M. Ram Murty · V. Kumar Murty

The Mathematical Legacy of Srinivasa Ramanujan

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On a height he stood that looked towards greater heights.
Our early approaches to the Infinite
Are sunrise splendours on a marvellous verge
While lingers yet unseen the glorious sun.
What now we see is a shadow of what must come.

Sri Aurobindo, Savitri *1.4*

How I wish I could show you the world through my eyes.

Vivekananda

Preface

22 December 2012 marks the 125th birth anniversary of the Indian mathematician Srinivasa Ramanujan. Being largely self-taught, he emerged from extreme poverty to become one of 20th century's most influential mathematicians. His story is a phenomenal "rags to mathematical riches" story. In his short life, he had a wealth of ideas that have transformed and reshaped 20th century mathematics. These ideas continue to shape mathematics of the 21st century.

This book is meant to be a panoramic view of his essential mathematical contributions. It is not an encyclopedic account of Ramanujan's work. Rather, it is an informal account of some of the major developments that emanated from his work in the 20th and 21st centuries. The twelve essays focus on a subset of his significant papers and show how these papers shaped the course of modern mathematics.

These essays are based on lectures given by the authors over the years at the Chennai Mathematical Institute, Harish-Chandra Research Institute, IISER (Kolkata), IISER (Bhopal), IIT (Powai), IIT (Chennai), Institute for Mathematical Sciences (Chennai), and the Tata Institute for Fundamental Research (Mumbai) as well as Queen's University, the Fields Institute, and the University of Toronto. The lectures were given so that the material is accessible to undergraduates and graduate students. We have striven to not be too technical. At the same time, we tried to convey some depth of the mathematical theories emerging from the work of Ramanujan. Surely, it is impossible to be comprehensive in such a mammoth task. Still, we hope that the reader will see how the vast landscape of Ramanujan's garden has blossomed over the past century.

Toronto, Canada

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Chapter 1
The Legacy of Srinivasa Ramanujan

Mathematics enjoys the freedom of art and the precision of science. There is freedom of combination of ideas and concepts, but there is also the precision of logic and the ring of truth. It is like a master symphony. The Soviet mathematician, I.R. Shafarevich [186] once remarked that "a superficial glance at mathematics may give an impression that it is a result of separate individual efforts of many scientists scattered about in continents and in ages. However, the inner logic of its development reminds one much more of the work of a single intellect, developing its thought systematically and consistently using the variety of human individualities only as a means. It resembles an orchestra performing a symphony composed by someone. A theme passes from one instrument to another, it is taken up by another and performed with irreproachable precision."

This is no doubt true and yet, the music reaches a crescendo in the hands of certain luminaries. One such luminary was Srinivasa Ramanujan. What is fascinating about Ramanujan is that he was largely self-taught and emerged from extreme poverty to become one of the 20th century's influential mathematicians. His story is a "rags to mathematical riches" story. In the cosmic symphony of mathematics, he played a major role.

The music of Ramanujan emanates both from his life and his work. Born on 22 December 1887 in humble and poor surroundings in the town of Erode situated in present day Tamil Nadu, India, Ramanujan cultivated his love for mathematics singlehandedly and in total isolation. As a child, he was quiet and often to himself. Those that knew him were impressed by his shining large eyes which were his most prominent features. He had a prodigious memory, and at school, he would entertain his friends by reciting the various declensions of Sanskrit roots and by repeating the value of the constant π to any number of decimal places.

