.3 Ideal Models in General Mechanics

Both *the general* (classical) *mechanics* and *the mechanics of deformable continuous media* (in particular, *the mechanics of deformable solids* and *the mechanics of fluids*) are studying the objective laws of the simplest form of motion, namely *the mechanical motion*.

First of all, for the study of the mechanical motion, a representation of *space* and *time* is necessary; thus, in classical mechanics, the physical space is the three-dimensional Euclidean space E_3, while the time (considered as universal and characterizing the duration, the succession and the simultaneity of the material processes) is still assimilable to the one-dimensional Euclidean space E_1. Thus, the *geometric models* of space and time, as used in general mechanics, are reflecting properties of the real space and time as *forms of existence of the matter*.

A real movable body is generally thought to be *rigid* and is often reduced to a *particle*. In the same manner, *the systems of particles* studied can be deformable. For various values of the time t in E_1, we are able to find the position of the body or of the system of particles in E_3.

Another intervening element is the cause of the mechanical motion (which, for the sake of simplicity, will be called “motion”). The bodies interact mechanically with one another and in many cases it is difficult to establish the physical nature of this action; generically, it was called *force*. This idea, which originates in the action of a human organism upon the external world, acquires a precise meaning in classical mechanics. It is the abstract expression of the measure of the transmission of motion. Without investigating the nature of the respective force, it was mathematically modelled with the help of vectors (sliding vectors in case of rigid solids and bound vectors in case of deformable systems of particles). We must mention that a mechanical motion could exist even in the absence of any force, namely *the inertial motion* (rectilinear and uniform).

The notion of *mass* of the particle must be introduced; this is a *fundamental property of matter*, exists objectively and is independent of the place where it is measured. I. Newton [56] conceived the mass as the measure of a quantity of matter. The notions of *gravitational mass* and *inertial mass* should be introduced; numerically, they are equal to each other, as it has been proved by L. Eötvös. In this manner, we dispose of various possibilities of measuring the mass.

In the classical Newtonian conception, space, time and mass are considered as independent from one another.

After these elements were introduced, the Newtonian model was born, by adopting the principles of Newton [56] (the principle of inertia, the principle of action of forces, the principle of action and reaction, the principle of the parallelogram of forces and the principle of initial conditions, in a modern view, to can put in evidence its deterministic aspect). The model was verified by direct practice in the case of bodies moving at relatively low velocities (negligible with respect to the velocity of light in vacuum).

1.1.2.4 Introduction of the Ideal Models in Mechanics or Deformable Solids

General mechanics endeavour in general to solve the problems of motion of solids on the assumption that they are rigid, although certain established general results are equally valid for deformable solids (particularly when applied to discrete deformable systems of particles).

In the study of deformable continuous media, the Newtonian model has to be completed; instead of a *rigid solid*, we introduce various models of deformable media. By examining the historic process of the development of these models, from a knowledge of the manner in which bodies undergoing certain actions are deformed and begin to flow, we shall have implicitly the process of the appearance of the theory of elasticity, of the theory of plasticity, of the perfect and viscous fluids, of the rheology etc. Hereafter, we shall deal with *deformable solids* in the frame of classical mechanics.

The most general problem which has arisen can be formulated as follows: Let be a given solid of volume V and boundary S in E_3. On S, the action of other bodies is known; on the other hand, one assumes that the action of other external loads (e.g., within the volume V) is known equally. Owing to these external loads, the given solid is deformed, the boundary S becoming S^*. We wish to find the new boundary S^* and to show how it varies in time; we look for the ratio between the dimensions of the body and the intensity of the external loads as to avoid the fracture of the body.

To this end, in view of a space-time representation, the same $E_3 + E_1$ geometric model is adopted. As new elements, a study is made of *the geometry and kinematics of deformations* and of *the mechanics of stresses*.

1.2 Fundamental Computation Hypotheses. Short Historical Account

As any other discipline, the mechanics of deformable solids (and especially the theory of elasticity) has a few *fundamental hypotheses* which allow us to simplify and generalize the phenomena, retaining the essential ones and rending them accessible to the mathematical computation. It is very important to know these hypotheses if we want to realize the possibility of practical application of the results thus obtained.

