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Geometric Aspects of General Topology

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We can imagine and consider many mathematical concepts, such as numbers, spaces, maps, dimensions, etc., that can be indefinitely extended beyond infinity in our minds. Contemplating our mathematical ability in such a manner, I can recall this phrase from the Scriptures:

Everything he has made pretty in its time. Even time indefinite he has put in their heart, that mankind may never find out the work that the true God has made from the start to the finish.—Ecclesiastes 3:11

May our Maker be glorified! Our brain is the work of his hands, as in Psalms 100:3, *Know that Jehovah is God. It is he that has made us, and not we ourselves.* There are many reasons to give thanks to God. Our mathematical ability is one of them.

Preface

This book is designed for graduates studying Dimension Theory, ANR Theory (Theory of Retracts), and related topics. As is widely known, these two theories are connected with various fields in Geometric Topology as well as General Topology. So, for graduate students who wish to research subjects in General and Geometric Topology, understanding these theories will be valuable. Some excellent texts on these theories are the following:

- W. Hurewicz and H. Wallman, *Dimension Theory* (Princeton Univ. Press, Princeton, 1941)
- K. Borsuk, *Theory of Retracts*, MM **44** (Polish Sci. Publ., Warsaw, 1966)
- S.-T. Hu, *Theory of Retracts* (Wayne State Univ. Press, Detroit, 1965)

However, these classical texts must be updated. This is the purpose of the present book.

A comprehensive study of Dimension Theory may refer to the following book:

- R. Engelking, *Theory of Dimensions, Finite and Infinite*, SSPM **10** (Heldermann Verlag, Lembo, 1995)

Engelking's book, however, lacks results relevant to Geometric Topology. In this or any other textbook, no proof is given that $\dim X \times \mathbf{I} = \dim X + 1$ for a metrizable space X,[1] and no example illustrates the difference between the small and large inductive dimensions or a hereditarily infinite-dimensional space (i.e., an infinite-dimensional space that has no finite-dimensional subspaces except for 0-dimensional subspaces).[2]

In the 1980s and 1990s, famous longstanding problems from Dimension Theory and ANR Theory were finally resolved. In the process, it became clear that

[1] This proof can be found in Kodama's appendix of the following book:

- K. Nagami, *Dimension Theory* (Academic Press, Inc., New York, 1970)

[2] As will be mentioned later, a hereditarily infinite-dimensional space is treated in the book of J. van Mill: *Infinite-Dimensional Topology*.

these theories are linked with others. In Dimension Theory, the Alexandroff Problem had long remained unsolved. This problem queried the existence of an infinite-dimensional space whose cohomological dimension is finite. On the other hand, the CE Problem arose as a fascinating question in Shape Theory that asked whether there exists a cell-like map of a finite-dimensional space onto an infinite-dimensional space. In the 1980s, it was shown that these two problems are equivalent. Finally, in 1988, by constructing an infinite-dimensional compact metrizable space whose cohomological dimension is finite, A.N. Dranishnikov solved the Alexandroff Problem.

On the other hand, in ANR Theory, for many years it was unknown whether a metrizable topological linear space is an AR (or more generally, whether a locally equi-connected metrizable space is an ANR). In 1994, using a cell-like map of a finite-dimensional compact manifold onto an infinite-dimensional space, R. Cauty constructed a separable metrizable topological linear space that is not an AR. These results are discussed in the latter half of the final chapter and provide an understanding of how deeply these theories are related to each other. This is also the purpose of this book.

The notion of simplicial complexes is useful tool in Topology, and indispensable for studying both Theories of Dimension and Retracts. There are many textbooks from which we can gain some knowledge of them. Occasionally, we meet non-locally finite simplicial complexes. However, to the best of the author's knowledge, no textbook discusses these in detail, and so we must refer to the original papers. For example, J.H.C. Whitehead's theorem on small subdivisions is very important, but its proof cannot be found in any textbook. This book therefore properly treats non-locally finite simplicial complexes. The homotopy type of simplicial complexes is usually discussed in textbooks on Algebraic Topology using CW complexes, but we adopt a geometrical argument using simplicial complexes, which is easily understandable.

As prerequisites for studying infinite-dimensional manifolds, Jan van Mill provides three chapters on simplicial complexes, dimensions, and ANRs in the following book:

- J. van Mill, *Infinite-Dimensional Topology, Prerequisites and Introduction*, North-Holland Math. Library **43** (Elsevier Sci. Publ. B.V., Amsterdam, 1989)

These chapters are similar to the present book in content, but they are introductory courses and restricted to separable metrizable spaces. The important results mentioned above are not treated except for an example of a hereditarily infinite-dimensional space. Moreover, one can find an explanation of the Alexandroff Problem and the CE Problem in Chap. 3 of the following book:

- A. Chigogidze, *Inverse Spectra*, North-Holland Math. Library **53** (Elsevier Sci. B.V., Amsterdam, 1996)

Unfortunately, this book is, however, inaccessible for graduate students.

The present text has been in use by the author for his graduate class at the University of Tsukuba. Every year, a lecture has been given based on some topic selected from this book except the final chapter, and the same material has been used for an undergraduate seminar. Readers are required to finish the initial courses of Set Theory and General Topology. Basic knowledge of Linear Algebra is also a prerequisite. Except for the latter half of the final chapter, this book is self-contained.

Chapter 2 develops the general material relating to topological spaces appropriate for graduate students. It provides a supplementary course for students who finished an undergraduate course in Topology. We discuss paracompact spaces and some metrization theorems for non-separable spaces that are not treated in a typical undergraduate course.[3] This chapter also contains Michael's theorem on local properties, which can be applied in many situations. We further discuss the direct limits of towers (increasing sequences) of spaces, which are appear in Geometric and Algebraic Topology.[4] A non-Hausdorff direct limit of a closed tower of Hausdorff spaces is included. The author has not found any literature representing such an example. The limitation topology on the function spaces is also discussed.

Chapter 3 is devoted to topological linear spaces and convex sets. There are many good textbooks on these subjects. This chapter represents a short course on fundamental results on them. First, we establish the existing relations between these objects and to General and Geometric Topology. Convex sets are then discussed in detail. This chapter also contains Michael's selection theorem. Moreover, we show the existence of free topological spaces.

