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Dedicated to Suchitra and Aryadev

Preface

Over the last four decades, intensive work has been carried out in the field of theory of nonautonomous third-order ordinary and delay differential equations. However, it is only recently that the global attractivity of third-order equations has been given a serious study. During these years, new investigations were developed and results of principal importance were obtained. In particular, suitable oscillation criteria for third-order linear and nonlinear differential equations were established, with emphasis on the oscillation of third-order nonhomogeneous differential equations with respect to oscillation and nonoscillation of third-order homogeneous differential equations. Existence and nonexistence conditions for oscillatory and nonoscillatory solutions of various types were found and asymptotic formulae for solutions of a sufficiently wide class of linear and nonlinear equations were derived. These studies notwithstanding, one would observe that oscillation and asymptotic behaviour of nonoscillatory solutions of third-order delay differential equations has rarely been studied.

In this book, an attempt is made to sum up these results. The necessity of such an attempt is felt especially as the well-known monograph of M. Greguš, Third Order Linear Differential Equations, Reidel, Dordrecht, The Netherlands, 1982, devoted to related topics, reflects results which are 30 years old.

This book consists of seven chapters. In the first one, we consider the third-order linear differential equation, with constant coefficients, of the form

$$x''' + ax'' + bx' + cx = 0, \tag{1}$$

where a, b and c are constants. Eight different cases of a, b and c were considered while dealing with (1). Further, it is observed that different structures of solution spaces of (1) appears for different cases on a, b and c. In this chapter, an introduction is given to the oscillation theory of the nonhomogeneous equation

$$x''' + ax'' + bx' + cx = f, \tag{2}$$

where a, b, c and f are constants. An apparatus of comparison theorems is developed, which allows us to establish criteria for (2) to have oscillatory or nonoscillatory solutions with the help of oscillation and nonoscillation of (1). An overall idea on oscillation of third-order delay differential equations is also included in this chapter. Additionally, some basic results are incorporated, which are needed in the forthcoming chapters.

Chapter 2 deals with the linear equation

$$x''' + a(t)x'' + b(t)x' + c(t)x = 0, \tag{3}$$

where $a, b \in C^1([\sigma, \infty), R)$ and $c \in C([\sigma, \infty), R), \sigma \in R$. This chapter contains seven sections, where most of the results obtained for (1) have been generalised to (3). Many problems have been proposed in the sections or at the end of the chapter. Apart from these, some new nontrivial sufficient conditions for the oscillation of (3) are given. Further, several sufficient conditions, different from earlier ones, are given which can be applied to the third-order Euler equation

$$x''' + \frac{a_0}{t}x'' + \frac{b_0}{t^2}x' + \frac{c_0}{t^3}x = 0,$$

where a_0, b_0 and c_0 are real constants. Several comparison principles have been used to establish the nonoscillation of (3). We have established criteria for (3) to have a family of solutions asymptotically, equivalent to solutions of (3), and we studied the properties of this family. Finally, existence criteria for solutions vanishing at infinity have been explored in this chapter.

Chapter 3 is concerned with the oscillation of the equation

$$x''' + a(t)x'' + b(t)x' + c(t)x = f(t), \tag{4}$$

where $a(t)$, $b(t)$ and $c(t)$ are as defined earlier and $f \in C([\sigma, \infty), R)$. Some of the eight different cases of $a(t)$, $b(t)$ and $c(t)$ have been considered while the remaining have been retained as open problems. We have applied the results of Chap. 2 to obtain sufficient conditions for the oscillation and nonoscillation of (4).

In Chap. 4, emphasis is given to the oscillation and nonoscillation of solutions and the asymptotic behaviour of the third-order nonlinear differential equations

$$x''' + b(t)x' + c(t)x^\alpha = 0,$$
$$x''' + b(t)x' + c(t)f(x) = 0,$$

and results of more general equations

$$x''' + a(t)x'' + b(t)x' + c(t)x^\alpha = 0,$$

and

$$x''' + a(t)x'' + b(t)x' + c(t)f(x) = 0,$$

where $a(t)$, $b(t)$ and $c(t)$ were defined earlier in (4) and $f : R \to R$ and some other restrictions on f. In this chapter, we solve the Kneser problem on monotone solutions and prove, under certain restrictions on f and α, that monotone solutions tend to zero eventually.

Chapter 5 is quite interesting. We have studied oscillation of solutions of the third-order nonlinear and nonhomogeneous equations of the form

$$\big(r(t)x''\big)' + q(t)\big(x'\big)^\beta + p(t)x^\alpha = f(t), \tag{5}$$

where r, p, q and $f \in C([\sigma, \infty), R)$ and $r(t) > 0$, α and β are quotient of odd integers. Along the way, we have given some interesting results on oscillation of the nonlinear equation

$$x''' + a(t)x'' + b(t)x' + c(t)x = f\big(t, x, x', x''\big), \tag{6}$$

where a, b and $c \in C([\sigma, \infty), R)$ and $f : [\sigma, \infty) \times R^3 \to R$.

In Chap. 6, we apply the results of Chap. 2 to obtain oscillation and asymptotic behaviour of nonoscillation of the third-order delay differential equation

$$x'''(t) + a(t)x''(t) + b(t)x'(t) + c(t)x(g(t)) = 0, \tag{7}$$

where $a(t)$, $b(t)$ and $c(t)$ are as defined earlier and $g : R \to R$ with the property that $0 \le g(t) \le t$ and $\lim_{t\to\infty} g(t) = \infty$. It has been observed that the presence of the delay term $g(t)$ makes it difficult to make a direct study of the asymptotic behaviour of solutions of (7). However, the canonical transformations makes it possible to study the properties of solutions of (7). Equations (3) and (7) have been transformed to equivalent canonical equations and then the asymptotic behaviour of solutions of (7) have been studied by knowing the oscillation and nonoscillation of (3). We have classified the nonoscillatory solutions of (7) as Property A and Property B, according to their asymptotic behaviour.

In the last chapter, we study the global attractivity of solutions of equations

$$x''' + \psi\big(x, x'\big)x'' + f\big(x, x'\big) = 0 \tag{8}$$

and

$$x''' + \psi\big(x, x'\big)x'' + f\big(x, x'\big) = p(t), \tag{9}$$

where ψ, f, ψ_x, $f_x \in C(R \times R, R)$ and $p \in C([0, \infty), R)$. Lyapunov function has been used to obtain the results. A direct step by step method has been used to find the asymptotic stability of solutions of the third-order equation

$$\begin{aligned} x'''(t) = p_1x''(t) + p_2x''(t-\tau) + q_1x'(t) + q_2x'(t-\tau) + r_1x(t) \\ + r_2x(t-\tau), \quad t \ge 0, \end{aligned} \tag{10}$$

with the initial condition

$$x(t) = \phi(t), \quad t \in [-\tau, 0], \tag{11}$$

where p_1, p_2, q_1, q_2, r_1 and r_2 are real constants, $\tau > 0$ is a real number and $\phi \in C([-\tau, 0), R)$ is an initial function. The results are then applied to obtain stability and asymptotic stability of solutions of (11).

Some results are given in the form of open problems to be solved by the readers. We also state some problems whose solutions we do not know.