At the age of 12, he borrowed a book on trigonometry from an older student and completely mastered its contents. This book was Loney's *Plane Trigonometry* published by Cambridge in 1894 and contains a great deal of information on summation of series, logarithms of complex numbers, calculation of π and Gregory's series. This certainly goes far beyond any modern curriculum of trigonometry taught in our high schools today. But the book that influenced him the most was Carr's *A synopsis*

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of elementary results in pure and applied mathematics. This book is a compilation of some 6165 theorems, systematically arranged but with practically no proofs. It is not a remarkable book and was used by students of Carr for their preparation for the Mathematical Tripos, the entrance examination of Cambridge University. But Ramanujan has made the book famous in that he set about demonstrating to himself each of the assertions enunciated therein. To do this, he used a slate, jotting down the formula to be proved, erasing it with his elbow, jotting down some more formulas that led to the proof, then erasing them again with his elbow and jotting down some more formulas. People used to speak about his bruised elbow, and we know how he got it. Thus he worked his way through the book. This experience influenced him profoundly, and his contact with this book marks the beginning of his exploration of the world of mathematics. Carr's synopsis was therefore a great blessing. But unfortunately, Ramanujan took this synopsis as his model for writing, and his famous notebooks consisting of over 4000 formulas are written down in this style without proofs. The intermediate results, the links of the chain, have been erased by the elbow of Ramanujan, and his legacy is simply a set of discoveries, a melody of formulas.

When we look through these formulas discovered by Ramanujan, it is like great music echoing through our consciousness and the music lingers. Each is pregnant with meaning and heralding further exploration. Perhaps it is not so unfortunate that Ramanujan had taken Carr as his model for writing. We all now have work to do. "When the kings are building, the carters have work to do."

Another thing that we learn from the early mathematical development of Ramanujan, is the importance of problem solving in the primary grades. The mathematician and educator, George Pólya, was right when he stated that mathematics cultivates logical and orderly thinking, and a precision for the expression of ideas. So Ramanujan mastered a large tract of college level mathematics simply through problem solving and working through Carr's synopsis.

A year later, in 1903, he secured a seat in the Government College in Kumbakonam. However, his passionate absorption in mathematics led him to neglect his other subjects, and the inevitable happened. He failed the exams at the end of his first year. Four years later, he entered another college in Madras (now called Chennai) but met with the same fate at the end of his first year.

In 1909, at the age of 22, he married Kumari Janaki, and with his new responsibility, it was necessary for him to secure a job. This he succeeded in doing in 1912, when he became a clerk in the Madras Port Trust Office. There his duties were light, and he found time to devote to his mathematical research. Moreover, as luck would have it, the manager of the office, S.N. Aiyar, was a mathematician who took kindly to him and his discoveries. With Aiyar's encouragement, Ramanujan communicated some of his results to several British mathematicians. (For a short biography of S.N. Aiyar and the role he played in Ramanujan's life, we refer the reader to a recent article by Berndt [23].) His first three attempts produced little or no response. But in 1913, he wrote to G.H. Hardy at Trinity College, Cambridge. This was a turning point since Hardy was a renowned expert in analysis and number theory.

We should say here that number theory should not be confused with numerology. There is no mysticism attached to number theory. The only mystifying element is that there are beautiful formulas and there is a logical, mathematical order in the apparent chaotic universe. Number theory is the study of hidden mathematical patterns among numbers. It is called the queen of mathematics because the problems of number theory have given birth to the diverse disciplines of mathematics. Problems are utilized as points of focus of concentration. In themselves, the problems are unimportant. But in the finding of their solution, new concepts arise, and new links and patterns are found with other concepts and disciplines of mathematics. It is the final mosaic that is the end in view and not the esoteric problem, which is used only as a means of motivation.

So when Hardy received the letter, he found himself a little confounded and could not at first decide whether it was written by a crank or a genius. To the letter were attached about 120 theorems of which a representative sample is given by the following 15:

$$1-\frac{3!}{(1!2!)^3}x^2+\frac{6!}{(2!4!)^3}x^4-\cdots$$
$$=\left(1+\frac{x}{1!^3}+\frac{x^2}{2!^3}+\cdots\right)\left(1-\frac{x}{1!^3}+\frac{x^2}{2!^3}-\cdots\right). \quad (1)$$

$$1-5\left(\frac{1}{2}\right)^3+9\left(\frac{1\cdot 3}{2\cdot 4}\right)^3-13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3+\cdots=\frac{2}{\pi}. \quad (2)$$