In Chap. 4, simplicial complexes are treated without assuming local finiteness. As mentioned above, we provide proof of J.H.C. Whitehead's theorem on small subdivisions. The simplicial mapping cylinder is introduced and applied to prove the Whitehead–Milnor theorem on the homotopy type of simplicial complexes. It is also applied to prove that every weak homotopy equivalence between simplicial complexes is a homotopy equivalence. The inverse limits of inverse sequences are also discussed, and it is shown that every completely metrizable space is homeomorphic to locally finite-dimensional simplicial complexes with the metric topology. These results cannot be found in any other book dealing with simplicial complexes but are buried in old journals. D.W. Henderson established the metric topology version of the Whitehead theorem on small subdivisions, but his proof is valid only for locally finite-dimensional simplicial complexes. Here we offer a complete proof without the assumption of local finite-dimensionality. Knowledge of homotopy groups is not required, even when weak homotopy equivalences are

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[3]These subjects are discussed in Munkres' book, now a very popular textbook at the *senior* or the *first-year graduate* level:

- J.R. Munkres, *Topology, 2nd ed.* (Prentice Hall, Inc., Upper Saddle River, 2000)

[4]The direct limits are discussed in Appendix of Dugundji's book:

- J. Dugundji, *Topology* (Allyn and Bacon, Inc., Boston, 1966)

But, they are not discussed even in Engelking's book, a comprehensive reference book for General Topology:

- R. Engelking, *General Topology, Revised and completed edition*, SSPM **6** (Heldermann Verlag, Berlin, 1989) discussed. However, we do review homotopy groups in Appendix 4.14 because they are helpful in the second half of Chap. 7.

Chapters 5 and 6 are devoted to Dimension Theory and ANR Theory, respectively. We prove basic results and fundamental theorems on these theories. The contents are very similar to Chaps. 5 and 6 of van Mill's "Infinite-Dimensional Topology". However, as mentioned previously, we do not restrict ourselves to separable metrizable spaces and instead go on to prove further results.

In Chap. 5, we describe a non-separable metrizable space such that the large inductive dimension does not coincide with the small inductive dimension. As mentioned above, such an example is not treated in any other textbook on Dimension Theory (not even Engelking's book). Here, we present Kulesza's example with Levin's proof. The transfinite inductive dimension is also discussed, which is not treated in van Mill's book. Further, we prove that every completely metrizable space with dimension $\leq n$ is homeomorphic to the inverse limit of an inverse sequence of metric simplicial complexes with dimension $\leq n$. Finally, hereditarily infinite-dimensional spaces are discussed based on van Mill's book.

In Chap. 6, we discuss several topics that are not treated in van Mill's book or in the two classical books by Hu and Borsuk mentioned above. Following are examples of such topics: uniform ANRs in the sense of Michael and its completion; Kozlowski's theorem that the metrizable range of a fine homotopy equivalence of an ANR is also an ANR; Cauty's characterization, with Sakai's proof, that a metrizable space is an ANR if and only if every open set has the homotopy type of an ANR; Haver's theorem that every countable-dimensional locally contractible metrizable space is an ANR; and Bothe's theorem, with Kodama's proof, that every n-dimensional metrizable space can be embedded in an $(n+1)$-dimensional AR as a closed set.

In Chap. 7, cell-like maps and related topics are discussed. The first half is self-contained, but the second half is not because some algebraic results are necessary. In the first half, we examine the existing relations between cell-like maps, soft maps, fine homotopy equivalences, etc. The second half is devoted to related topics. In particular, the CE Problem is explained and Cauty's example is presented. Note that Chigogidze's "Inverse Spectra" is the only book dealing with soft maps and provides an explanation of the Alexandroff Problem and the CE Problem.

In the second half of Chap. 7, using the K-theory result of Adams, we present the Taylor example. Eilenberg–MacLane spaces are usually constructed as CW complexes, but here they are constructed as simplicial complexes. To avoid using cohomology, we define the cohomological dimension geometrically. By applying the cohomological dimension, we can prove the equality $\dim X \times \mathbf{I} = \dim X + 1$ for every metrizable space X. We also discuss the Alexandroff Problem and the CE Problem as mentioned above. The equivalence of these problems is proved. Next, we describe the Dydak–Walsh example that gives an affirmative answer to the Alexandroff Problem. However, this part of the text is not self-contained. As a corollary, we can answer the CE Problem, i.e., we can obtain a cell-like mapping of a finite-dimensional compact manifold onto an infinite-dimensional compactum. We also present Cauty's example, i.e., a metrizable topological linear space that is not an absolute extensor. In the proof, we need the above cell-like mapping to be open, and we therefore use Walsh's open mapping approximation theorem. A proof of Walsh's theorem is beyond the scope of this book.

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Contents

Chapter 1
Preliminaries

The reader should have finished a first course in Set Theory and General Topology; basic knowledge of Linear Algebra is also a prerequisite. In this chapter, we introduce some terminology and notation. Additionally, we explain the concept of Banach spaces contained in the product of real lines.

1.1 Terminology and Notation

For the standard sets, we use the following notation:

- $\mathbb{N}$ — the set of natural numbers (i.e., positive integers);
- $\omega = \mathbb{N} \cup \{0\}$ — the set of non-negative integers;
- $\mathbb{Z}$ — the set of integers;
- $\mathbb{Q}$ — the set of rationals;
- $\mathbb{R} = (-\infty, \infty)$ — the real line with the usual topology;
- $\mathbb{C}$ — the complex plane;
- $\mathbb{R}_+ = [0, \infty)$;
- $\mathbf{I} = [0, 1]$ — the unit closed interval.

A (topological) **space** is assumed to be **Hausdorff** and a **map** is a **continuous** function. A **singleton** is a space consisting of one point, which is also said to be **degenerate**. A space is said to be **non-degenerate** if it is not a singleton. Let X be a space and $A \subset X$. We denote

- $\mathrm{cl}_X A$ (or $\mathrm{cl}\, A$) — the closure of A in X;
- $\mathrm{int}_X A$ (or $\mathrm{int}\, A$) — the interior of A in X;
- $\mathrm{bd}_X A$ (or $\mathrm{bd}\, A$) — the boundary of A in X;
- id_X (or id) — the identity map of X.

For spaces X and Y,

- $X \approx Y$ means that X and Y are homeomorphic.

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Given subspaces $X_1, \dots, X_n \subset X$ and $Y_1, \dots, Y_n \subset Y$,

- $(X, X_1, \dots, X_n) \approx (Y, Y_1, \dots, Y_n)$ means that there exists a homeomorphism $h : X \to Y$ such that $h(X_1) = Y_1, \dots, h(X_n) = Y_n$;
- $(X, x_0) \approx (Y, y_0)$ means $(X, \{x_0\}) \approx (Y, \{y_0\})$.

We call (X, x_0) a **pointed space** and x_0 its **base point**.

For a set Γ, the cardinality of Γ is denoted by $\operatorname{card} \Gamma$. The **weight** $w(X)$, the **density** $\operatorname{dens} X$, and the **cellularity** $c(X)$ of a space X are defined as follows:

- $w(X) = \min\{\operatorname{card} \mathcal{B} \mid \mathcal{B}$ is an open basis for $X\}$;
- $\operatorname{dens} X = \min\{\operatorname{card} D \mid D$ is a dense set in $X\}$;
- $c(X) = \sup\{\operatorname{card} \mathcal{G} \mid \mathcal{G}$ is a pair-wise disjoint open collection$\}$.

As is easily observed, $c(X) \leq \operatorname{dens} X \leq w(X)$ in general. If X is metrizable, all these cardinalities coincide.