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Contents

Chapter 1
Introduction

1.1 Preliminaries

In recent years, a great deal of work has been done on various aspects of differential equations of third order. There are different problems concerning third-order differential equations which have drawn the attention of researchers throughout the world. Third-order differential equations describe many mathematical models of great interest in engineering, biology and physics. Equations of the form

$$x''' + a(t)x'' + b(t)x' + c(t)x = f(t)$$

arise in the study of entry-flow phenomenon, a problem of hydrodynamics which is of considerable importance in many branches of engineering (see [29]). Reynolds [51] has studied a model equation of the type

$$x'''(t) - \lambda x'(t) - 2x(t)x'(t) = \mu_1 \big(Dx'\big)(t) + \mu_2 \sin t, \tag{1.1a}$$

$$x(t + 2\pi) = x(t), \qquad \int_{-\pi}^{\pi} x(t)\,dt = 0 \tag{1.1b}$$

which describes the steady flow of water in a long rectangular tank, oscillating horizontally near a resonant frequency. The integral damping operator D in (1.1a) is formally defined by

$$(Du)(t) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} u(t-s)s^{-1/2}\,ds$$

for all functions u with period 2π and mean zero. Cox and Mortell [8] have considered (1.1a) and (1.1b) with $(Dx')(t) = x(t)$. In [28], Jackiewicz et al. have investigated the asymptotic behaviour of the solutions of Volterra integro-differential equations of the form

$$x'(t) = \gamma x(t) + \int_0^1 (\lambda + \mu t + \upsilon s)x(s)\,ds, \quad t \geq 0,$$
$$x(0) = 1$$

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with the help of third-order differential equations of the type

$$x''' = \gamma x'' + \big(\lambda + (\mu + \upsilon)t\big)x' + (2\mu + \upsilon)x,$$

where λ, γ, μ and υ are real parameters and $\upsilon + \mu \neq 0$. In the development of a mathematical theory of thyroid-pituitary interaction, Danziger and Elemergreen (see [18], p. 133) have obtained the following third-order linear differential equations:

$$\alpha_3\theta''' + \alpha_2\theta'' + \alpha_1\theta' + (1+k)\theta = kc, \quad \theta < c$$

and

$$\alpha_3\theta''' + \alpha_2\theta'' + \alpha_1\theta' + \theta = 0, \quad \theta > c.$$

These equations describe the variation of thyroid hormone with time. Here $\theta = \theta(t)$ is the concentration of thyroid hormone at time t and $\alpha_3, \alpha_2, \alpha_1, k$ and c are constants. For other application of third-order differential equations, one may refer to Chap. 4 in [21].

In early 1950s, Alan Llyod Hodgkin and Andrew Huxley developed a mathematical model for the propagation of electrical pulses in the nerve of a squid. The original model describes the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axom. The Hodgkin–Huxley model is a set of nonlinear differential equations that approximates the electrical characteristics of excitable cells such as neurons and cardiac myocytes. The model has played a vital role in biophysics and neuronal modelling. Alan Llyod Hodgkin and Andrew Huxley were awarded a Nobel prize in 1963 for this work. A reduced version of the Hodgkin–Huxley model was proposed by Nagumo. He suggested a third-order differential equation of the form

$$x''' - cx'' + f'(x)x' - \frac{b}{c}x = 0$$

as a model exhibiting many of the features of the Hodgkin–Huxley equations, where the function f is an entire function. For more details of Nagumo's equations, we refer the reader to the paper by McKeen [44] who gave some of the background of these equations and summarised some of the numerical results of this model. For application in physics, Vreeke and Sandquist [62] proposed the following system of differential equations (which is equivalent to a system of third-order differential equations)

$$\begin{aligned}
\frac{dx_1}{dt} &= x_1\big(\gamma_1(1 - x_2) + \gamma_2(1 - x_3)\big),\\
\frac{dx_2}{dt} &= \gamma_3(x_1 - x_2),\\
\frac{dx_3}{dt} &= \gamma_4(x_1 - x_3)
\end{aligned}$$

to describe the two temperature feedback nuclear reactor problem, where x_1 is normalised neutron density, x_2 and x_3 are normalised temperatures, x_2 being associated with fuel and x_3 with the moderator or coolant, γ_3 and γ_4 are positive heat transfer coefficients, γ_1 and γ_2 are normalised effective neutron lifetime parameters associated with the temperature feedbacks, and the expression $\gamma_1(1-x_2)+\gamma_2(1-x_3)$ in the first equation is called the *reactivity* and is a measure of multiplication factor of the neutrons in the fusion reactor.

The Kuramoto–Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + \frac{1}{2}u^2 = 0$$

arises in a wide variety fascinating physical phenomena. For instance, we recall that the Kuramoto–Sivashinsky equation is introduced to describe pattern formulation in reaction diffusion systems, and to model the instability of flame front propagation (see Y. Kuramoto and T. Yamada [37] and D. Michelson [45]). To find the travelling wave solutions of this partial differential equation, we may use the substitution of the form $u(x, ct) = u(x - ct)$ with period c and one has to solve the nonlinear third-order differential equation of the form

$$\lambda u'''(x) + u'(x) + f(u) = 0,$$

where λ is a parameter and f is an even function. On the other hand, by the qualitative behaviour of the above third-order differential equation, one can predict the behaviour of the above-mentioned partial differential equation. Hence, it would be interesting to study the qualitative behaviour of solutions of third-order linear and nonlinear differential equations. One may observe that the oscillation theory of second-order differential equations is well developed. It is only recently that the oscillation theory of third-order equations has been given a serious study.

In 1908, Birkhoff [6] applied a method of projective geometry and started the study of separation and comparison theorems for third-order equations. While Birkhoff's paper is a necessary reference to many papers on differential equations, his results or methods are seldom quoted. In 1961, Hanan [25] studied the oscillation and nonoscillation of two different types of third-order differential equation. The works due to Hanan [25] and Lazer [42] are the starting point of many investigations on the oscillation and asymptotic behaviour of solutions of third-order differential equations. In 1969, Barrett [5] made a self contained study on development of third-order equations. We may note that the third order is the lowest order where truly non-self-adjoint equations do occur. For example, $x''' + tx' + t^2x = 0$ is a non-self-adjoint third-order differential equation. The self-adjoint form of the third-order equation is

$$x''' + q(t)x' + \frac{1}{2}q'(t)x = 0, \tag{1.2}$$

where $q \in C^1([\sigma, \infty), R)$ and $\sigma \in R$. In the study of second- and fourth-order equations, the self-adjoint form of an equation plays a significant role. However, the self-adjoint form of the third-order equation does not play any special role in the theory. The general solution of (1.2) is given by

$$x(t) = c_1 u^2(t) + c_2 u(t)v(t) + c_3 v^2(t),$$

where $u(t)$ and $v(t)$ are linearly independent solutions of the second-order equation

$$z'' + \frac{1}{4} q(t) z = 0,$$

and c_1, c_2 and c_3 are constants. Obviously $u^2(t), u(t)v(t)$ and $v^2(t)$ are linearly independent solutions of (1.2).

During last three decades, a considerable amount of work has been done on oscillation theory of linear homogeneous third-order differential equations with variable coefficients of the form

$$x''' + a(t)x'' + b(t)x' + c(t)x = 0, \tag{LH1}$$

under various sign restrictions on the coefficient functions $a(t)$, $b(t)$ and $c(t)$, where a, b and $c \in C([\sigma, \infty), R)$. If a is twice continuously differentiable, then the transformation

$$x = y \exp\left(-\frac{1}{3} \int_\sigma^t a(s)\, ds\right)$$

transforms (LH1) to an equation of the form

$$y''' + Q(t)y' + P(t)y = 0, \tag{1.3}$$

where

$$P(t) = \frac{1}{27}\left[2a^3(t) - 9a(t)b(t) + 27c(t) - 9a''(t)\right]$$

and

$$Q(t) = \frac{1}{3}\left[3b(t) - 3a'(t) - a^2(t)\right].$$

Third-order differential equations of the form

$$\left(r(t)x''\right)' + q(t)x' + p(t)x = 0, \tag{LH2}$$

where $p, q, r \in C([\sigma, \infty), R)$ such that $r(t) > 0$, are more general than (LH1). In fact, (LH1) may be written as (LH2), where $r(t) = \exp(\int_\sigma^t a(s)\, ds)$, $q(t) = r(t)b(t)$ and $p(t) = r(t)c(t)$.