$$1+9\left(\frac{1}{4}\right)^4+17\left(\frac{1\cdot 5}{4\cdot 8}\right)^4+25\left(\frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}\right)^4+\cdots=\frac{2^{3/2}}{\sqrt{\pi}\,\Gamma(3/4)^2}. \quad (3)$$

$$1-5\left(\frac{1}{2}\right)^5+9\left(\frac{1\cdot 3}{2\cdot 4}\right)^5-13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^5+\cdots=\frac{2}{\Gamma(3/4)^4}. \quad (4)$$

$$\int_0^\infty \frac{1+(\frac{x}{b+1})^2}{1+(\frac{x}{a})^2}\cdot\frac{1+(\frac{x}{b+2})^2}{1+(\frac{x}{a+1})^2}\cdots dx$$
$$=\frac{\sqrt{\pi}}{2}\frac{\Gamma(a+\frac{1}{2})\Gamma(b+1)\Gamma(b-a+\frac{1}{2})}{\Gamma(a)\Gamma(b+\frac{1}{2})\Gamma(b-a+1)}. \quad (5)$$

$$\int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\ldots}=\frac{\pi}{2(1+r+r^3+r^6+r^{10}+\cdots)}. \quad (6)$$

If $\alpha\beta=\pi^2$, then

$$\alpha^{-1/4}\left(1+4\alpha\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}dx\right)=\beta^{-1/4}\left(1+4\beta\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}dx\right). \quad (7)$$

$$\int_0^a e^{-x^2}dx = \frac{1}{2}\sqrt{\pi} - \frac{e^{-a^2}}{2a+}\frac{1}{a+}\frac{2}{2a+}\frac{3}{a+}\frac{4}{2a+\cdots}. \tag{8}$$

$$4\int_0^\infty \frac{xe^{-x\sqrt{5}}}{\cosh x}dx = \frac{1}{1+}\frac{1^2}{1+}\frac{1^2}{1+}\frac{2^2}{1+}\frac{2^2}{1+}\frac{3^2}{1+}\frac{3^2}{1+\cdots}. \tag{9}$$

$$\text{If}\quad u = \frac{x}{1+}\frac{x^5}{1+}\frac{x^{10}}{1+}\frac{x^{15}}{1+\cdots},\qquad v = \frac{x^{1/5}}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+\cdots},$$
$$\text{then}\quad v^5 = u\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}. \tag{10}$$

$$\frac{1}{1+}\frac{e^{-2\pi}}{1+}\frac{e^{-4\pi}}{1+\cdots} = \left\{\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)} - \frac{\sqrt{5}+1}{2}\right\}e^{2\pi/5}. \tag{11}$$

$$\frac{1}{1+}\frac{e^{-2\pi\sqrt{5}}}{1+}\frac{e^{-4\pi\sqrt{5}}}{1+\cdots} = \left[\frac{\sqrt{5}}{1+(5^{3/4}(\frac{\sqrt{5}-1}{2})^{5/2}-1)^{1/5}} - \frac{\sqrt{5}+1}{2}\right]e^{2\pi/\sqrt{5}} \tag{12}$$

$$\text{If}\quad F(k) = 1 + \left(\frac{1}{2}\right)^2 k + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^2 + \cdots \quad\text{and}$$
$$F(1-k) = \sqrt{210}F(k),\quad\text{then}$$
$$k = (\sqrt{2}-1)^4(2-\sqrt{3})^2(\sqrt{7}-\sqrt{6})^4(8-3\sqrt{7})^2(\sqrt{10}-3)^4$$
$$\times(4-\sqrt{15})^4(\sqrt{15}-\sqrt{14})^2(6-\sqrt{35})^2. \tag{13}$$

The coefficient of x^n in $(1-2x+2x^4-2x^9+\cdots)^{-1}$ is the integer nearest to

$$\frac{1}{4n}\left(\cosh\pi\sqrt{n} - \frac{\sinh\pi\sqrt{n}}{\pi\sqrt{n}}\right). \tag{14}$$

The number of numbers between A and x which are either squares or sums of two squares is

$$K\int_A^x \frac{dt}{\sqrt{\log t}} + \theta(x), \tag{15}$$

where $K = 0.764\ldots$, and $\theta(x)$ is very small compared with the previous integral.