Indeed, let D be a dense set in X with $\operatorname{card} D = \operatorname{dens} X$, and $\mathcal{G}$ be a pairwise disjoint collection of non-empty open sets in X. Since each $G \in \mathcal{G}$ meets D, we have an injection $g : \mathcal{G} \to D$, hence $\operatorname{card} \mathcal{G} \leq \operatorname{card} D = \operatorname{dens} X$. It follows that $c(X) \leq \operatorname{dens} X$. Now, let $\mathcal{B}$ be an open basis for X with $\operatorname{card} \mathcal{B} = w(X)$. By taking any point $x_B \in B$ from each $B \in \mathcal{B}$, we have a dense set $\{x_B \mid B \in \mathcal{B}\}$ in X, which implies $\operatorname{dens} X \leq w(X)$.

When X is metrizable, we show the converse inequality. The case $\operatorname{card} X < \aleph_0$ is trivial. We may assume that $X = (X, d)$ is a metric space with $\operatorname{diam} X \geq 1$ and $\operatorname{card} X \geq \aleph_0$. Let D be a dense set in X with $\operatorname{card} D = \operatorname{dens} X$. Then, $\{\mathrm{B}(x, 1/n) \mid x \in D,\ n \in \mathbb{N}\}$ is an open basis for X, which implies $w(X) \leq \operatorname{dens} X$. For each $n \in \mathbb{N}$, using Zorn's Lemma, we can find a maximal 2^{-n}-discrete subset $X_n \subset X$, i.e., $d(x, y) \geq 2^{-n}$ for every pair of distinct points $x, y \in X_n$. Then, $\mathcal{G}_n = \{\mathrm{B}(x, 2^{-n-1}) \mid x \in X_n\}$ is a pairwise disjoint open collection, and hence we have $\operatorname{card} X_n = \operatorname{card} \mathcal{G}_n \leq c(X)$. Observe that $X_* = \bigcup_{n \in \mathbb{N}} X_n$ is dense in X, which implies $\sup_{n \in \mathbb{N}} \operatorname{card} X_n = \operatorname{card} X_* \geq \operatorname{dens} X$. Therefore, $c(X) \geq \operatorname{dens} X$.

For the product space $\prod_{\gamma \in \Gamma} X_\gamma$, the γ-coordinate of each point $x \in \prod_{\gamma \in \Gamma} X_\gamma$ is denoted by $x(\gamma)$, i.e., $x = (x(\gamma))_{\gamma \in \Gamma}$. For each $\gamma \in \Gamma$, the projection $\mathrm{pr}_\gamma : \prod_{\gamma \in \Gamma} X_\gamma \to X_\gamma$ is defined by $\mathrm{pr}_\gamma(x) = x(\gamma)$. For $\Lambda \subset \Gamma$, the projection $\mathrm{pr}_\Lambda : \prod_{\gamma \in \Gamma} X_\gamma \to \prod_{\lambda \in \Lambda} X_\lambda$ is defined by $\mathrm{pr}_\Lambda(x) = x|\Lambda \ (= (x(\lambda))_{\lambda \in \Lambda})$. In the case that $X_\gamma = X$ for every $\gamma \in \Gamma$, we write $\prod_{\gamma \in \Gamma} X_\gamma = X^\Gamma$. In particular, $X^{\mathbb{N}}$ is the product space of countable infinite copies of X. When $\Gamma = \{1, \dots, n\}$, $X^\Gamma = X^n$ is the product space of n copies of X. For the product space $X \times Y$, we denote the projections by $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$.

A compact metrizable space is called a **compactum** and a connected compactum is called a **continuum**.[1] For a metrizable space X, we denote

- $\mathrm{Metr}(X)$ — the set of all admissible metrics of X.

Now, let $X = (X, d)$ be a metric space, $x \in X$, $\varepsilon > 0$, and $A, B \subset X$. We use the following notation:

[1]Their plurals are **compacta** and **continua**, respectively.

- $\mathrm{B}_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$ — the ε-neighborhood of x in X (or the open ball with center x and radius ε);
- $\overline{\mathrm{B}}_d(x, \varepsilon) = \{y \in X \mid d(x, y) \leq \varepsilon\}$ — the closed ε-neighborhood of x in X (or the closed ball with center x and radius ε);
- $\mathrm{N}_d(A, \varepsilon) = \bigcup_{x \in A} \mathrm{B}_d(x, \varepsilon)$ — the ε-neighborhood of A in X;
- $\operatorname{diam}_d A = \sup\{d(x, y) \mid x, y \in A\}$ — the diameter of A;
- $d(x, A) = \inf\{d(x, y) \mid y \in A\}$ — the distance of x from A;
- $\operatorname{dist}_d(A, B) = \inf\{d(x, y) \mid x \in A,\ y \in B\}$ — the distance of A and B.

It should be noted that $\mathrm{N}_d(\{x\}, \varepsilon) = \mathrm{B}_d(x, \varepsilon)$ and $d(x, A) = \operatorname{dist}_d(\{x\}, A)$. For a collection $\mathcal{A}$ of subsets of X, let

- $\operatorname{mesh}_d \mathcal{A} = \sup\{\operatorname{diam}_d A \mid A \in \mathcal{A}\}$ — the mesh of $\mathcal{A}$.

If there is no possibility of confusion, we can drop the subscript d and write $\mathrm{B}(x, \varepsilon)$, $\overline{\mathrm{B}}(x, \varepsilon)$, $\mathrm{N}(A, \varepsilon)$, $\operatorname{diam} A$, $\operatorname{dist}(A, B)$, and $\operatorname{mesh} \mathcal{A}$.

The standard spaces are listed below:

- $\mathbb{R}^n$ — the n-dimensional Euclidean space with the norm

$$\|x\| = \sqrt{x(1)^2 + \cdots + x(n)^2},$$

  $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$ — the origin, the zero vector or the zero element,
  $\mathbf{e}_i \in \mathbb{R}^n$ — the unit vector defined by $\mathbf{e}_i(i) = 1$ and $\mathbf{e}_i(j) = 0$ for $j \neq i$;
- $\mathbf{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ — the unit $(n-1)$-sphere;
- $\mathbf{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ — the unit closed n-ball;
- $\Delta^n = \{x \in (\mathbb{R}_+)^{n+1} \mid \sum_{i=1}^{n+1} x(i) = 1\}$ — the standard n-simplex;
- $\boldsymbol{Q} = [-1, 1]^{\mathbb{N}}$ — the Hilbert cube;
- $\boldsymbol{s} = \mathbb{R}^{\mathbb{N}}$ — the space of sequences;
- $\mu^0 = \{\sum_{i=1}^{\infty} 2x_i/3^i \mid x_i \in \{0, 1\}\}$ — the Cantor (ternary) set;
- $\nu^0 = \mathbb{R} \setminus \mathbb{Q}$ — the space of irrationals;
- $\mathbf{2} = \{0, 1\}$ — the discrete space of two points.

Note that $\mathbf{S}^{n-1}$, $\mathbf{B}^n$, and Δ^n are not product spaces, even though the same notations are used for product spaces. The indexes $n-1$ and n represent their dimensions (the indexes of μ^0 and ν^0 are identical).