In [21], Greguš discussed oscillatory and asymptotic behaviour of solutions of equations of the type

$$x''' + 2A(t)x' + \left(A'(t) + B(t)\right)x = 0. \tag{1.4}$$

Equation (1.4) is obtained from (1.3) by putting $A(t) = \frac{1}{2}Q(t)$ and $B(t) = P(t) - A'(t)$. Greguš has assumed that $a \in C^2([\sigma, \infty), R)$ and $b \in C^1([\sigma, \infty), R)$. The form (1.4) of a differential equation is called the *normal form* and the function $B(t)$ is referred to as the *Laguerre invariant*. Although many authors (see [4, 11, 13, 14, 17, 19, 20, 25–27, 30–33, 36, 42, 55–57, 59]) have considered Eq. (1.3), it is always interesting to obtain sufficient conditions for oscillation/nonoscillation of solutions of (LH1) explicitly in terms of the coefficient functions $a(t)$, $b(t)$ and $c(t)$. For this reason, Greguš has discussed Eq. (LH1) in Chap. 2 in his monograph (see [21]). However, very little work has been done (see [9, 10, 47–49]) on the linear nonhomogeneous differential equations of third order of the type

$$x''' + a(t)x'' + b(t)x' + c(t)x = f(t), \tag{NH1}$$

or

$$\left(r(t)x''\right)' + q(t)x' + p(t)x = f(t), \tag{NH2}$$

where $a, b, c, p, q, r \in C([\sigma, \infty), R)$ and $r(t) > 0$. Unlike second-order equations, (LH2) cannot be transformed to an equation of the form (1.3), and (NH1) cannot be transformed to an equation of the type

$$x''' + Q(t)x' + P(t)x = f(t).$$

Thus, the oscillation theory for Eqs. (LH2) and (NH2) is to be developed systematically.

A function $x \in C([\sigma, \infty), R)$ is said to be nonoscillatory, if there exists a $t_1 \geq \sigma$ such that $x(t) > 0$ or < 0 for $t \geq t_1$; otherwise, $x(t)$ is called oscillatory. By a solution of any of the above-mentioned differential equations, we mean a real-valued solution of that equation which exists on $[T, \infty)$ and is nontrivial in any neighbourhood of infinity, where $T \geq \sigma$ depends on the solution. Equation (LH1) is said to be nonoscillatory if all its solutions are nonoscillatory. It is said to be oscillatory if it admits an oscillatory solution. Equation (LH1) is said to be strongly oscillatory if all of its solutions are oscillatory. These definitions also hold good for other differential equations without delay. Such a classification of definitions is necessary because we have third-order differential equations which admit both oscillatory and nonoscillatory solutions. For example, $x''' - x = 0$ admits solutions $x_1(t) = e^t$ and $x_2(t) = e^{-t/2} \sin \frac{\sqrt{3}}{2}t$, $t \geq 0$. It is easy to see that all solutions of $x''' - x' = 0$ are nonoscillatory. The nonhomogeneous equation

$$x''' - 2x' + e^t x = 1 + e^{-t}, \quad t \geq 0$$

has a nonoscillatory solution $x(t) = e^{-t}$. From Lazer's work (Theorem 1.3 in [42]), it follows that $x''' - 2x' + e^t x = 0$ is oscillatory. Consequently, the above nonhomogeneous equation is oscillatory (see Corollary 4.2.4 in [9]). The equation

$$x''' - x = -\frac{9}{16}e^{-\frac{t}{2}}, \quad t \geq 0$$

admits a nonoscillatory solution, $x_1(t) = \frac{1}{2}e^{-t/2}$. Clearly, $x_2(t) = e^{-t/2}(\frac{1}{2} + \sin\frac{\sqrt{3}}{2}t)$ is an oscillatory solution of the equation. Further, the nonhomogeneous delay differential equation

$$x''' - e^{\pi/2}x\left(t - \frac{\pi}{2}\right) = \left(e^{\pi/2} - 1\right)\cos t, \quad t \geq 0$$

admits an oscillatory solution $x_1(t) = \sin t$ and a nonoscillatory solution $x_2(t) = \sin t + e^t$. The homogeneous delay differential equation

$$x'''(t) + 2x'(t) - x\left(t - \frac{3\pi}{2}\right) = 0$$

has an oscillatory solution $x_1(t) = \sin t$. If $\phi(m) = m^3 + 2m - e^{-(3\pi/2)m} = 0$, then $\phi(0) = -1 < 0$ and $\phi(m) \to \infty$ as $m \to \infty$. Hence there exists $m_1 > 0$ such that $\phi(m_1) = 0$. Consequently, $x_2(t) = e^{m_1 t}$ is a nonoscillatory solution of the equation.

It would be interesting to obtain an example of third-order differential equation which has only oscillatory solutions. Equation (LH1) is said to be disconjugate in $[T, \infty)$, $T \geq \sigma$ if no solution of (LH1) has more than two zeros in $[T, \infty)$, counting multiplicities. However, it carries no meaning for nonhomogeneous and nonlinear equations.

Now, we state oscillation of a solution of (LH1) in two different ways and show that these definitions are equivalent. Both definitions are used freely for linear homogeneous equations of the form (LH1) or (LH2). However, these definitions differ in case of linear nonhomogeneous and nonlinear equations.

Definition 1.1.1 A solution $x(t)$ of (LH1) is said to be oscillatory on $[T_x, \infty)$, $T_x \geq \sigma$, if it has arbitrarily large zeros in $[T_x, \infty)$, that is, there exists a sequence $\{t_n\} \subset [T_x, \infty)$ such that $t_n \to \infty$ as $n \to \infty$ and $x(t_n) = 0$ for $n = 1, 2, 3, \ldots$.

Definition 1.1.2 A solution $x(t)$ of (LH1) is said to be oscillatory on $[T_x, \infty)$, $T_x \geq \sigma$ if it has an infinite number of zeros in $[T_x, \infty)$.

These two definitions are equivalent. Suppose that a solution $x(t)$ of (LH1) is oscillatory on $[T_x, \infty)$ in the sense of Definition 1.1.2, then $x(t)$ has infinite number of zeros in $[T_x, \infty)$. We claim that the infinite set $A \subset [T_x, \infty)$ of zeros of $x(t)$ is unbounded. If not, then A is bounded and hence there exist real numbers a and b ($T_x \leq a < b < \infty$) such that $A \subset [a, b] \subset [T_x, \infty)$. From the Weierstrass theorem, it follows that A has a limit point, say t_0. Hence there exists a sequence $\{t_n\} \subset A$ such that $t_n \to t_0$ as $n \to \infty$. Thus $t_0 \in [a, b]$ and $x(t_n) = 0$ for $n = 1, 2, 3, \ldots$. Since x is continuous, $x(t_0) = 0$. If $x'(t_0) \neq 0$, then $x'(t_0) > 0$ or < 0. We may assume, without any loss of generality, that $x'(t_0) > 0$. From the continuity of $x'(t)$, it follows that there exists a $\delta > 0$ such that $x'(t) > 0$ for $t \in (t_0 - \delta, t_0 + \delta)$. Hence $x(t)$ is increasing in $(t_0 - \delta, t_0 + \delta)$. Since $x(t_0) = 0$, we have $x(t) < 0$ for $t \in (t_0 - \delta, t_0)$ and $x(t) > 0$ for $t \in (t_0, t_0 + \delta)$. On the other hand, there exists a $t^* \in A$, $t^* \neq t_0$ such that $t^* \in (t_0 - \delta, t_0 + \delta)$ as t_0 is a limit point of A. Hence $x(t^*) = 0$, a contradiction. Thus $x'(t_0) = 0$. Further, if $x''(t_0) \neq 0$, then we may assume, without any loss of generality, that $x''(t_0) > 0$. Since $x''(t)$ is continuous, there exists an $\eta > 0$ such that $x''(t) > 0$ for $t \in (t_0 - \eta, t_0 + \eta)$. Hence $x'(t)$ is continuous and increasing in $(t_0 - \eta, t_0 + \eta)$. As $x'(t_0) = 0$, then $x'(t) < 0$ for $t \in (t_0 - \eta, t_0)$ and $x'(t) > 0$ for $t \in (t_0, t_0 + \eta)$. Hence $x(t)$ is decreasing in $(t_0 - \eta, t_0)$ and increasing in $(t_0, t_0 + \eta)$. Since $x(t_0) = 0$, we have $x(t) > 0$ for $t \in (t_0 - \eta, t_0) \cup (t_0, t_0 + \eta)$. However, there exists a $t^{**} \neq t_0$, $t^{**} \in A$ such that $t^{**} \in (t_0 - \eta, t_0 + \eta)$ because t_0 is a limit point of A. Consequently, $0 < x(t^{**}) = 0$, a contradiction. Hence $x''(t_0) = 0$. Thus from uniqueness it follows that $x(t) \equiv 0$ on $[T_x, \infty)$, a contradiction. Then our claim holds, that is, A is unbounded. Hence there exists a sequence $\{t_n\} \subset A$ such that $t_n \to \infty$ as $n \to \infty$. As $x(t_n) = 0$ for $n = 1, 2, 3, \ldots$, then $x(t)$ is oscillatory in the sense of Definition 1.1.1.