After reviewing such hypotheses, we shall supply some general data concerning the development of the mechanics of deformable solids and then insist on the development of the theory of elasticity.

1.2.1 Fundamental Computation Hypotheses

First of all, we present the bodies with which one deals in mechanics of deformable solids, as well as the fundamental computation hypotheses in the theory of elasticity. Then we specify the position of the theory of elasticity in the field of the mechanics of deformable solids.

1.2.1.1 Bodies in the Mechanics of Deformable Solids

Hereafter, we shall give a classification of the solid bodies according to the ratio of their dimensions.

Bodies with a much greater dimension (length) than the other two ones (corresponding to their cross section) shall be called *bars*. To make the notion clearer, we shall give a constructive definition.

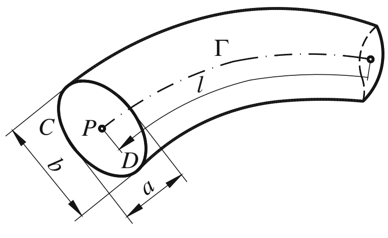
Let there be a curve Γ of finite length l, namely *the bar axis*. In the plane normal to the curve Γ, at one of its points P, we shall consider a closed curve C bounding a plane domain D (*the bar cross section*); the centre of gravity of this domain is assumed to be on the curve at the point P. If the point P travels through the curve Γ, the curve C, which can also be deformable, will generate a surface bounding a three-dimensional domain which will be called *bar* (Fig. 1.1). According to the form of the axis, the bars can be *straight* or *curved*. Moreover, the curved bars can be classified as *skew curved bars* and *plane curved bars*.

Both the (average) dimensions a and b of the cross section shall be considered as being of the same order, provided the condition $a, b \ll l$ is fulfilled. If all three dimensions are of different order ($a \ll b \ll l$), we shall have to consider *thin wall bars* (Fig. 1.2a). Lastly, if the cross section dimensions are negligible with regard to the bar length, so that the bar should be perfectly flexible (unable to retain a bending), then we have a *string* (Fig. 1.2b).

Bodies having one dimension (*the thickness*) much smaller than the other two-dimension (corresponding to *the middle surface*) shall be called *plates*.

Let be a surface S of finite area, *the middle surface* of the plate. On the normal to the surface S at a point P of the latter one, we shall consider a segment of a line of length h (*the thickness*), the middle of which is assumed to be on the surface S, at the point P. If the point P travels through the surface S, the extremities of the segment of a line, which can be of variable length, will generate two surfaces

Fig. 1.1 Bar



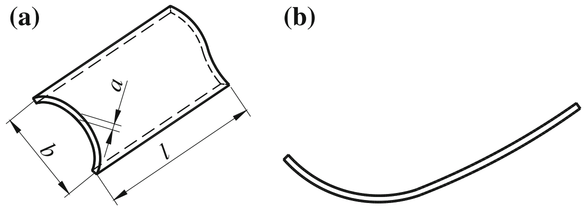


Fig. 1.2 Thin wall bar (**a**). String (**b**)

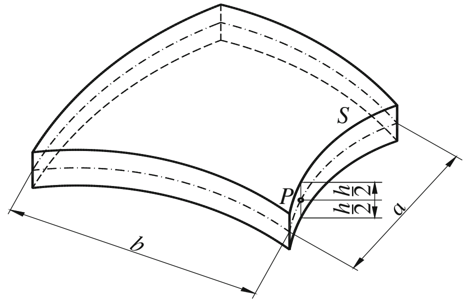


Fig. 1.3 Plate

bounding a three-dimensional domain, which we shall call *plate* (Fig. 1.3). According to the form of the middle surface, the plates may be *plane* or *curved*; the plane plates are called simply plates.

According to the ratio between the (average) dimensions a and b of the plate, contained in the middle surface, and its thickness h, we shall consider *thin plates* (for $h \ll a, b$) (Fig. 1.4a), *moderately thick plates* (Fig. 1.4b) and *thick plates* (for $h < a, b$) (Fig. 1.4c); the curved thin plates are called *shells* too. Let us remark that accurate delimitations between these plate categories cannot be made; they depend on the computation possibilities of the states of strain and stress and can vary from one case to another. Finally, if the thickness h is quite negligible with regard to the other two dimensions, so that the plate should be perfectly flexible, we shall obtain a *membrane* (Fig. 1.4d).