As is well-known, the countable product $\mathbf{2}^{\mathbb{N}}$ of the discrete space $\mathbf{2} = \{0, 1\}$ is homeomorphic to the Cantor set μ^0 by the correspondence:

$$x \mapsto \sum_{i \in \mathbb{N}} \frac{2x(i)}{3^i}.$$

On the other hand, the countable product $\mathbb{N}^{\mathbb{N}}$ of the discrete space $\mathbb{N}$ of natural numbers is homeomorphic to the space ν^0 of irrationals. In fact, $\mathbb{N}^{\mathbb{N}} \approx (0, 1) \setminus \mathbb{Q} \approx (-1, 1) \setminus \mathbb{Q} \approx \nu^0$. These three homeomorphisms are given as follows:

$$x \mapsto \cfrac{1}{x(1) + \cfrac{1}{x(2) + \cfrac{1}{x(3) + \cfrac{1}{\ddots}}}}; \quad t \mapsto 2t - 1; \quad s \mapsto \frac{s}{1 - |s|}.$$

That the first correspondence is a homeomorphism can be verified as follows: for each $n \in \mathbb{N}$, let $a_n : \mathbb{N}^{\mathbb{N}} \to \mathbf{I}$ be a map defined by

$$a_n(x) = \cfrac{1}{x(1) + \cfrac{1}{x(2) + \cfrac{1}{\ddots + \cfrac{1}{x(n)}}}}.$$

Then, $0 < a_2(x) < a_4(x) < \cdots < a_3(x) < a_1(x) \leq 1$. Using the fact shown below, we can conclude that the first correspondence $\mathbb{N}^{\mathbb{N}} \ni x \mapsto \alpha(x) = \lim_{n\to\infty} a_n(x) \in (0, 1)$ is well-defined and continuous.

Fact. *For every $m > n$, $|a_n(x) - a_m(x)| < (n + 1)^{-1}$.*

This fact can be shown by induction on $n \in \mathbb{N}$. First, observe that

$$|a_1(x) - a_2(x)| = \frac{1}{x(1)(x(1)x(2) + 1)} < 1/2,$$

which implies the case $n = 1$. When $n > 1$, for each $x \in \mathbb{N}^{\mathbb{N}}$, define $x^* \in \mathbb{N}^{\mathbb{N}}$ by $x^*(i) = x(i + 1)$. By the inductive assumption, $|a_{n-1}(x^*) - a_{m-1}(x^*)| < n^{-1}$ for $m > n$, which gives us

$$\begin{aligned}
|a_n(x) - a_m(x)| &= \frac{|a_{n-1}(x^*) - a_{m-1}(x^*)|}{(x(1) + a_{n-1}(x^*))(x(1) + a_{m-1}(x^*))} \\
&\leq \frac{|a_{n-1}(x^*) - a_{m-1}(x^*)|}{(1 + a_{n-1}(x^*))(1 + a_{m-1}(x^*))} \\
&< \frac{|a_{n-1}(x^*) - a_{m-1}(x^*)|}{1 + |a_{n-1}(x^*) - a_{m-1}(x^*)|} \\
&\leq \frac{n^{-1}}{1 + n^{-1}} = \frac{1}{n+1}.
\end{aligned}$$

Let $t = q_1/q_0 \in (0, 1) \cap \mathbb{Q}$, where $q_1 < q_0 \in \mathbb{N}$. Since $q_0/q_1 = t^{-1} > 1$, we can choose $x_1 \in \mathbb{N}$ so that $x_1 \leq q_0/q_1 < x_1 + 1$. Then, $1/(x_1 + 1) < t \leq 1/x_1$. If $t \neq 1/x_1$, then $x_1 < q_0/q_1$, and hence $t^{-1} = q_0/q_1 = x_1 + q_2/q_1$ for some $q_2 \in \mathbb{N}$ with $q_2 < q_1$. Now, we choose $x_2 \in \mathbb{N}$ so that $x_2 \leq q_1/q_2 < x_2 + 1$. Thus, $x_1 + 1/(x_2 + 1) < x_1 + q_2/q_1 \leq x_1 + 1/x_2$, so $1/(x_1 + 1/x_2) \leq t < 1/(x_1 + 1/(x_2 + 1))$. If $t \neq 1/(x_1 + 1/x_2)$, then $x_2 < q_1/q_2$. Similarly, we write $q_1/q_2 = x_2 + q_3/q_2$, where $q_3 \in \mathbb{N}$ with $q_3 < q_2$ $(< q_1)$, and choose $x_3 \in \mathbb{N}$ so that $x_3 \leq q_2/q_3 < x_3 + 1$. Then, $1/(x_1 + 1/(x_2 + 1/x_3)) \leq t < 1/(x_1 + 1/(x_2 + 1/(x_3 + 1)))$. This process has only a finite number of steps (at most q_1 steps). Thus, we have the following unique representation:

$$t = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots + \cfrac{1}{x_n}}}}, \quad x_1, \ldots, x_n \in \mathbb{N}.$$

It follows that $\alpha(\mathbb{N}^{\mathbb{N}}) \subset (0,1) \setminus \mathbb{Q}$.

For each $t \in (0,1) \setminus \mathbb{Q}$, choose $x_1 \in \mathbb{N}$ so that $x_1 < t^{-1} < x_1 + 1$. Then, $1/(x_1+1) < t < 1/x_1$ and $t^{-1} = x_1 + t_1$ for some $t_1 \in (0,1) \setminus \mathbb{Q}$. Next, choose $x_2 \in \mathbb{N}$ so that $x_2 < t_1^{-1} < x_2 + 1$. Thus, $x_1 + 1/(x_2+1) < x_1 + t_1 < x_1 + 1/x_2$, and so $1/(x_1 + 1/x_2) < t < 1/(x_1 + 1/(x_2+1))$. Again, write $t_1^{-1} = x_2 + t_2$, $t_2 \in (0,1) \setminus \mathbb{Q}$, and choose $x_3 \in \mathbb{N}$ so that $x_3 < t_2^{-1} < x_3 + 1$. Then, $1/(x_1 + 1/(x_2 + 1/(x_3+1))) < t < 1/(x_1 + 1/(x_2 + 1/x_3))$. We can iterate this process infinitely many times. Thus, there is the unique $x = (x_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $a_{2n}(x) < t < a_{2n+1}(x)$ for each $n \in \mathbb{N}$, where $\alpha(x) = \lim_{n\to\infty} a_n(x) = t$. Therefore, $\alpha : \mathbb{N}^{\mathbb{N}} \to (0,1) \setminus \mathbb{Q}$ is a bijection.

In the above, let $a_{2n}(x) < s < a_{2n-1}(x)$ and define $y = (y_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ for this s similar to x for t. Then, $\alpha(y) = s$ and $x_i = y_i$ for each $i \leq 2n-1$, i.e., the first $2n-1$ coordinates of x and y are all the same. This means that α^{-1} is continuous.