Conversely, if $x(t)$ is oscillatory in the sense of Definition 1.1.1, then obviously, it is oscillatory in the sense of Definition 1.1.2.

These two definitions are equivalent for solutions of linear homogeneous second-order differential equations of the form

$$\big(r(t)x'\big)' + p(t)x = 0,$$

but not equivalent for a function $x \in C([\sigma, \infty), R)$. We may note that the equation

$$x'' = t^2\big(30t^2 - 1\big)\sin\frac{1}{t} - 10t^3\cos\frac{1}{t} + 21t^5, \quad t > 0$$

admits a solution $x(t) = t^6(\sin\frac{1}{t} + \frac{t}{2})$, which has an infinite number of zeros in $(0, 1]$, a finite number of zeros in [1, 2] and it is positive on $(2, \infty)$. Thus $x(t)$ is oscillatory on $(0, \infty)$ in the sense of Definition 1.1.2, but is nonoscillatory by Definition 1.1.1. This is because the equation is nonhomogeneous one. Further, one may observe that the zeros of an oscillatory solution of (LH1) are isolated. Indeed, if $x(t)$ is an oscillatory solution of (LH1) on $[T_x, \infty)$, then there exists a sequence $\{t_n\} \subset [T_x, \infty)$ such that $t_n \to \infty$ as $n \to \infty$ and $x(t_n) = 0, n = 1, 2, 3, \ldots$. We claim that t_n is isolated for all n. If not, then t_m is not isolated for some $m \geq 1$. Thus t_m is a limit point of $\{t_n; n \geq 1\}$. Hence there exists a subsequence $\{t_{n_j}\}$ of $\{t_n\}$ such that $t_{n_j} \to t_m$ as $j \to \infty$. But $t_n \to \infty$ as $n \to \infty$ implies that $t_{n_j} \to \infty$ as $j \to \infty$, a contradiction. Hence our assertion stands.

The adjoint of (LH1) is given by

$$\Big(\big(z' - a(t)z\big)' + b(t)z\Big)' - c(t)z = 0. \tag{LH1*}$$

Definitions 1.1.1 and 1.1.2 are equivalent for (LH1) {(LH1*)}. However, if we consider (LH1) {(LH1*)} on $(0, d)$, where $d < \infty$, then Definition 1.1.1 has no meaning. Here we repeat that a nontrivial solution $x(t)$ of (LH1) {(LH1*)} is said to be nonoscillatory if it is not oscillatory. If Eq. (LH1) {(LH1*)} has a nontrivial oscillatory solution, then it is said to be oscillatory; otherwise, Eq. (LH1) {(LH1*)} is said to be nonoscillatory.

Let S and S^* denote the solution space of (LH1) and (LH1*), respectively. Thus each of them is a three-dimensional vector space over the field of real numbers. Let S_1 $\{S_1^*\}$ denote a nontrivial subspace of S $\{S^*\}$. Then S_1 $\{S_1^*\}$ is said to be nonoscillatory, if every nonzero member of S_1 $\{S_1^*\}$ is nonoscillatory. S_1 $\{S_1^*\}$ is said to be weakly oscillatory, if it contains a nontrivial oscillatory and nonoscillatory solution. S_1 $\{S_1^*\}$ is said to be strongly oscillatory, if every nonzero member of S_1 $\{S_1^*\}$ oscillates, and S_1 $\{S_1^*\}$ is said to be oscillatory if S_1 $\{S_1^*\}$ is either weakly oscillatory or strongly oscillatory. It may be noted that weakly oscillatory definition applies only to subspaces of dimension greater than or equal to two. If S $\{S^*\}$ is nonoscillatory, weakly oscillatory or strongly oscillatory, then (LH1) $\{$(LH1*)$\}$ is said to be nonoscillatory, weakly oscillatory or strongly oscillatory, respectively.

1.2 Third-Order Linear Differential Equations with Constant Coefficients

Consider the linear third-order differential equation with constant coefficients of the form

$$x''' + ax'' + bx' + cx = 0, \quad t \geq \sigma, \tag{1.5}$$

where a, b and c are real constants. The auxiliary or the characteristic equation of (1.5) is given by

$$m^3 + am^2 + bm + c = 0. \tag{1.6}$$

The transformation $n = m + \frac{a}{3}$ transforms (1.6) to

$$n^3 + 3Hn + G = 0, \tag{1.7}$$

where $H = \frac{1}{3}(b - \frac{a^2}{3})$ and $G = c - \frac{ab}{3} + \frac{2a^3}{27}$.

From “Theory of Equations” (see [7]), it is well known that $G^2 + 4H^3 > 0$ implies that (1.7) has two imaginary roots and a real root. If $G^2 + 4H^3 \leq 0$, then all the three roots of (1.7) are real. In particular, $G^2 + 4H^3 = 0$ implies that two of these three roots are equal. If $G = 0$ and $H = 0$, then all the three roots are equal. Two imaginary roots $\alpha + i\beta$ and $\alpha - i\beta$ of (1.7) give rise to two oscillatory solutions

$$x_1(t) = e^{(\alpha - \frac{a}{3})t} \cos \beta t \quad \text{and} \quad x_2(t) = e^{(\alpha - \frac{a}{3})t} \sin \beta t$$

of (1.5). It is well known that an algebraic equation of an odd degree has at least one real root of sign opposite to that of its last term. Consequently, Eq. (1.5) always admits a nonoscillatory solution of the type $x_3(t) = e^{mt}$, where $m < 0$ if $c > 0$ and $m > 0$ if $c < 0$. If $c = 0$, then $m = 0$. Thus there is no linear homogeneous third-order differential equation with constant coefficients, which is strongly oscillatory.

According to the signs of a, b and c, one may consider the following eight different cases, viz., (i) $a \geq 0$, $b \leq 0$, $c > 0$, (ii) $a \leq 0$, $b \leq 0$, $c > 0$, (iii) $a \leq 0$, $b \leq 0$, $c < 0$, (iv) $a \geq 0$, $b \leq 0$, $c < 0$, (v) $a \geq 0$, $b \geq 0$, $c > 0$, (vi) $a \leq 0$, $b \geq 0$, $c > 0$, (vii) $a \geq 0$, $b \geq 0$, $c < 0$ and (viii) $a \leq 0$, $b \geq 0$, $c < 0$.

Proposition 1.2.1 *Suppose that $a \geq 0$, $b \leq 0$ and $c > 0$. Then the following holds*:

(i) *Equation* (1.5) *admits oscillatory solutions if and only if*

$$\frac{2a^3}{27} - \frac{ab}{3} + c - \frac{2}{3\sqrt{3}}\left(\frac{a^2}{3} - b\right)^{3/2} > 0 \tag{1.8}$$

holds, *and*

(ii) *When* (1.8) *is satisfied*, *all solutions of* (1.5) *are oscillatory except constant multiples of one solution which does not vanish on* $[\sigma, \infty)$ *and which together with all of its derivatives is monotonic on* $[\sigma, \infty)$ *and approaches zero as t tends to* ∞. *In fact*, *when* (1.8) *is satisfied*, *nonoscillatory solutions of* (1.5) *form a one-dimensional subspace of the solution space of* (1.5).

Proof (i) Clearly $G > 0$ and $H \leq 0$. Further, $G^2 + 4H^3 > 0$ if and only if $(G - 2(-H)^{3/2})(G + 2(-H)^{3/2}) > 0$ if and only if $G - 2(-H)^{3/2} > 0$, that is (1.8) holds. Hence (1.8) holds if and only if (1.5) admits oscillatory solutions.