These fifteen entries from Ramanujan's letter to Hardy give a representative sample of the formulas contained there. The first four belong to the theory of infinite series. The next three are new definite integrals. Formulas (8) to (12) are in the theory of continued fractions. Formula (13) belongs to the theory of complex multiplication and singular moduli. Formula (14) is the first suggestion of Ramanujan's knowledge of the circle method (about which we say more in a later chapter). Finally, (15) belongs to analytic number theory.

Hardy took over two hours to analyse the letter to ascertain whether the author was a crank or a genius. Hardy's reaction is expressed in his own words: "I should like you to begin by trying to reconstruct the immediate reactions of an ordinary professional mathematician who receives a letter like this from an unknown Indian clerk."

"The first question was whether I could recognise anything. I had proved things rather like (7) myself and seemed vaguely familiar with (8). Actually (8) is classical; it is a formula of Laplace first proved properly by Jacobi and (9) occurs in a paper published by Rogers in 1907."

So the conclusion was that Ramanujan had rediscovered all of these theorems amidst the impoverished mathematical background of his rustic surroundings.

Hardy continues, "I thought, that as an expert in definite integrals, I could probably prove (5) and (6) and did so, though with a good deal more trouble than I had expected. ... The series formulas (1)–(4) I found much more intriguing and it soon became obvious that Ramanujan must possess much more general theorems and was keeping a great deal up his sleeve. ... The formulas (10)–(13) are on a different level and obviously both difficult and deep. An expert in elliptic functions can see at once that (13) is derived somehow from the theory of complex multiplication, but (10)–(12) defeated me completely; I had never seen anything in the least like them before. ... The last two formulas stand apart. ... The function in (14) is a genuine approximation to the coefficient, though not at all close as Ramanujan imagined and Ramanujan's false statement was one of the most fruitful he ever made, since it ended by leading us to all our joint work on partitions."

Indeed, (14) could only be derived by the circle method, a powerful technique developed later by Hardy and Ramanujan in their work on the partition function. The entry in the letter shows that Ramanujan had already thought about the circle method in India, before he had met Hardy.

It seems that Hardy invited his colleague Littlewood and showed him the letter. They sat with it for three hours, from 9pm to midnight and finally concluded that this indeed was the work of a genius. Hardy [67] wrote later, "A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true, because if they were not true, no one would have had the imagination to invent them."

Soon thereafter, Hardy invited Ramanujan to come to Cambridge, which he felt could provide a better environment in which his mathematical genius could flourish. So Ramanujan sailed for England in March 1914.

Against the background of the first world war, from 1914 to 1917, Hardy and Ramanujan spent time in wonderful mathematical collaboration. Hardy remarks that every day Ramanujan would show him about half a dozen new theorems. These three years saw prodigious mathematical activity by both Hardy and Ramanujan. In that period, Ramanujan wrote over 30 papers, in which were laid the foundations of three fundamental methods in number theory.

The first of them concerns the circle method which, as noted earlier, already has its genesis in Eq. (14) of his first letter to Hardy. This method concerns an ingenious idea for computing the integral of a function by studying its behaviour at rational points and sufficiently small neighborhoods. The method can be developed further and enables one to attack classical unsolved problems such as Goldbach's conjecture, Waring's problem and explicit formulas for the Fourier coefficients of modular forms and modular functions. These problems have defied solution for a long time. Due to Ramanujan's early demise, this work was carried on by Hardy and Littlewood, and today it is called the circle method, or the Hardy–Littlewood method. In their paper on the partition function, Hardy and Ramanujan laid the groundwork of the method. But it was clear that this was a viable technique for attacking many age-old problems. Subsequently, this technique was developed and improved by Hardy and Littlewood and I.M. Vinogradov.