Let $f : A \to Y$ be a map from a closed set A in a space X to another space Y. The **adjunction space** $Y \cup_f X$ is the quotient space $(X \oplus Y)/\sim$, where $X \oplus Y$ is the topological sum and $\sim$ is the equivalence relation corresponding to the decomposition of $X \oplus Y$ into singletons $\{x\}$, $x \in X \setminus A$, and sets $\{y\} \cup f^{-1}(y)$, $y \in Y$ (the latter is a singleton $\{y\}$ if $y \in Y \setminus f(A)$). In the case that Y is a singleton, $Y \cup_f X \approx X/A$. One should note that, in general, the adjunction spaces are *not Hausdorff*. Some further conditions are necessary for the adjunction space to be Hausdorff.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of X and $Y \subset X$. We define

- $\mathcal{A} \wedge \mathcal{B} = \{A \cap B \mid A \in \mathcal{A},\ B \in \mathcal{B}\}$;
- $\mathcal{A}|Y = \{A \cap Y \mid A \in \mathcal{A}\}$;
- $\mathcal{A}[Y] = \{A \in \mathcal{A} \mid A \cap Y \neq \emptyset\}$.

When each $A \in \mathcal{A}$ is contained in some $B \in \mathcal{B}$, it is said that $\mathcal{A}$ **refines** $\mathcal{B}$ and denoted by:

$$\mathcal{A} \prec \mathcal{B} \text{ or } \mathcal{B} \succ \mathcal{A}.$$

It is said that $\mathcal{A}$ **covers** Y (or $\mathcal{A}$ is a **cover** of Y in X) if $Y \subset \bigcup \mathcal{A}$ $(= \bigcup_{A \in \mathcal{A}} A)$. When $Y = X$, a cover of Y in X is simply called a cover of X. A cover of Y in X is said to be **open** or **closed** in X depending on whether its members are open or closed in X. If $\mathcal{A}$ is an open cover of X then $\mathcal{A}|Y$ is an open cover of Y and $\mathcal{A}[Y]$ is an open cover of Y in X. When $\mathcal{A}$ and $\mathcal{B}$ are open covers of X, $\mathcal{A} \wedge \mathcal{B}$ is also an open cover of X. For covers $\mathcal{A}$ and $\mathcal{B}$ of X, it is said that $\mathcal{A}$ is a **refinement** of $\mathcal{B}$ if $\mathcal{A} \prec \mathcal{B}$, where $\mathcal{A}$ is an **open** (or **closed**) **refinement** if $\mathcal{A}$ is an open (or closed) cover. For a space X, we denote

- $\operatorname{cov}(X)$ — the collection of all open covers of X.

Let $(X_\gamma)_{\gamma\in\Gamma}$ be a family of (topological) spaces and $X = \bigcup_{\gamma\in\Gamma} X_\gamma$. The **weak topology** on X with respect to $(X_\gamma)_{\gamma\in\Gamma}$ is defined as follows:

$$U \subset X \text{ is open in } X \;\Leftrightarrow\; \forall\gamma\in\Gamma, U\cap X_\gamma \text{ is open in } X_\gamma$$

$$\Big(A \subset X \text{ is closed in } X \;\Leftrightarrow\; \forall\gamma\in\Gamma, A\cap X_\gamma \text{ is closed in } X_\gamma\Big).$$

Suppose that X has the weak topology with respect to $(X_\gamma)_{\gamma\in\Gamma}$, and that the topologies of X_γ and $X_{\gamma'}$ agree on $X_\gamma \cap X_{\gamma'}$ for any $\gamma,\gamma'\in\Gamma$. If $X_\gamma\cap X_{\gamma'}$ is closed (resp. open) in X_γ for any $\gamma,\gamma'\in\Gamma$ then each X_γ is closed (resp. open) in X and the original topology of each X_γ is a subspace topology inherited from X. In the case that $X_\gamma\cap X_{\gamma'}=\emptyset$ for $\gamma\neq\gamma'$, X is the **topological sum** of $(X_\gamma)_{\gamma\in\Gamma}$, denoted by $X=\bigoplus_{\gamma\in\Gamma} X_\gamma$.

Let $f : X \to Y$ be a map. For $A \subset X$ and $B \subset Y$, we denote

$$f(A) = \big\{f(x) \bigm| x\in A\big\} \text{ and } f^{-1}(B) = \big\{x\in X \bigm| f(x)\in B\big\}.$$

For collections $\mathcal{A}$ and $\mathcal{B}$ of subsets of X and Y, respectively, we denote

$$f(\mathcal{A}) = \big\{f(A) \bigm| A\in\mathcal{A}\big\} \text{ and } f^{-1}(\mathcal{B}) = \big\{f^{-1}(B) \bigm| B\in\mathcal{B}\big\}.$$

The restriction of f to $A\subset X$ is denoted by $f|A$. It is said that a map $g : A\to Y$ **extends over** X if there is a map $f : X\to Y$ such that $f|A = g$. Such a map f is called an **extension** of g.

Let $[a,b]$ be a closed interval, where $a<b$. A map $f : [a,b]\to X$ is called a **path** (from $f(a)$ to $f(b)$) in X, and we say that two points $f(a)$ and $f(b)$ are connected by the path f in X. An embedding $f : [a,b]\to X$ is called an **arc** (from $f(a)$ to $f(b)$) in X, and the image $f([a,b])$ is also called an **arc**. Namely, a space is called an **arc** if it is homeomorphic to $\mathbf{I}$. It is known that each pair of distinct points $x,y\in X$ are connected by an arc if and only if they are connected by a path.[2]

For spaces X and Y, we denote

- $\mathrm{C}(X,Y)$ — the set of (continuous) maps from X to Y.

For maps $f,g : X\to Y$ (i.e., $f,g\in\mathrm{C}(X,Y)$),

- $f\simeq g$ means that f and g are **homotopic** (or f is **homotopic** to g),

that is, there is a map $h : X\times\mathbf{I}\to Y$ such that $h_0 = f$ and $h_1 = g$, where $h_t : X\to Y$, $t\in\mathbf{I}$, are defined by $h_t(x) = h(x,t)$, and h is called a **homotopy** from f to g (between f and g). When g is a constant map, it is said that f is **null-homotopic**, which we denote by $f\simeq 0$. The relation $\simeq$ is an equivalence relation on $\mathrm{C}(X,Y)$. The equivalence class $[f] = \{g\in\mathrm{C}(X,Y) \mid g\simeq f\}$ is called the **homotopy class** of f. We denote

[2] This will be shown in Corollary 5.14.6.

- $[X, Y] = \{[f] \mid f \in \mathrm{C}(X, Y)\} = \mathrm{C}(X, Y)/\simeq$
  — the set of the homotopy classes of maps from X to Y.