(ii) Since $G > 0$ and (1.8) holds, (1.7) admits a negative root γ and two imaginary roots $\alpha + i\beta$ and $\alpha - i\beta$. From "Theory of equations", it is clear that $\gamma + (\alpha + i\beta) + (\alpha - i\beta) = 0$ holds and hence $\gamma = -2\alpha$. This in turn implies that $\alpha > 0$. Thus,

$$\left\{e^{(\alpha - \frac{a}{3})t}\cos\beta t, e^{(\alpha - \frac{a}{3})t}\sin\beta t, e^{(\gamma - \frac{a}{3})t}\right\}$$

is a basis of the solution space of (1.5). Clearly, constant multiples of $e^{(\gamma - \frac{a}{3})t}$ are nonoscillatory solutions of (1.5). These solutions and their derivatives are monotonic and tend to zero as t tends to ∞. To complete the proof of the proposition, it is enough to show that

$$x(t) = e^{(\alpha - \frac{a}{3})t}(\lambda_1\cos\beta t + \lambda_2\sin\beta t) + \lambda_3 e^{(\gamma - \frac{a}{3})t}$$

is oscillatory, where λ_1, λ_2 and λ_3 are constants such that λ_1 and λ_2 are not simultaneously equal to zero. If $\lambda_3 = 0$, then $x(t)$ is oscillatory. Without any loss of generality, we assume that $\lambda_3 > 0$. If possible, let $x(t) < 0$ for $t > t_0$. Since $u(t) = \lambda_1\cos\beta t + \lambda_2\sin\beta t$ is oscillatory, there exists a $t_1 > t_0$ such that $u(t_1) = 0$. So $x(t_1) > 0$, a contradiction. Suppose that $x(t) > 0$ for $t > t_0$. So $e^{(\alpha-\gamma)t}u(t) > -\lambda_3$ for $t > t_0$, a contradiction, because $\alpha - \gamma > 0$ implies that $e^{(\alpha-\gamma)t}u(t)$ oscillates between $-\infty$ and ∞. This completes the proof. □

Proposition 1.2.2 *Suppose that $a \leq 0$, $b \leq 0$ and $c > 0$. Then*

(i) *Equation* (1.5) *admits oscillatory solutions if and only if* (1.8) *is true*, *and*

(ii) *Further*, *if* (1.8) *is satisfied*, *then the conclusion of Proposition* 1.2.1 *remains true*.

Proof (i) Clearly, $H \leq 0$. (1.8) implies that $G > 0$, $G - 2(-H)^{3/2} > 0$ and hence $G^2 + 4H^3 > 0$. Consequently, (1.5) admits oscillatory solutions. On the other hand, if (1.5) admits oscillatory solutions, then (1.6) has two imaginary roots and a real root. So (1.7) has two imaginary roots and a real root. This in turn implies that $G^2 + 4H^3 > 0$. Since $c > 0$, the real root of (1.6) is negative. Thus $\gamma - \frac{a}{3} < 0$. Since $a < 0$, the real root γ of (1.7) is negative. Consequently, $G > 0$ and (1.8) holds.

(ii) If $\alpha + i\beta$ and $\alpha - i\beta$ are imaginary roots of (1.7), then $\gamma + (\alpha + i\beta) + (\alpha - i\beta) = 0$ implies that $\alpha > 0$ and

$$\left\{e^{(\alpha-\frac{a}{3})t}\cos\beta t, e^{(\alpha-\frac{a}{3})t}\sin\beta t, e^{(\gamma-\frac{a}{3})t}\right\}$$

defines a basis of the solution space of (1.5). The rest of the proof is same as that of Proposition 1.2.1. □

Proposition 1.2.3 *Suppose that $a \le 0$, $b \le 0$ and $c < 0$. Then*

(i) *Equation* (1.5) *admits oscillatory solutions if and only if*

$$-\frac{2a^3}{27} + \frac{ab}{3} - c - \frac{2}{3\sqrt{3}}\left(\frac{a^2}{3} - b\right)^{3/2} > 0 \tag{1.9}$$

holds,

(ii) *When* (1.9) *holds, oscillatory solutions of* (1.5) *form a two-dimensional subspace of the solutions space of* (1.5), *the zeros of any two oscillatory solutions of* (1.5) *separate on* $[\sigma, \infty)$ *and the absolute values of the successive maxima and minima form a decreasing sequence, these oscillatory solutions tend to zero as* $t \to \infty$, *and*

(iii) *Equation* (1.5) *admits a positive solution which tends to infinity as* t *tends to* ∞ *and whose successive derivatives are positive.*

Proof (i) In this case $G < 0$ and $H \le 0$. Clearly, (1.9) holds if and only if (1.5) admits oscillatory solutions.

(ii) If (1.9) holds, then (1.7) has two imaginary roots, say $\alpha + i\beta$ and $\alpha - i\beta$ and a positive root γ, say. So $\gamma + (\alpha + i\beta) + (\alpha - i\beta) = 0$ implies that $\alpha < 0$ and

$$\left\{e^{(\alpha-\frac{a}{3})t}\cos\beta t, e^{(\alpha-\frac{a}{3})t}\sin\beta t, e^{(\gamma-\frac{a}{3})t}\right\}$$

defines a basis of the solution space of (1.5). The general solution of (1.5) is given by

$$x(t) = \lambda_1 e^{(\alpha-\frac{a}{3})t}\cos\beta t + \lambda_2 e^{(\alpha-\frac{a}{3})t}\sin\beta t + \lambda_3 e^{(\gamma-\frac{a}{3})t},$$

where λ_1, λ_2 and λ_3 are reals. We may note that $x(t)$ is oscillatory, if and only if $\lambda_3 = 0$. Indeed, $\lambda_3 = 0$ implies that $x(t)$ is oscillatory. If $\lambda_3 \ne 0$, then

$$\left|x(t)\right| \ge |\lambda_3| e^{(\gamma-\frac{a}{3})t} - e^{(\alpha-\frac{a}{3})t}\left(|\lambda_1| + |\lambda_2|\right)$$

implies that $x(t)$ is nonoscillatory. So oscillatory solutions of (1.5) form a two-dimensional subspace of the solution space of (1.5) with the basis

$$\left\{e^{(\alpha-\frac{a}{3})t}\cos\beta t, e^{(\alpha-\frac{a}{3})t}\sin\beta t\right\}.$$

Since the elements of the above basis are solutions of the second-order linear homogeneous equation

$$z'' - 2\left(\alpha - \frac{a}{3}\right)z' + \left[\left(\alpha - \frac{a}{3}\right)^2 + \beta^2\right] = 0, \tag{1.10}$$

from Sturm's separation theorem, it follows that the zeros of any two oscillatory solutions of (1.5) separate on $[\sigma, \infty)$. Further,

$$2\left(\alpha - \frac{a}{3}\right)\left(\gamma - \frac{a}{3}\right) + \left(\alpha - \frac{a}{3}\right)^2 + \beta^2 = b$$

and $b \le 0$ implies that $(\alpha - \frac{a}{3}) < 0$. This, in turn, implies that the absolute values of the successive maxima and minima of an oscillatory solution of (1.5) form a decreasing sequence, and these oscillatory solutions tend to zero as $t \to \infty$.

(iii) Clearly, $e^{(\gamma - \frac{a}{3})t}$ is the required nonoscillatory solution of (1.5) which together with all its derivatives tends to ∞ as $t \to \infty$. This completes the proof of the proposition. □

Proposition 1.2.4 *Suppose that $a \ge 0$, $b \le 0$ and $c < 0$. Then the following hold*:

(i) *Equation* (1.5) *admits oscillatory solutions*, *if and only if* (1.9) *holds*,
(ii) *if* (1.9) *holds*, *then oscillatory solutions of* (1.5) *form a two-dimensional subspace of the solution space of* (1.5). *The zeros of any two linearly independent oscillatory solutions of* (1.5) *separate each other on* $[\sigma, \infty)$, *and these oscillatory solutions tend to zero as $t \to \infty$*,
(iii) *Equation* (1.5) *admits an oscillatory solution*, *if and only if all nonoscillatory solutions of* (1.5) *tend to $\pm\infty$ as $t \to \infty$*, *and*
(iv) *Equation* (1.5) *admits a positive solution*, *which tends to ∞ as $t \to \infty$ and whose successive derivative are positive and tend to ∞ as $t \to \infty$*.