Bodies with all three dimensions a, b, c of the same order of magnitude shall be called *blocks* (Fig. 1.5).

Generally, the real bodies occupy finite three-dimensional domains. We shall, however, consider also infinite domains, the study of which is of particular interest; as a matter of fact, these domains idealize real cases, often occurring in practice, or else they can be used as intermediate stages in solving problems corresponding to other domains.

1.2.1.2 Fundamental Computation Hypotheses in the Theory of Elasticity

The fundamental computation hypotheses that can be made and that—sometimes—particularize the body to be studied are the following:

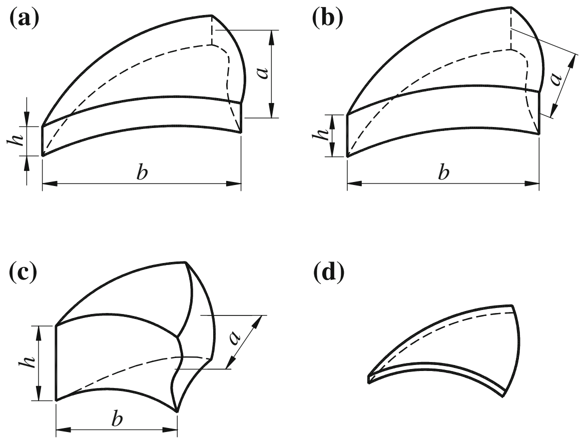


Fig. 1.4 Plate: thin (**a**), moderately thick (**b**), thick (**c**), Membrane (**d**)

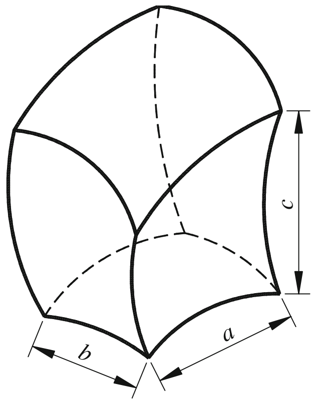


Fig. 1.5 Block

(i) *The solid body* we are studying (considered to be at rest with respect to an inertial (fixed) frame of reference) *is subjected to the action of balanced external loads*. If the body is in motion, then we introduce moreover the inertial forces; therefore, the external loads are in *dynamic equilibrium*. On the other hand, each part of the body and each particle detached from it will be subjected to loads statically or dynamically balanced (load system equivalent to zero), This hypothesis allows to write the partial differential equations verified by the stresses within the body and to express the boundary and initial conditions.

(ii) *The solid body* is considered as *a continuous medium* (without holes, internal microscopic cracks etc.). This encourages us to assume that the body deformation will be equally continuous and therefore the strains and the stresses will be mathematically expressed by *continuous functions*.

For some points of the body (*singular points*), where the strains and the stresses tend to infinity, an additional study must be made; it is important to use the methods of the theory of distributions in this case. Likewise, the case of the bodies with *internal holes* (continuous bodies to which correspond *multiply connected domains*) and the case of the bodies with *internal cuts* (particular cases of multiple connection) must be considered separately. In the case of *the disperse media* (that do not verify the continuity hypotheses) it is necessary to have recourse to *aleatory variables*.

(iii) *The solid body is isotropic*, i.e. it has the same mechanical (and physical) properties in all directions in the neighbourhood of each of its points. This property is expressed by the relations between strains and stresses. When this hypothesis is not observed, then we have to deal with *anisotropic* (*aeolotropic*) *bodies*.
(iv) *The solid body is homogeneous*, i.e. it has the same mechanical (and physical) properties at each of its points. On the basis of this property, the mechanical coefficients of the material, occurring in the constitutive law, are constant with respect to the space variables. When these coefficients are variable, the body is *non-homogeneous*.

The properties of isotropy and homogeneity do not condition one another. A solid body can be *homogeneous and isotropic* (as we shall generally admit it) or *homogeneous and anisotropic* (i.e. having the same properties at each point for a given direction) or *isotropic and nonhomogeneous* (i.e. having the same properties for any direction, but different from a point to another one) or even *non-homogeneous and anisotropic*.