The second major contribution was the normal order method which proved that almost all natural numbers have $\log\log n$ prime factors. These investigations were later developed into the beautiful probabilistic theory of numbers, starting with the work of Turán and culminating in the celebrated Erdös–Kac theorem. Afterwards, it was taken up by Kubilius, who infused finer improvements into the theory. We feel that there is a further direction for these investigations into the domain of Fourier coefficients of modular forms, and we discuss this theme in this monograph. This brings us to the third great contribution of Ramanujan.

In 1916, he wrote a classic paper entitled "On certain arithmetical functions" in which he investigated the Fourier coefficients of various modular forms. There, he noticed patterns of congruences and made three significant conjectures concerning the behaviour of these Fourier coefficients. The most famous of these concerns the Ramanujan τ-function. He conjectured that this function satisfies a multiplicative law and that its growth is controlled by a simple polynomial function. At the time, the conjecture did not have much meaning other than as an esoteric problem in analytic number theory. Indeed, regarding the τ-function, Hardy [67] wrote, "We may seem to be straying into one of the backwaters of mathematics, but the genesis of $\tau(n)$ as a coefficient in so fundamental a function compels us to treat it with respect."

Putting respect aside, the fact is that the function occupies a central place in the pantheon of coefficients of modular forms. So the legacy left by Ramanujan's conjecture is vast and deep. For it slowly transpired that these conjectures had intimate connection with profound aspects of number theory and algebraic geometry.

The first significant step towards the conjectures of Ramanujan was taken by Mordell, who proved the multiplicative law, the first part of Ramanujan's conjectures. But Mordell only treated the case of τ and did not realize that it was prototypical of a spectrum of functions, each in its own right of central importance. This realization came twenty years later, in the work of the German mathematician Erich Hecke (who incidently was also born in the same year as Ramanujan), and the meaning of the multiplicative properties conjectured by Ramanujan was unravelled. This work of Hecke is considered a masterpiece of mathematics.

But the connections to algebraic geometry were deeper still. Indeed, after the pioneering work of Artin and Hasse, Weil formulated in 1949 general conjectures about solutions of systems of equations over finite fields. In the 1950s, it was suspected that the τ-function of Ramanujan enumerates the number of these solutions for a certain system of equations (called a variety). Several Japanese mathematicians constructed the possible candidate, but there were technical problems related to compactification. These problems were resolved in 1974 by a Belgian mathematician Pierre Deligne, who completely settled the conjectures of Ramanujan. In 1976, Deligne was awarded the Fields medal (which is the mathematical analogue of the Nobel prize) for this achievement. This is quite a legacy!

The connection between Ramanujan's conjecture and Weil's conjecture could not be so easily foreseen. Both conjectures reflect elements of the master symphony that Shafarevich spoke about. Indeed, in his retrospective essay on number theory, André Weil wrote [201], "In 1947, in Chicago, I felt bored and depressed, and not knowing what to do, I started reading Gauss's two memoirs on biquadratic residues, which I had never read before. The Gaussian integers occur in the second paper. The first one deals essentially with the number of solutions of equations $ax^4 - by^4 = 1$ in the prime field modulo p, and with the connection between these and certain Gaussian sums. ... then I noticed that similar principles can be applied to all equations of the form $ax^m + by^n + cz^r + \cdots = 0$, and that this implies the truth of the so-called "Riemann hypothesis" for all curves $ax^n + by^n + cz^n = 0$ over finite fields, and also a "generalized Riemann hypothesis" for varieties in projective space with a "diagonal" equation $\sum a_i x_i^n \equiv 0$. This led me in turn to conjectures about varieties over finite fields."

It was only a matter of time before several notable mathematicians realized that Ramanujan's conjecture was really a "Riemann hypothesis" for a certain zeta function of a variety over a finite field and that it would follow from Weil's conjecture. This was the achievement of Pierre Deligne in 1974.