Proof (i) We may notice that $H \le 0$ and $-G - 2(-H)^{3/2} > 0$ if and only if (1.9) holds. Clearly, (1.9) implies that $G < 0$. Since $G^2 + 4H^3 = (-G - 2(-H)^{3/2})(-G + 2(-H)^{3/2})$, (1.9) implies that $G^2 + 4H^3 > 0$. Thus, (1.7) has two imaginary roots and a real root. Consequently, (1.6) has two imaginary roots, say $(\alpha - \frac{a}{3}) + i\beta$ and $(\alpha - \frac{a}{3}) - i\beta$, and a real root $(\gamma - \frac{a}{3})$, where $\alpha + i\beta, \alpha - i\beta$ and γ are the roots of (1.7). Thus (1.5) admits oscillatory solutions. On the other hand, if (1.5) admits an oscillatory solution, then (1.6) has two imaginary roots and a real root. This real root is positive because $c < 0$. Consequently, $\gamma - \frac{a}{3} > 0$ because $c < 0$. Thus (1.7) has two imaginary roots and a positive root. Hence $G^2 + 4H^3 > 0$ and $G < 0$ because $\gamma - \frac{a}{3} > 0$. This in turn implies that (1.9) holds. Thus Eq. (1.5) admits an oscillatory solution, if and only if (1.9) holds.

(ii) If (1.9) holds, then a basis of the solution space of (1.5) is given by

$$\left\{e^{(\alpha - \frac{a}{3})t}\cos\beta t, e^{(\alpha - \frac{a}{3})t}\sin\beta t, e^{(\gamma - \frac{a}{3})t}\right\}.$$

If we set

$$x(t) = \lambda_1 e^{(\alpha-\frac{a}{3})t}\cos\beta t + \lambda_2 e^{(\alpha-\frac{a}{3})t}\sin\beta t + \lambda_3 e^{(\gamma-\frac{a}{3})t}, \tag{1.11}$$

where λ_1, λ_2 and λ_3 are reals such that $\lambda_3 \neq 0$, then $x(t)$ is nonoscillatory because $\gamma - \frac{a}{3} > 0$. Again

$$\left(\alpha - \frac{a}{3}\right) + i\beta + \left(\alpha - \frac{a}{3}\right) - i\beta + \left(\gamma - \frac{a}{3}\right) = -a$$

implies that $\alpha < 0$ and, we may write $x(t)$ as

$$x(t) = e^{(\gamma-\frac{a}{3})t}\big[(\lambda_1\cos\beta t + \lambda_2\sin\beta t)e^{(\alpha-\gamma)t} + \lambda_3\big].$$

Hence, if (1.9) holds, then the oscillatory solutions of (1.5) form a two-dimensional subspace of the solution space of (1.5). Further, these oscillatory solutions of (1.5) tend to zero as $t \to \infty$, because $\alpha - \frac{a}{3} < 0$. Since $e^{(\alpha-\frac{a}{3})t}\cos\beta t$ and $e^{(\alpha-\frac{a}{3})t}\sin\beta t$ are solutions of (1.10), from Sturm's separation theorem, it follows that the zeros of any two linearly independent oscillatory solutions of (1.5) separate each other on $[\sigma, \infty)$.

(iii) If (1.5) admits an oscillatory solution and if $x(t)$ is a nonoscillatory solution of (1.5), then we may write

$$x(t) = e^{(\alpha-\frac{a}{3})t}(\mu_1\cos\beta t + \mu_2\sin\beta t) + \mu_3 e^{(\gamma-\frac{a}{3})t},$$

where μ_1, μ_2 and μ_3 are reals such that $\mu_3 \neq 0$. As

$$\big|x(t)\big| \geq |\mu_3|e^{(\gamma-\frac{a}{3})t} - \big(|\mu_1| + |\mu_2|\big)e^{(\alpha-\frac{a}{3})t},$$

then $|x(t)| \to \infty$ as $t \to \infty$. Conversely, if every nonoscillatory solution of (1.5) tend to $\pm\infty$ as $t \to \infty$, then (1.5) admits an oscillatory solution. Indeed, if all solutions of (1.5) are nonoscillatory, then all roots of (1.6) are positive. Hence the sum of the product of these roots taken two at a time is positive. But from Eq. (1.6), it follows that this sum is equal to $b \leq 0$, a contradiction.

(iv) Clearly, $x(t) = e^{m_3 t}, m_3 > 0$ is a nonoscillatory solution of (1.5) which together with all its derivatives tends to ∞ as $t \to \infty$. We note that $m_3 > 0$, because $c < 0$. Hence the proposition is proved. □

Proposition 1.2.5 *Suppose that $a \geq 0$, $b \geq 0$ and $c > 0$. Then*

(i) *Equation (1.5) admits a nonoscillatory solution which together with all of its derivatives tend to zero as $t \to \infty$,*
(ii) *if $3b > a^2$, then (1.5) is oscillatory,*
(iii) *if $3b \leq a^2$, then (1.5) is oscillatory if and only if*

$$\left|\frac{2a^3}{27} - \frac{ab}{3} + c\right| - \frac{2}{3\sqrt{3}}\left(\frac{a^2}{3} - b\right)^{3/2} > 0 \tag{1.12}$$

holds,

(iv) *if* $2c \geq ab$, *then* (1.5) *is oscillatory*,

(v) *if* (1.5) *admits an oscillatory solution and*

$$\frac{2a^3}{27} - \frac{ab}{3} + c > 0,$$

then nonoscillatory solutions of (1.5) *form a one-dimensional subspace of the solution space of* (1.5); *moreover, nonoscillatory solutions of* (1.5) *together with their derivatives tend to zero as* $t \to \infty$,

(vi) *if* (1.5) *admits an oscillatory solution and*

$$\frac{2a^3}{27} - \frac{ab}{3} + c < 0,$$

then oscillatory solutions of (1.5) *form a two-dimensional subspace of the solution space of* (1.5), *the zeros of any two linearly independent oscillatory solutions of* (1.5) *separate each other and every oscillatory solution of* (1.5) *tends to zero as* $t \to \infty$, *and*

(vii) *if*

$$\frac{2a^3}{27} - \frac{ab}{3} + c = 0 \quad \textit{and} \quad 3b \leq a^2,$$

then Eq. (1.5) *is nonoscillatory.*

Proof (i) Since $c > 0$, Eq. (1.5) admits a solution of the form $x(t) = e^{mt}$, where $m < 0$ is a root of Eq. (1.6). Thus (1.5) admits a nonoscillatory solution which together with all of its derivatives tends to zero as $t \to \infty$.

(ii) If $3b > a^2$, then $H > 0$ and hence $G^2 + 4H^3 > 0$. Consequently, (1.5) admits an oscillatory solution.

(iii) Let $3b \leq a^2$. Clearly (1.5) admits an oscillatory solution, if and only if $G^2 + 4H^3 > 0$, if and only if $(|G| + 2(-H)^{3/2})(|G| - 2(-H)^{3/2}) > 0$, if and only if $|G| - 2(-H)^{3/2} > 0$. That is, (1.12) holds.

(iv) If $2c \geq ab$, then

$$\frac{2a^3}{27} - \frac{ab}{3} + c \geq \frac{ab}{6} + \frac{2a^3}{27} \geq 0.$$

Hence, for $a > 0$ and $b > 0$,

$$\begin{aligned} G^2 + 4H^3 &= \left(\frac{2a^3}{27} - \frac{ab}{3} + c\right)^2 + \frac{4}{27}\left(b - \frac{a^2}{3}\right)^3 \\ &\geq \frac{b}{108}\left(8a^4 + 16b^2 - 13a^2 b\right) \\ &\geq \frac{343b^3}{108 \times 32} > 0, \end{aligned}$$

because $8a^4 + 16b^2 - 13a^2b$ attains the minimum $\frac{343}{32}b^2$ at $\frac{13}{16}b$. In each of the other three cases, viz., $a = 0, b = 0$; $a > 0, b = 0$ and $a = 0, b > 0$, we observe that $G^2 + 4H^3 > 0$. Thus $2c \geq ab$ implies that (1.5) is oscillatory.