We mention that, by changing the system of co-ordinates in the case of an anisotropic body, we can obtain, formally, a body with non-homogeneous properties. We shall admit, in this case, that an anisotropic body is non-homogeneous if it remains non-homogeneous in any system of curvilinear coordinates (in other cases, it can be homogeneous, with a *curvilinear anisotropy*).

We also mention that the *composite bodies* can be modelled as anisotropic ones.

(v) *The body* under study *is perfectly elastic*. Under the action of external loads, the body is deformed, but when the action of these loads ceases to operate, it resumes its initial position (in the case of static loads) and its initial form (without any hysteresis phenomenon taking place); the deformation phenomenon is *reversible*. Therefore, the deformed form of a solid body will only be influenced by the external loads that act on it at that moment. A one-to-one relationship exists between strains and stresses; as already mentioned, the state of stress at a point of the solid body will only depend on the state of strain in the neighbourhood of this point. In this manner, we happen to be within the framework of *the theory of elasticity*.

In the case of many bodies currently used, if the stresses go beyond the elastic limit, after unloading *residual deformations* remain. Beyond the elastic limit, the body presents elastic plastic properties. When this is the case, whether the elastic deformation can or cannot be neglected, the computation of the state of strain and stress must be made with the help of *the theory of plasticity*.

Generally, the one-to-one relation between strains and stresses is expressed mathematically by a *linear relation* (Hooke's law). This hypothesis corresponds—often—well enough to the physical phenomenon and, on the other hand, leads to considerable simplifications of the mathematical computation. Sometimes, a more intricate mathematical relation is considered; this leads to a *theory* of elasticity, *non-linear* from a *physical* standpoint.

Moreover, special relations can be considered, e.g., as in the case of *the hypo-elasticity theory*, created by C. Truesdell, or in the case of *hyperelastic bodies*.

If, besides the stresses, we also take into consideration the couple-stresses (micromoments), we shall find a theory of *asymmetrical elasticity* for bodies of Cosserat type (after the name of the brothers E. and F. Cosserat [13] who were the first to study, in 1907, such a problem), i.e. bodies with microstructure. In such a case, a volume has six degrees of freedom and the volume moments can also act as external loads; the constitutive law—even in the linear case—necessitates the introduction of several mechanical constants of the material.

(vi) *The deformations* and *displacements* of the solid bodies, submitted to the action of external loads, *are very small* with regard to the general dimensions of the body; it follows that *the strains are negligible* with regard to unity. If *the rigid body local rotations* are equally *negligible* with regard to unity, then we have to discuss a *linear theory* from a *geometrical standpoint* (the terms of second order are *negligible* with respect to the terms of first order), as well as from a *mechanical standpoint* (the equations of static and dynamic equilibrium can be written for the undeformed form of the body). Therefore, concerning the strains, we can apply *the principle of superposition of effects*.

There are, however, cases where it is necessary to apply non-linear methods from a mechanical standpoint or a non-linear theory from both geometrical and mechanical standpoint (the case of finite deformations). These considerations are absolutely necessary in the case of *stability problems* (a solid body loses its stability when, for a certain intensity of the external loads (critical values), at least two distinct states of strain and stress result). Phenomena of bifurcation and chaos can also occur.

Generally, the deformation of the bodies is concomitant with a change of volume (the *solid* bodies are *compressible*). Particularly, the case of *incompressible solids* and of bodies subjected to *isochoric deformations* are considered equally; this can also constitute a first computing approximation.

Let us mention also that we shall not take into account *the rate of deformation of the body*; thus, we shall not consider *bodies with rheological properties* (for which one takes into account the influence of viscosity), therefore bodies of Maxwell or Voigt type, mixed or of a more complex nature, for which the constitutive law (generally, integro-differential) depends moreover on a time variable. Consequently, we shall not consider *creep* and *relaxation* phenomena.

(vii) When the deformations are propagated, within the solid body, at a very high velocity (a case recently appeared in a unified theory of elementary particles), the phenomenon should be studied from the standpoint of *the theory of relativity*.

(viii) The influence of the *temperature variation* is not taken into consideration neither in the constitutive laws nor in the computation of the deformations. But it will be introduced in the considerations concerning *thermodynamical principles*.