For each $f, f' \in \mathrm{C}(X, Y)$ and $g, g' \in \mathrm{C}(Y, Z)$, we have the following:

$$f \simeq f',\ g \simeq g' \Rightarrow gf \simeq g'f'.$$

Thus, we have the composition $[X, Y] \times [Y, Z] \to [X, Z]$ defined by $([f], [g]) \mapsto [g][f] = [gf]$. Moreover,

- $X \simeq Y$ means that X and Y are **homotopy equivalent** (or X is **homotopy equivalent** to Y),[3]

that is, there are maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$, where f is called a **homotopy equivalence** and g is a **homotopy inverse** of f.

Given subspaces $X_1, \ldots, X_n \subset X$ and $Y_1, \ldots, Y_n \subset Y$, a map $f : X \to Y$ is said to be a map from $(X, X_1, \ldots, X_n)$ to $(Y, Y_1, \ldots, Y_n)$, written

$$f : (X, X_1, \ldots, X_n) \to (Y, Y_1, \ldots, Y_n),$$

if $f(X_1) \subset Y_1, \ldots, f(X_n) \subset Y_n$. We denote

- $\mathrm{C}((X, X_1, \ldots, X_n), (Y, Y_1, \ldots, Y_n))$
  — the set of maps from $(X, X_1, \ldots, X_n)$ to $(Y, Y_1, \ldots, Y_n)$.

A homotopy h between maps $f, g \in \mathrm{C}((X, X_1, \ldots, X_n), (Y, Y_1, \ldots, Y_n))$ requires the condition that $h_t \in \mathrm{C}((X, X_1, \ldots, X_n), (Y, Y_1, \ldots, Y_n))$ for every $t \in \mathbf{I}$, i.e., h is regarded as the map

$$h : (X \times \mathbf{I}, X_1 \times \mathbf{I}, \ldots, X_n \times \mathbf{I}) \to (Y, Y_1, \ldots, Y_n).$$

Thus, $\simeq$ is an equivalence relation on $\mathrm{C}((X, X_1, \ldots, X_n), (Y, Y_1, \ldots, Y_n))$. We denote

- $[(X, X_1, \ldots, X_n), (Y, Y_1, \ldots, Y_n)] = \mathrm{C}((X, X_1, \ldots, X_n), (Y, Y_1, \ldots, Y_n))/\simeq$.

When there exist maps

$$f : (X, X_1, \ldots, X_n) \to (Y, Y_1, \ldots, Y_n),$$
$$g : (Y, Y_1, \ldots, Y_n) \to (X, X_1, \ldots, X_n)$$

such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$, we denote

- $(X, X_1, \ldots, X_n) \simeq (Y, Y_1, \ldots, Y_n)$.

[3]It is also said that X and Y have the **same homotopy type** or X has the **homotopy type** of Y.

Similarly, for each pair of pointed spaces (X, x_0) and (Y, y_0),

- $C((X, x_0), (Y, y_0)) = C((X, \{x_0\}), (Y, \{y_0\}))$;
- $[(X, x_0), (Y, y_0)] = C((X, x_0), (Y, y_0))/\simeq$;
- $(X, x_0) \simeq (Y, y_0)$ means $(X, \{x_0\}) \simeq (Y, \{y_0\})$.

For $A \subset X$, a homotopy $h : X \times \mathbf{I} \to Y$ is called a **homotopy relative to** A if $h(\{x\} \times \mathbf{I})$ is degenerate (i.e., a singleton) for every $x \in A$. When a homotopy from f to g is a homotopy relative to A (where $f|A = g|A$), we denote

- $f \simeq g$ rel. A.

Let $f, g : X \to Y$ be maps and $\mathcal{U}$ a collection of subsets of Y (in usual, $\mathcal{U} \in \operatorname{cov}(Y)$). It is said that f and g are **$\mathcal{U}$-close** (or f is **$\mathcal{U}$-close** to g) if

$$\big\{\{f(x), g(x)\} \bigm| x \in X\big\} \prec \mathcal{U} \cup \big\{\{y\} \bigm| y \in Y\big\},$$

which implies that $\mathcal{U}$ covers the set $\{f(x), g(x) \mid f(x) \neq g(x)\}$. A homotopy h is called a **$\mathcal{U}$-homotopy** if

$$\big\{h(\{x\} \times \mathbf{I}) \bigm| x \in X\big\} \prec \mathcal{U} \cup \big\{\{y\} \bigm| y \in Y\big\},$$

which implies that $\mathcal{U}$ covers the set

$$\bigcup \big\{h(\{x\} \times \mathbf{I}) \bigm| h(\{x\} \times \mathbf{I}) \text{ is non-degenerate}\big\}.$$

We say that f and g are **$\mathcal{U}$-homotopic** (or f is **$\mathcal{U}$-homotopic** to g) and denoted by $f \simeq_{\mathcal{U}} g$ if there is a $\mathcal{U}$-homotopy $h : X \times \mathbf{I} \to Y$ such that $h_0 = f$ and $h_1 = g$.

When $Y = (Y, d)$ is a metric space, we define the distance between $f, g \in C(X, Y)$ as follows:

$$d(f, g) = \sup \big\{d(f(x), g(x)) \bigm| x \in X\big\}.$$

In general, it may be possible that $d(f, g) = \infty$, in which case d is not a metric on the set $C(X, Y)$. If Y is bounded or X is compact, then this d is a metric on the set $C(X, Y)$, called the **sup-metric**. For $\varepsilon > 0$, we say that f and g are **ε-close** or f is **ε-close** to g if $d(f, g) < \varepsilon$. A homotopy h is called an **ε-homotopy** if $\operatorname{mesh}\{h(\{x\} \times \mathbf{I}) \mid x \in X\} < \varepsilon$, where $f = h_0$ and $g = h_1$ are said to be **ε-homotopic** and denoted by $f \simeq_\varepsilon g$.

In the above, even if d is not a metric on $C(X, Y)$ (i.e., $d(f, g) = \infty$ for some $f, g \in C(X, Y)$), it induces a topology on $C(X, Y)$ such that each f has a neighborhood basis consisting of

$$B_d(f, \varepsilon) = \big\{g \in C(X, Y) \bigm| d(f, g) < \varepsilon\big\}, \ \varepsilon > 0.$$

This topology is called the **uniform convergence topology**.

The **compact-open topology** on $\mathrm{C}(X,Y)$ is generated by the sets

$$\langle K;U\rangle = \{f \in \mathrm{C}(X,Y) \mid f(K) \subset U\},$$

where K is any compact set in X and U is any open set in Y. With respect to this topology, we have the following:

Proposition 1.1.1. *Every map $f : Z \times X \to Y$ (or $f : X \times Z \to Y$) induces the map $\bar{f} : Z \to \mathrm{C}(X,Y)$ defined by $\bar{f}(z)(x) = f(z,x)$ (or $\bar{f}(z)(x) = f(x,z)$).*

Proof. For each $z \in Z$, it is easy to see that $\bar{f}(z) : X \to Y$ is continuous, i.e., $\bar{f}(z) \in \mathrm{C}(X,Y)$. Thus, $\bar{f}$ is well-defined.