(v) If (1.5) has an oscillatory solution and $G > 0$, then (1.7) admits two imaginary roots $\alpha + i\beta$ and $\alpha - i\beta$ and a negative root γ. As $\alpha + i\beta + \alpha - i\beta + \gamma = 0$, then $\alpha > 0$. Clearly,

$$\left\{e^{(\alpha-\frac{a}{3})t}\cos\beta t, e^{(\alpha-\frac{a}{3})t}\sin\beta t, e^{(\gamma-\frac{a}{3})t}\right\}$$

is a basis of the solution space of (1.5). We claim that nonoscillatory solutions of (1.5), along with the trivial solution, form a one-dimensional subspace of the solution space of (1.5). For this, it is enough to show that the function $x(t)$ given in (1.11) is oscillatory, where λ_1, λ_2 and λ_3 are constants such that $\lambda_1^2 + \lambda_2^2 \neq 0$ and $\lambda_3 \neq 0$. Without any loss of generality, we may assume that $\lambda_3 > 0$. If possible, let $x(t) < 0$ for $t > t_0 > \sigma$. Since $u(t) = \lambda_1 \cos\beta t + \lambda_2 \sin\beta t$ is oscillatory, there exists a $t_1 > t_0$ such that $u(t_1) = 0$. Hence $x(t_1) > 0$, a contradiction. Next, suppose that $x(t) > 0$ for $t > t_0 > \sigma$. Then $e^{(\alpha-\gamma)t}u(t) > -\lambda_3$ for $t > t_0$. This is impossible, since $u(t)$ changes sign and $\alpha - \gamma > 0$. Hence our claim holds.

(vi) If (1.5) has an oscillatory solution and $G < 0$, then (1.7) has two imaginary roots $\alpha + i\beta$ and $\alpha - i\beta$ and a positive root γ. Since the sum of these roots is equal to zero, $\alpha < 0$. We wish to show that oscillatory solutions of (1.5), along with the trivial solution, form a two-dimensional subspace of the solution space of (1.5). Indeed, for this it is enough to prove that $x(t)$ given by (1.11) is nonoscillatory, if $\lambda_3 \neq 0$. Since

$$\left|x(t)\right| \geq e^{(\alpha-\frac{a}{3})t}\left[|\lambda_3|e^{(\gamma-\alpha)t} - \left(|\lambda_1| + |\lambda_2|\right)\right],$$

we have $|x(t)| > 0$ for large t. Thus $x(t)$ is nonoscillatory. We may observe that, in this case, the zeros of any two linearly independent oscillatory solutions of (1.5) separate each other, and every oscillatory solution of (1.5) tends to zero as $t \to \infty$.

(vii) Let $G = 0$. Then $G^2 + 4H^3 > 0$ or ≤ 0 if and only if $3b > a^2$ or $3b \leq a^2$. Thus $G = 0$ and $3b \leq a^2$ imply that (1.5) is nonoscillatory. This completes the proof of the proposition. □

Proposition 1.2.6 *Suppose that $a \leq 0$, $b \geq 0$ and $c > 0$. Then*

(i) *Equation (1.5) admits a nonoscillatory solution which together with all of its derivative tends to zero as $t \to \infty$,*
(ii) *if*

$$\frac{2a^3}{27} - \frac{ab}{3} + c < 0,$$

then all the solutions of (1.5) are nonoscillatory,
(iii) *if $3b > a^2$, then (1.5) is oscillatory,*
(iv) *if $3b \leq a^2$, then (1.5) is oscillatory if and only if (1.8) holds, and*

(v) *if* (1.5) *admits an oscillatory solution and*

$$\frac{2a^3}{27} - \frac{ab}{3} + c > 0,$$

then nonoscillatory solutions of (1.5) *form a one-dimensional subspace of the solution space of* (1.5). *Moreover, nonoscillatory solution space of* (1.5), *together with their derivatives, tend to zero as* $t \to \infty$.

Proof (i) The proof of the first observation is similar to that of Proposition 1.2.5.

(ii) Suppose that $G < 0$. If possible, let (1.5) admits an oscillatory solution. Then (1.7) admits two imaginary roots and a positive real root γ. Consequently, (1.6) admits two imaginary roots and a positive real root $\gamma - \frac{a}{3}$. On the other hand, $c > 0$ implies that $\gamma - \frac{a}{3} < 0$, a contradiction. Hence all solutions of (1.5) are nonoscillatory.

(iii) If $3b > a^2$, then $G^2 + 4H^3 > 0$ and hence (1.5) admits an oscillatory solution.

(iv) Suppose that $3b \le a^2$. If (1.5) admits oscillatory solutions, then $G^2 + 4H^3 > 0$. $G = 0$ implies that $G^2 + 4H^3 \le 0$, a contradiction. If $G < 0$, then from the above discussion, it follows that all solutions of (1.5) are nonoscillatory, a contradiction again. So, $G > 0$. Writing $G^2 + 4H^3 > 0$ as $(G - (-H)^{3/2})(G + (-H)^{3/2}) > 0$, we conclude that (1.8) holds. On the other hand, (1.8) implies that $G > 0$ and $G^2 + 4H^3 > 0$. Thus (1.5) admits oscillatory solutions.

One may proceed as in Proposition 1.2.5 to prove the last observation. Hence the proof is complete. □

Proposition 1.2.7 *Suppose that* $a \ge 0$, $b \ge 0$ *and* $c < 0$. *Then*

(i) *Equation* (1.5) *admits a positive solution which tends to* ∞ *as* $t \to \infty$ *and whose successive derivatives are positive*,
(ii) *if*

$$\frac{2a^3}{27} - \frac{ab}{3} + c > 0,$$

then all solutions of (1.5) *are nonoscillatory*,
(iii) *if* $3b > a^2$, *then* (1.5) *admits an oscillatory solution*, *and*
(iv) *if* $3b \le a^2$, *then* (1.5) *is oscillatory if and only if* (1.9) *holds*. *In this case, oscillatory solutions of* (1.5) *form a two-dimensional subspace of the solution space of* (1.5).

Proof (i) $c < 0$ implies that (1.6) has a positive root. This, in turn, implies that (1.5) admits a positive solution which tend to ∞ as $t \to \infty$ and whose successive derivatives are positive.

(ii) Given that $G > 0$. If possible, suppose that (1.5) has an oscillatory solution. So (1.7) has two imaginary roots and a negative root γ. Consequently, (1.6) has two imaginary roots and a negative root $\gamma - \frac{a}{3}$. But $c < 0$ implies that $\gamma - \frac{a}{3} > 0$, a contradiction. Hence all solutions of (1.5) are nonoscillatory.

(iii) Clearly, $3b > a^2$ implies that $G^2 + 4H^3 > 0$ and hence (1.5) admits oscillatory solutions.

(iv) Suppose that $3b \le a^2$. If (1.5) admits oscillatory solutions, then $G^2 + 4H^3 > 0$. From the above discussion, it follows that $G \not> 0$. Further, $G = 0$ implies that $G^2 + 4H^3 \le 0$, a contradiction. Thus $G < 0$. Writing $G^2 + 4H^3 > 0$ as $(-G - (-H)^{3/2})(-G + (-H)^{3/2}) > 0$, we obtain (1.9). Again (1.9) implies that $G < 0$, $-G + 2(-H)^{3/2} > 0$ and hence $G^2 + 4H^3 > 0$. Consequently, (1.5) admits oscillatory solutions. Now, one may proceed as in Proposition 1.2.5 to prove that oscillatory solutions of (1.5) form a two-dimensional subspace of the solution space of (1.5). The proposition is proved. □

Proposition 1.2.8 *Suppose that $a \le 0$, $b \ge 0$ and $c < 0$. Then*

(i) *Equation (1.5) admits a positive solution which tends to ∞ as $t \to \infty$ and whose successive derivatives are positive and tend to ∞ as $t \to \infty$,*
(ii) *if $3b > a^2$, then (1.5) is oscillatory,*
(iii) *if $3b \le a^2$, then (1.5) is oscillatory if and only if (1.12) holds,*
(iv) *if $2c \le ab$, then (1.5) is oscillatory,*
(v) *if (1.5) admits an oscillatory solution and*

$$\frac{2a^3}{27} - \frac{ab}{3} + c > 0,$$

then nonoscillatory solutions of (1.5) form a one-dimensional subspace of the solution space of (1.5); moreover, every nonoscillatory solution of (1.5) together with their derivatives tends to $\pm\infty$ as $t \to \infty$,
(vi) *if (1.5) admits an oscillatory solution and*

$$\frac{2a^3}{27} - \frac{ab}{3} + c < 0,$$

then oscillatory solutions of (1.5) form a two-dimensional subspace of the solution space of (1.5) and the zeros of any two linearly independent oscillatory solutions of (1.5) separate each other, and
(vii) *if*

$$\frac{2a^3}{27} - \frac{ab}{3} + c = 0 \quad \textit{and} \quad 3b \le a^2,$$

then Eq. (1.5) is nonoscillatory.