No one could have foreseen such a cosmic connection. Yet, Weil is quite harsh on Hardy and in the same essay [201] wrote "Hardy's remarkable comment is: "We seem to have drifted into one of the backwaters of mathematics." To him it was just another inequality; he found it curious that anyone could get deeply interested in it. In fact, he becomes apologetic and explains that, in spite of the apparent lack of interest of this problem it might still have some features which made it not unworthy of Ramanujan's attention."

The problem with Weil's assessment of Hardy is that it is inaccurate. Hardy's original quotation is that "We may seem to be straying into one of the backwaters of mathematics, but the genesis of $\tau(n)$ as a coefficient in so fundamental a function compels us to treat it with respect." The reader will note that Weil replaced "We may seem" with "We seem" which gives quite a twist to the meaning. Moreover, Hardy does not say we will study it simply because Ramanujan had studied it, but rather that it is the coefficient of a fundamental function, namely $\Delta(z)$ in the theory of modular forms. If Hardy thought that the τ-function was in the backwaters of mathematics, then it would have been unreasonable to give it as a doctoral thesis problem to one of his celebrated students, R.A. Rankin, who created what is now called Rankin's method (also called the Rankin–Selberg method) an analogue of which was instrumental in the resolution of the Weil conjectures. So it is quite presumptuous to make absolute pronouncements on the significance of various mathematical ideas since we never know how ideas are interconnected. And this is part of the legacy.

But this legacy does not end here, and these investigations form just the tip of an iceberg. The central problem of number theory revolves around what is called the reciprocity law. The function of Ramanujan, and Fourier coefficients in general and their congruence properties reflect some aspects of the non-abelian reciprocity law. The theory of modular forms was further generalized by Jacquet and Langlands, and higher-dimensional versions of Ramanujan's conjectures were formulated as part of the Langlands program. Some of these conjectures go beyond the Ramanujan conjecture. Finer distribution conjectures concerning the τ-function inspired by the work of Sato and Tate in the theory of elliptic curves and first enunciated by Serre have now been proved. These results represent Himalayan peaks in the mathematical landscape of the 21st century.

Returning to our narrative of Ramanujan, we find that at the end of his three years work in England, he left behind a tremendous mathematical legacy. In the summer of 1917, he fell ill with what was suspected to be tuberculosis. He never recovered. Nevertheless, he continued to work unabated. Hardy relates an interesting story during the time that Ramanujan was staying in the hospital in Putney. He went to visit him in a taxicab, and as he entered Ramanujan's room, remarked that he had just ridden in a taxicab with number 1729 which seemed to be to him a rather dull number and hoped that this was not the indication of a bad omen. Ramanujan replied that on the contrary it is a very interesting number. It is the smallest number which can be expressed as the sum of two cubes in exactly two different ways:

$$1729 = 1^3 + 12^3 \quad = 10^3 + 9^3.$$

This is not without significance. Recently, a beautiful theorem in the theory of elliptic curves was proved involving this taxicab number. If for each k, there is a squarefree natural number n that can be expressed in at least k different ways as the sum of two cubes, then an outstanding problem in the theory of elliptic curves has an affirmative solution, namely ranks of elliptic curves over $\mathbb{Q}$ tend to infinity.

Ramanujan had great intuition into what was important and central. His facile mind revealed an artistic symbiosis between intellect and inspiration. We are reminded of a statement made by a great Indian sage, Swami Vivekananda, over a century ago. He said: "Just as the intellect is the instrument of knowledge, so is the heart the instrument of inspiration. In a lower state, the heart is a much weaker instrument than the intellect. ... Properly cultivated, the heart can be changed and will go beyond intellect; it will function through inspiration. Man will have to go beyond intellect in the end. The knowledge of man, his powers of perception, of reasoning and intellect and heart, all are busy churning this milk of the world. Out of long churning comes butter ... Men of [cultivated] heart get the butter and the buttermilk is left for the intellectual."