To verify the continuity of $\bar{f} : Z \to \mathrm{C}(X,Y)$, it suffices to show that $\bar{f}^{-1}(\langle K;U\rangle)$ is open in Z for each compact set K in X and each open set U in Y. Let $z \in \bar{f}^{-1}(\langle K;U\rangle)$, i.e., $f(\{z\} \times K) \subset U$. Using the compactness of K, we can easily find an open neighborhood V of z in Z such that $f(V \times K) \subset U$, which means that $V \subset \bar{f}^{-1}(\langle K;U\rangle)$. □

With regards to the relation $\simeq$ on $\mathrm{C}(X,Y)$, we have the following:

Proposition 1.1.2. *Each $f, g \in \mathrm{C}(X,Y)$ are connected by a path in $\mathrm{C}(X,Y)$. When X is metrizable or locally compact, the converse is also true, that is, $f \simeq g$ if and only if f and g are connected by a path in $\mathrm{C}(X,Y)$ if $f \simeq g$.*[4]

Proof. By Proposition 1.1.1, a homotopy $h : X \times \mathbf{I} \to Y$ from f to g induces the path $\bar{h} : \mathbf{I} \to \mathrm{C}(X,Y)$ defined as $\bar{h}(t)(x) = h(x,t)$ for each $t \in \mathbf{I}$ and $x \in X$, where $\bar{h}(0) = f$ and $\bar{h}(1) = g$.

For a path $\varphi : \mathbf{I} \to \mathrm{C}(X,Y)$ from f to g, we define the homotopy $\tilde{\varphi} : X \times \mathbf{I} \to Y$ as $\tilde{\varphi}(x,t) = \varphi(t)(x)$ for each $(x,t) \in X \times \mathbf{I}$. Then, $\tilde{\varphi}_0 = \varphi(0) = f$ and $\tilde{\varphi}_1 = \varphi(1) = g$. It remains to show that $\tilde{\varphi}$ is continuous if X is metrizable or locally compact.

In the case that X is locally compact, for each $(x,t) \in X \times \mathbf{I}$ and for each open neighborhood U of $\tilde{\varphi}(x,t) = \varphi(t)(x)$ in Y, x has a compact neighborhood K in X such that $\varphi(t)(K) \subset U$, i.e., $\varphi(t) \in \langle K;U\rangle$. By the continuity of φ, t has a neighborhood V in $\mathbf{I}$ such that $\varphi(V) \subset \langle K;U\rangle$. Thus, $K \times V$ is a neighborhood of $(x,t) \in X \times \mathbf{I}$ and $\tilde{\varphi}(K \times V) \subset U$. Hence, $\tilde{\varphi}$ is continuous.

In the case that X is metrizable, let us assume that $\tilde{\varphi}$ is not continuous at $(x,t) \in X \times \mathbf{I}$. Then, $\tilde{\varphi}(x,t)$ has some open neighborhood U in Y such that $\tilde{\varphi}(V) \not\subset U$ for any neighborhood V of (x,t) in $X \times \mathbf{I}$. Let $d \in \mathrm{Metr}(X)$. For each $n \in \mathbb{N}$, we have $x_n \in X$ and $t_n \in \mathbf{I}$ such that $d(x_n,x) < 1/n$, $|t_n - t| < 1/n$ and $\tilde{\varphi}(x_n,t_n) \notin U$. Because $x_n \to x$ $(n \to \infty)$ and $\varphi(t)$ is continuous, we have $n_0 \in \mathbb{N}$ such that $\varphi(t)(x_n) \in U$ for all $n \geq n_0$. Note that $K = \{x_n, x \mid n \geq n_0\}$ is compact and $\varphi(t)(K) \subset U$. Because $t_n \to t$ $(n \to \infty)$ and φ is continuous at t, $\varphi(t_{n_1})(K) \subset U$ for some $n_1 \geq n_0$. Thus, $\tilde{\varphi}(x_{n_1}, t_{n_1}) \in U$, which is a contradiction. Consequently, $\tilde{\varphi}$ is continuous. □

Remark 1. It is easily observed that Proposition 1.1.2 is also valid for

[4]More generally, this is valid for every k-space X, where X is a $\boldsymbol{k}$**-space** provided U is open in X if $U \cap K$ is open in K for every compact set $K \subset X$. A k-space is also called a **compactly generated space**.

$$\mathrm{C}((X, X_1, \ldots, X_n), (Y, Y_1, \ldots, Y_n)).$$

Some Properties of the Compact-Open Topology 1.1.3.

The following hold with respect to the compact-open topology:

(1) For $f \in \mathrm{C}(Z, X)$ and $g \in \mathrm{C}(Y, Z)$, the following are continuous:

$$f^* : \mathrm{C}(X, Y) \to \mathrm{C}(Z, Y),\ f^*(h) = h \circ f;$$
$$g_* : \mathrm{C}(X, Y) \to \mathrm{C}(X, Z),\ g_*(h) = g \circ h.$$

(2) When Y is locally compact, the following (composition) is continuous:

$$\mathrm{C}(X, Y) \times \mathrm{C}(Y, Z) \ni (f, g) \mapsto g \circ f \in \mathrm{C}(X, Z).$$

Sketch of Proof. Let K be a compact set in X and U an open set in Z with $f \in \mathrm{C}(X, Y)$ and $g \in \mathrm{C}(Y, Z)$ such that $g \circ f(K) \subset U$. Since Y is locally compact, we have an open set V in Y such that $\mathrm{cl}\, V$ is compact, $f(K) \subset V$ and $g(\mathrm{cl}\, V) \subset U$. Then, $f'(K) \subset V$ and $g'(\mathrm{cl}\, V) \subset U$ imply $g' \circ f'(K) \subset U$.

(3) For each $x_0 \in X$, the following (evaluation) is continuous:

$$\mathrm{C}(X, Y) \ni f \mapsto f(x_0) \in Y.$$

(4) When X is locally compact, the following (evaluation) is continuous:

$$\mathrm{C}(X, Y) \times X \ni (f, x) \mapsto f(x) \in Y.$$

In this case, for every map $f : Z \to \mathrm{C}(X, Y)$, the following is continuous:

$$Z \times X \ni (z, x) \mapsto f(z)(x) \in Y.$$

(5) In the case that X is locally compact, we have the following inequalities:

$$w(Y) \le w(\mathrm{C}(X, Y)) \le \aleph_0 w(X) w(Y).$$

Sketch of Proof. By embedding Y into $\mathrm{C}(X, Y)$, we obtain the first inequality. For the second, we take open bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ for X and Y, respectively, such that $\mathrm{card}\, \mathcal{B}_X = w(X)$, $\mathrm{card}\, \mathcal{B}_Y = w(Y)$, and $\mathrm{cl}\, A$ is compact for every $A \in \mathcal{B}_X$. The following is an open sub-basis for $\mathrm{C}(X, Y)$:

$$\mathcal{B} = \big\{ \langle \mathrm{cl}\, A, B \rangle \ \big|\ (A, B) \in \mathcal{B}_X \times \mathcal{B}_Y \big\}.$$

Indeed, let K be a compact set in X, U be an open set in Y, and $f \in \mathrm{C}(X, Y)$ with $f(K) \subset U$, i.e., $f \in \langle K, U \rangle$. First, find $B_1, \ldots, B_n \in \mathcal{B}_Y$ so that $f(K) \subset B_1 \cup \cdots \cup B_n \subset U$. Next, find $A_1, \ldots, A_m \in \mathcal{B}_X$ so that $K \subset A_1 \cup \cdots \cup A_m$ and each $\mathrm{cl}\, A_i$ is contained in some $f^{-1}(B_{j(i)})$. Then, $f \in \bigcap_{i=1}^m \langle \mathrm{cl}\, A_i, B_{j(i)} \rangle \subset \langle K, U \rangle$.