Proof (i) As $c < 0$, Eq. (1.5) admits a solution of the form $x(t) = e^{mt}$, where $m > 0$ is a root of (1.6). Thus (1.5) admits a positive solution which together with all its derivatives tends to ∞ as $t \to \infty$.

(ii) If $3b > a^2$, then $H > 0$ and hence $G^2 + 4H^3 > 0$. Consequently, (1.5) is oscillatory.

(iii) Suppose that $3b \le a^2$. Then $H \le 0$. Hence, Eq. (1.5) is oscillatory, if and only if $G^2 + 4H^3 > 0$, if and only if $(|G| + 2(-H)^{3/2})(|G| - 2(-H)^{3/2}) > 0$, if and only if $|G| - 2(-H)^{3/2} > 0$, that is, (1.12) holds.

(iv) Let $2c \le ab$. Then

$$\frac{2a^3}{27} - \frac{ab}{3} + c \le \frac{ab}{6} + \frac{2a^3}{27} \le 0.$$

Hence, for $a < 0$ and $b > 0$,

$$\begin{aligned} G^2 + 4H^3 &= \left(\frac{2a^3}{27} - \frac{ab}{3} + c\right)^2 + \frac{4}{27}\left(b - \frac{a^2}{3}\right)^3 \\ &= \frac{b}{108}\left(8a^4 + 16b^2 - 13a^2 b\right) \\ &= \frac{343b^3}{108 \times 32} > 0, \end{aligned}$$

because $8a^4 + 16b^2 - 13a^2b$ attains the minimum $\frac{343}{32}b^2$ at $\frac{13}{16}b$. In each of the other three cases, viz., $a = 0, b = 0; a < 0, b = 0$ and $a = 0, b < 0$, we have $G^2 + 4H^3 > 0$. Thus $2c \le ab$ implies that (1.5) is oscillatory.

(v) If (1.5) admits an oscillatory solution and $G > 0$, then (1.7) admits two imaginary roots $\alpha + i\beta$ and $\alpha - i\beta$ and a real negative root γ. As $\alpha + i\beta + \alpha - i\beta + \gamma = 0$, then $\alpha > 0$. Clearly

$$\left\{e^{(\alpha - \frac{a}{3})t} \cos \beta t, e^{(\alpha - \frac{a}{3})t} \sin \beta t, e^{(\gamma - \frac{a}{3})t}\right\}$$

is a basis of the solution space of (1.5). We show that nonoscillatory solutions of (1.5), along with the trivial solution, form a one-dimensional subspace of the solution space of (1.5). For this, it is enough to show that $x(t)$ given in (1.11) is oscillatory, where λ_1, λ_2 and λ_3 are constants such that $\lambda_1^2 + \lambda_2^2 \neq 0$ and $\lambda_3 \neq 0$. Without any loss of generality, we may assume that $\lambda_3 > 0$. If possible, let $x(t) < 0$ for $t > t_0 > \sigma$. Since $u(t) = \lambda_1 \cos \beta t + \lambda_2 \sin \beta t$ is oscillatory, there exists a $t_1 > t_0$ such that $u(t_1) = 0$. Hence $x(t_1) > 0$, a contradiction. Suppose that $x(t) > 0$ for $t > t_0 > \sigma$. Then $e^{(\alpha - \gamma)t} u(t) > -\lambda_3$ for $t > t_0$. This is impossible, because $u(t)$ is oscillatory and $\alpha - \gamma > 0$. Hence nonoscillatory solutions of (1.5) form a one-dimensional subspace of the solution space of (1.5). Since $\gamma - \frac{a}{3}$ is the only root of (1.6) and $c < 0$, we have $\gamma - \frac{a}{3} > 0$. As every nonoscillatory solution of (1.5) is a constant multiple of $e^{(\gamma - \frac{a}{3})t}$, the conclusion holds.

(vi) If Eq. (1.5) admits an oscillatory solution and $G < 0$, then (1.7) has two imaginary roots $\alpha + i\beta$ and $\alpha - i\beta$ and a positive root γ. Since $\alpha + i\beta + \alpha - i\beta + \gamma = 0$, we have $\alpha < 0$. We claim that oscillatory solutions of (1.5) form a two-dimensional subspace of the solution space of (1.5). For this, it is enough to show that $x(t)$ given by (1.11) is nonoscillatory, if $\lambda_3 \neq 0$. Since

$$\left|x(t)\right| \ge e^{(\alpha - \frac{a}{3})t}\left[|\lambda_3| e^{(\gamma - \alpha)t} - \left(|\lambda_1| + |\lambda_2|\right)\right],$$

we have $|x(t)| > 0$ for large t. Thus $x(t)$ is nonoscillatory. We may observe that in this case, the zeros of any two linearly independent oscillatory solutions of (1.5) separate each other.

(vii) Let $G = 0$. Then $G^2 + 4H^3 > 0$ or ≤ 0 if and only if $3b > a^2$ or $3b \leq a^2$. Thus, $G = 0$ and $3b \leq a^2$ imply that Eq. (1.5) is nonoscillatory. This completes the proof of the proposition. □

Now, we consider the third-order nonhomogeneous linear differential equation

$$y''' + ay'' + by' + cy = f, \quad t \geq \sigma, \tag{1.13}$$

where a, b, c and f are real constants and $f \neq 0$. We shall consider the following two cases:

(i) $a \geq 0$ (≤ 0), $b \leq 0$, $c > 0$ and
(ii) $a \geq 0$, $b \geq 0$, $c > 0$.

First suppose that $a \geq 0$ (≤ 0), $b \leq 0$ and $c > 0$. Clearly, all solutions of (1.5) are nonoscillatory, if and only if the characteristic equation (1.6) has only real roots, say γ_i, $i = 1, 2, 3$. Consequently, the general solution of (1.13) is given by

$$y(t) = \frac{f}{c} + \sum_{i=1}^{3} \lambda_i e^{\gamma_i t}, \quad \lambda_i \in R, \; c \neq 0,$$

which is nonoscillatory. Hence nonoscillation of (1.5) implies the nonoscillation of (1.13), that is, the oscillation of (1.13) implies the oscillation of (1.5). On the other hand, oscillation of (1.5) need not imply the oscillation of (1.13). Indeed, the solutions of

$$y''' - 2y' - 4y = 0$$

is given by $e^{-t} \cos t$, $e^{-t} \sin t$ and e^{2t}. So, the general solution of the corresponding nonhomogeneous equation

$$y''' - 2y' - 4y = f,$$

where $f \in R$ and $f \neq 0$, is of the form

$$y(t) = -\frac{f}{4} + \lambda_1 e^{-t} \cos t + \lambda_2 e^{-t} \sin t + \lambda_3 e^{2t},$$

which is nonoscillatory for all real λ_i, $i = 1, 2, 3$. However, for the above considered two cases (i) and (ii), the situation is different. The oscillation of (1.5) implies the oscillation of (1.13). Indeed, oscillation of (1.5) implies that (1.6) admits two imaginary roots $\alpha + i\beta$ and $\alpha - i\beta$ and a negative root γ. Using Descartes's rule of signs, the addition given by $b = \gamma(\alpha + i\beta) + \gamma(\alpha - i\beta) + (\alpha + i\beta)(\alpha - i\beta)$ implies that $\alpha > 0$. Consequently, $y(t) = \frac{f}{c} + \lambda e^{\alpha t} \cos \beta t$ is an oscillatory solution of (1.13). This yields the following proposition:

Proposition 1.2.9 *Suppose that $a \geq 0$ (≤ 0), $b \leq 0$ and $c > 0$. Then (1.5) is oscillatory, if and only if (1.13) is oscillatory.*