We do not know how Ramanujan discovered his theorems. On this point, Hardy [67] wrote, "It was his insight into algebraical formulas, transformations of infinite series and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi. He worked far more than the majority of modern mathematicians, by induction from numerical examples; all his congruence properties of partitions, for example, were discovered in this way. But with his memory, his patience, and his power of calculation he combined a power of generalisation, a feeling for form, a capacity for rapid modification of his hypothesis, that were often really startling, and made him, in his own peculiar field, without a rival in his day."

These comments were made by Hardy in 1936 when he delivered his famous Harvard lectures on the work of Ramanujan. He began them with a rather sentimental tone. "I have to help you," he said, "to form some sort of reasoned estimate of the most romantic figure in the recent history of mathematics ... Ramanujan was, in a way, my discovery. I did not invent him—like other great men, he invented himself—but I was the first really competent person who had a chance to see some of his work, and I can still remember with satisfaction that I could recognize at once what a treasure I had found ... And my association with him is the one romantic incident in my life." These are powerful feelings indeed describing one of the great collaborations of mathematics!

We conclude this introduction by reflecting upon what we call the cultural legacy left behind by Ramanujan. We can do this no better than to relate the feelings expressed by the Nobel laureate, Subramanian Chandrasekhar, at the Ramanujan Centennial Conference in Urbana in 1987. He wrote [31]: "It must have been a day in April 1920, when I was not quite ten years old, when my mother told me of an item in the newspaper of the day that a famous Indian mathematician, Ramanujan by name, had died the preceding day; and she told me further that Ramanujan had gone to England some years earlier, had collaborated with some famous English mathematicians and that he had returned only very recently, and was well known internationally for what he had achieved. Though I had no idea at that time of what kind of a mathematician Ramanujan was, or indeed what scientific achievement meant, I can still recall the gladness I felt at the assurance that one brought up under circumstances similar to my own, could have achieved what I could not grasp. I am sure that others were equally gladdened. I hope that it is not hard for you to imagine what the example of Ramanujan could have provided for young men and women of those times, beginning to look at the world with increasingly different perceptions.

"The fact that Ramanujan's early years were spent in a scientifically sterile atmosphere, that his life in India was not without hardships, that under circumstances that appeared to most Indians as nothing short of miraculous, he had gone to Cambridge, supported by eminent mathematicians and had returned to India with every assurance that he would be considered, in time, as one of the most original mathematicians of the century—these facts were enough—more than enough—for aspiring young Indian students to break their bonds of intellectual confinement and perhaps soar the way that Ramanujan did.

"It may be argued, perhaps with some justice, that this was a sentimental attitude: Ramanujan represents so extreme a fluctuation from the norm that his being born an Indian must be considered to a large extent as accidental. But to the Indians of the time, Ramanujan was not unique in the way we think of him today. He was one of others who had, during that same period, achieved, in their judgement, comparably in science and in other areas of human activity. Gandhi, Nehru, Rabindranath Tagore, J.C. Bose, C.V. Raman, M.N. Saha, S.N. Bose and a host of others, were in the forefront of the then fermenting scene."

In these words of Chandrasekhar, we see the legacy of Ramanujan. For the life of Chandrasekhar was equally full of hardships. Born in the same village surroundings as Ramanujan, he went to study at Cambridge and there as a graduate student discovered the mathematical implications of the theory of relativity in the collapse of certain massive stars. These he predicted degenerate into black holes. The high priests of physics of that time rejected his calculations as meaningless. He had to wait for another thirty years before the theory came into the forefront of modern physics and finally in 1983, he was awarded the Nobel prize in physics as recognition of his work. The life of Subramanian Chandrasekhar itself reveals to some extent the grandeur of the legacy of Ramanujan.

But a scientist belongs to no nation. Many of the mathematicians of distinction that we have met and talked with have all told us that Ramanujan directly or indirectly inspired their mathematical life. This is not surprising. For as we have seen, Ramanujan embodies that marvelous miracle of the human mind to frame concepts and to use formulas and symbols as tools of thought to probe deeper into the mysteries of one's own being. As long as the spirit of science is alive, his legacy will live, and the music will pass from one luminary to another. And all of us who think and work with mathematical ideas are participants in that wonderful symphony.