(ix) *The body* under study *does not have initial stresses*, which could be due to initial deformations of the material. They could result from machinery operations (rolling, drawing etc.), from assembling operations or could be due to phenomena occurring during the working, before the action of the external loads, contraction phenomena of concrete etc.

Although these phenomena do exist and cannot be eliminated, to simplify the computation, one assumes that *the initial stresses* are missing. Thus, to an unloaded body (case of vanishing external loads) corresponds a *null state of stress*. If the initial stresses cannot be neglected, then one makes a supplementary study to take into account their influence.

We will assume—in general—that all these hypotheses are taken into account; if, in particular cases, one of these hypotheses is not respected and if we are placed in a more general situation (a less restrictive hypothesis), then we make a special mention about it.

1.2.1.3 Position of the Theory of Elasticity in the Frame of Mechanics of Deformable Solids

As we have seen, in the mechanics of deformable solids, the geometrical and mechanical aspect of the considered problems must be completed with an aspect of physical, experimental nature, which specifies the nature of the solid body. One introduces a certain *constitutive law* of the deformable solid, a certain relation between strains and stresses.

In general, one can establish (especially on an experimental way) relations between strains and stresses in a one-dimensional case. The experiments in two- or three-dimensional cases are much more difficult (e.g., in mechanics of earth, the experiments with the three-axial apparatuses are of great interest); the most time—on the basis of various hypotheses—one extends the results obtained in the linear case.

As well, the fundamental law of the deformable solids can be established on the basis of various considerations and hypotheses of *energetical* and *thermodynamical nature*. In particular, the relations which are obtained (and in which intervene—the most time—the invariants of the strain and stress tensors) depend on certain coefficients (constant or variable), which are determined on an experimental way.

We have seen that an *elastic body* is that one for which the deformation phenomenon is *reversible*. As a chapter of the mechanics of deformable solids, the theory of elasticity has as object of study the determination of the state of strain and stress of an elastic solid body subjected to the action of certain external loads in static or dynamic equilibrium. To the fundamental equations of the mechanics of deformable solids one must add a constitutive law, which characterizes the respective elastic body.

We mention that, the most times, a solid body may be considered as being elastic only till a certain intensity of the stresses or till a certain magnitude of the strains (till the limit of elasticity); after this, it becomes elastic plastic properties. Thus, our study will be valid for *the* so-called *elastic zone* of the respective body.

1.2.2 Short Historical Account. Development Trends

Certainly, since most ancient times, since men began to build, some problems of mechanics arose—in general—and of strength of materials—in particular; various problems about deformable solid bodies (although this property was not quite clear in their minds) were imposed by daily practice.

We shall supply some general data concerning the development of the mechanics of deformable solids and then we shall insist on the development of the theory of elasticity. We shall mention also the researches made in Romania.

As well, we will put in evidence some development trends and the new problems which are put.

1.2.2.1 Mechanics of Deformable Solids

Strength of materials problems occured and were solved—in a certain form—by famous builders, such as Archytas of Tarentum (approx. 440–360 A.C.) or architects such as Vitruvius (the second half of the first century A.C.). The Egyptians had, probably, empirical rules that they took into account. The Greeks, more advanced—it is sufficient to mention Archimedes (287–212 A.C.)—developed statics, that is the very basis of the study of real bodies. The Romans were equally great builders, many of their structures being even now in use; we mention thus the famous "Pont du Gard" in the south of France.

Much of the empirical knowledge acquired A.C. or at the beginning A.D. was lost in the course of the feudal epoch, being rediscovered only during the Renaissance. Leonardo da Vinci (1452–1519) was the first to make fracture tests on a steel string. So, at the end of a suspended string, a vessel is hung into which a quantity of sand is slowly flowing. The string extends until it breaks. Leonardo da Vinci also mentioned that it is good practice to make a great many experiments and to take the average of these results.

Later on, Galileo Galilei (1564–1642) showed moreover how tensile strength experiments can be made on a bar; he was the first to admit a certain stress distribution in a cantilever bar, the free end of which is subjected to a concentrated force. According to Galileo, in the built-in cross section should appear normal stresses uniformly distributed on the whole cross section, which later on proved to be only a rather vague approximation of the physical reality. Architect Fontana (1543–1607) applied these results to hoist an obelisk at the Vatican.

Robert Hooke (1635–1703) [32], on the basis of various experiments, stated in 1678 his famous law "ut tensio sic vis" (such is the force, as the extension is). Edmé Mariotte (1620–1684) verified this law on wood tensile test samples. Valuable ideas about the notion of elasticity were set forth by Mikhail Vasilievich Lomonosov (1711–1765).

In the seventeenth and eighteenth centuries, progress was realized, especially in the field of the mechanics of structures and of the strength of materials. Many of the acquired results have applications corresponding to the new problems met within civil engineering and machine construction. We mention moreover some theoretical results, obtained by Jacob (1654–1705) [4] and Jean (1667–1748) Bernoulli brothers (to the former is due the hypothesis of the plane cross sections in straight bars). The latter's son Daniel Bernoulli (1700–1782), Daniel's nephew Jacques Bernoulli (1759–1789) and chiefly Leonhard Euler (1707–1783) and Joseph-Louis Lagrange (1736–1813) dealt with problems of deformation of thin elastic bars.

Other theoretical and experimental studies are due to Charles-Augustin de Coulomb (1736–1806), Jean-Victor Poncelet (1788–1867) and Franz Grashof (1826–1893). Jean le Rond d'Alembert (1717–1783) and Peter Gustav Lejeune-Dirichlet (1803–1859) contributed by their studies in the field of mechanics to the birth of the theory of elasticity.

The first studies about sliding lines were made in the soil mechanics by Ch.-A. de Coulomb, William-John Macquorn Rankine (1820–1872) and Maurice Lévy (1839–1910). Their importance was grasped by Henri-Édouard Tresca (1814–1885) who, in 1864 and 1872, published several notes, concerning a condition of plasticity in which the maximal shear stress is constant at each point of the elastic-plastic zone, and by Christian Otto Mohr (1835–1918) who, on this basis, formulated in 1882 a theory of the strength of materials. Adhémar-Jean-Claude Barré de Saint-Venant (1797–1886) reviewed H.-É. Tresca's works at the Academy of Science (Paris) and, on this occasion, developed the fundamental equations of the plasticity theory, by admitting that the cubical dilatation vanishes during the plastic deformation (which is thus incompressible), that the principal directions of the tensor of the rate of strain coincide with the principal directions of the stress tensor and that the maximal shear stress is constant at each point of the solid body; and we can assert that it is with this development that *the theory of plasticity* was born in 1870.

And, still in the second half of the ninteenth century, the problem of the study of the solid bodies for which the deformations rate has been taken into account began to be formulated. Thus, James Clerk Maxwell (1831–1879), William Thomson (Lord Kelvin) (1842–1907) and Woldemar Voigt (1850–1919) define particular viscoelastic bodies (bodies of Maxwell or of Kelvin (or Voigt) type; the last two arrived at analogous results by independent studies), corresponding to certain real bodies. Nevertheless, from a formal standpoint, the rheology was born only at an international congress in 1929 (it is on this occasion that the name "rheology" (from "panta rei") was introduced); especially, *the theory of viscoelasticity* has been developed.

Thus, the bases are set of a new branch of mechanics that deals with the study of deformable solids.

1.2.2.2 Theory of Elasticity

The bases of *the mathematical theory of elasticity* were set forth, at the beginning of nineteenth century, by the works of Louis-Marie-Henri Navier (1785–1836) [54] and Mikhail Vasilievich Ostrogradskĭ (1801–1861); but it was Augustin-Louis Cauchy (1789–1857) who, making use of these results, established the fundamental equations of elastic bodies (the equations of static and dynamic equilibrium and the relations between displacements and strains, valid for any deformable solid bodies) in the form still used today. Siméon-Denis Poisson (1781–1841) contributed to elucidate these problems, by introducing also the notion of transverse contraction, while Benoît-Paul-Émile Clapeyron (1799–1869) establishes a theorem of equivalence between the internal work (of deformation) and the external work. The formulation in displacements of the fundamental problems of the theory of elasticity as well as an important treatise on the theory of elasticity, where is equally formulated the famous *problem of the elastic parallelepipedon*, which only in the last years obtained solution, are chiefly due to Gabriel Lamé (1795–1870) [39]; on the other hand, starting from the studies of Karl Friedrich Gauss (1777–1855), he introduced [40] the systematic use of curvilinear co-ordinates in solving problems of the theory of elasticity.

During the nineteenth century and the beginning of twentieth century, many researches in the domain of the theory of elasticity have been performed, results which are very difficult to pass, even summarily, in review.

At the same time, the strength of materials acquires equally a large development concerning the study of plates related to the design of railway bridges. As to the problem of plates, we may recall, besides the name of J.-L. Lagrange, those of Sophie Germain (1776–1831) and Gustav Robert Kirchhoff (1824–1887) [36], who continued—theoretically—the experimental research begun by Ernst Friederich Chladni (1756–1827).

A controvercy (which lasted several decades) about the elastic constants of the material confronted the supporters of the conception according to which a single elastic constant is sufficient to characterize the mechanical properties of the material (L.-M.-H. Navier [54], A.-L. Cauchy and their pupils) against those who held that two elastic constants are necessary to achieve this end. George Green (1793–1841), starting from considerations of an energetic nature, showed that, in the general case of anisotropy, 21 constants are necessary, which in case of isotropic bodies are reduced to two distinct constants. Adolf Yakovlevich Kupffer (1799–1865), at the Central Laboratory of Weights and Measures in Sankt Petersburg, Wilhelm

Wertheim (1815–1861), in Paris, and—later—Franz Ernst Neumann (1798–1895) [55] and his pupils (G. R. Kirchhoff and W. Voigt) made experimental research by which they verified the theoretical considerations of G. Green. The two elastic constants, usually used, are the modulus of longitudinal elasticity, named after Thomas Young (1773–1829) but introduced by L. Euler, and the coefficient of transverse contraction of S.-D. Poisson. Among the elastic constants characterizing the anisotropic bodies, we mention the coefficients of N. G. Chentsov.

In England, George-Bidell Airy (1801–1892) established in 1862–1863 the bases of the plane theory of elasticity, by introducing *the stress function* (the first important stress potential function in the theory of elasticity), that bears his name today. George Gabriel Stokes (1819–1900) found many important results, chiefly in the field of vibrations of elastic bodies and introduced, moreover, in 1849, some *displacement potential functions*.

In France, we must chiefly mention the studies of B. de Saint-Venant, whose ideas—as it is interesting to mention—did not appear in a volume, but were published as articles, papers, notes or appendices to the successive editions of L.-M.-H. Navier's [54] course and to the French translation of the treatise of Rudolf Friederich Alfred Clebsch (1833–1872) [12]; even the equations of continuity, although bearing his name, cannot be found except in these works. Among the many and various studies of B. de Saint-Venant, we mention especially those referring to the problem of torsion of a straight bar of any cross section however, which are at the basis of modern research in this field; here *the semi-inverse computation method was applied first*.

The first formulation of the thermoelasticity problem, in the case of uncoupled equations, was stated in the field of displacements by Jean-Marie-Constant Duhamel (1797–1872) and Fr. E. Neumann. A great many results of Neumann [55] are to be found in his treatise on the problems of the theory of elasticity and optics; let us mention his studies in the field of double refraction by which he continues and specifies the results acquired by David Brewster (1781–1868) and Ludwig Friederich Seebeck (1805–1849). These studies constitute the physical basis of photoelasticity. The coupled equations of thermoelasticity and the formulation of its problems in its most general form are due to other researchers who subsequently completed the equation of heat propagation of François-Marie-Charles Fourier (1772–1837) with the terms corresponding to the elastic deformation of the body.

G. R. Kirchhoff demonstrated, under certain conditions, the theorem of uniqueness of the solution of the elasticity theory problem. In what concerns the existence of the solution of the elasticity equations, the first studies were made by Lord Kelvin; the study of the problem was resumed by Arthur Korn, using the method of successive approximations, and by Erik Ivar Fredholm (1866–1927), using the method of integral equations.

R. F. A. Clebsch brought important contributions to the plane problem of the theory of elasticity (conditions on the boundary, representation of the displacements etc.). On the other hand, he deals with the vibrations of the elastic sphere; particularly, in the static case, he obtained results which later were thoroughly discussed by Lord Kelvin. The latter brought many contributions in the theory of