(6) If X is compact and $Y = (Y, d)$ is a metric space, then the sup-metric on $\mathrm{C}(X, Y)$ is admissible for the compact-open topology on $\mathrm{C}(X, Y)$.

Sketch of Proof. Let K be a compact set in X and U be an open set in Y with $f \in \mathrm{C}(X, Y)$ such that $f(K) \subset U$. Then, $\delta = \operatorname{dist}(f(K), Y \setminus U) > 0$, and $d(f, f') < \delta$ implies $f'(K) \subset U$. Conversely, for each $\varepsilon > 0$ and $f \in \mathrm{C}(X, Y)$, we have $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^n f^{-1}(\mathrm{B}(f(x_i), \varepsilon/4)$. Observe that

$$\begin{aligned} &f'(f^{-1}(\overline{\mathrm{B}}(f(x_i), \varepsilon/4))) \subset \mathrm{B}(f(x_i), \varepsilon/2) \ (\forall i = 1, \ldots, n) \\ &\Rightarrow d(f, f') < \varepsilon. \end{aligned}$$

(7) Let $X = \bigcup_{n \in \mathbb{N}} X_n$, where X_n is compact and $X_n \subset \operatorname{int} X_{n+1}$. If $Y = (Y, d)$ is a metric space, then $\mathrm{C}(X, Y)$ with the compact-open topology is metrizable.

Sketch of Proof. We define a metric ρ on $\mathrm{C}(X, Y)$ as follows:

$$\rho(f, g) = \sup_{n \in \mathbb{N}} \min \left\{ n^{-1}, \sup_{x \in X_n} d(f(x), g(x)) \right\}.$$

Then, ρ is admissible for the compact-open topology on $\mathrm{C}(X, Y)$. To see this, refer to the proof of (6).

1.2 Banach Spaces in the Product of Real Lines

Throughout this section, let Γ be an infinite set. We denote

- $\operatorname{Fin}(\Gamma)$ — the set of all non-empty finite subsets of Γ.

Note that $\operatorname{card} \operatorname{Fin}(\Gamma) = \operatorname{card} \Gamma$. The product space $\mathbb{R}^\Gamma$ is a linear space with the following scalar multiplication and addition:

$$\mathbb{R}^\Gamma \times \mathbb{R} \ni (x, t) \mapsto tx = (tx(\gamma))_{\gamma \in \Gamma} \in \mathbb{R}^\Gamma;$$

$$\mathbb{R}^\Gamma \times \mathbb{R}^\Gamma \ni (x, y) \mapsto x + y = (x(\gamma) + y(\gamma))_{\gamma \in \Gamma} \in \mathbb{R}^\Gamma.$$

In this section, we consider various (complete) norms defined on linear subspaces of $\mathbb{R}^\Gamma$. In general, the unit closed ball and the unit sphere of a normed linear space $X = (X, \|\cdot\|)$ are denoted by $\mathbf{B}_X$ and $\mathbf{S}_X$, respectively. Namely, let

$$\mathbf{B}_X = \{x \in X \mid \|x\| \leq 1\} \text{ and } \mathbf{S}_X = \{x \in X \mid \|x\| = 1\}.$$

The zero vector (the zero element) of X is denoted by $\mathbf{0}_X$, or simply $\mathbf{0}$ if there is no possibility of confusion.

Before considering norms, we first discuss the product topology of $\mathbb{R}^\Gamma$. The scalar multiplication and addition are continuous with respect to the product topology. Namely, $\mathbb{R}^\Gamma$ with the product topology is a topological linear space.[5] Note that $w(\mathbb{R}^\Gamma) = \operatorname{card}\Gamma$.

Let $\mathcal{B}_0$ be a countable open basis for $\mathbb{R}$. Then, $\mathbb{R}^\Gamma$ has the following open basis:

$$\left\{ \bigcap_{\gamma\in F} \mathrm{pr}_\gamma^{-1}(B_\gamma) \;\middle|\; F \in \mathrm{Fin}(\Gamma),\ B_\gamma \in \mathcal{B}_0\ (\gamma \in F) \right\}.$$

Thus, we have $w(\mathbb{R}^\Gamma) \le \aleph_0 \operatorname{card}\mathrm{Fin}(\Gamma) = \operatorname{card}\Gamma$. Let $\mathcal{B}$ be an open basis for $\mathbb{R}^\Gamma$. For each $B \in \mathcal{B}$, we can find $F_B \in \mathrm{Fin}(\Gamma)$ such that $\mathrm{pr}_\gamma(B) = \mathbb{R}$ for every $\gamma \in \Gamma \setminus F_B$. Then, $\operatorname{card}\bigcup_{B\in\mathcal{B}} F_B \le \aleph_0 \operatorname{card}\mathcal{B}$. If $\operatorname{card}\mathcal{B} < \operatorname{card}\Gamma$ then $\operatorname{card}\bigcup_{B\in\mathcal{B}} F_B < \operatorname{card}\Gamma$, so we have $\gamma_0 \in \Gamma \setminus \bigcup_{B\in\mathcal{B}} F_B$. The open set $\mathrm{pr}_{\gamma_0}^{-1}((0,\infty)) \subset \mathbb{R}^\Gamma$ contains some $B \in \mathcal{B}$. Then, $\mathrm{pr}_{\gamma_0}(B) \subset (0,\infty)$, which means that $\gamma_0 \in F_B$. This is a contradiction. Therefore, $\operatorname{card}\mathcal{B} \ge \operatorname{card}\Gamma$, and thus we have $w(\mathbb{R}^\Gamma) \ge \operatorname{card}\Gamma$.

For each $\gamma \in \Gamma$, we define the unit vector $\mathbf{e}_\gamma \in \mathbb{R}^\Gamma$ by $\mathbf{e}_\gamma(\gamma) = 1$ and $\mathbf{e}_\gamma(\gamma') = 0$ for $\gamma' \neq \gamma$. It should be noted that $\{\mathbf{e}_\gamma \mid \gamma \in \Gamma\}$ is not a Hamel basis for $\mathbb{R}^\Gamma$, and the linear span of $\{\mathbf{e}_\gamma \mid \gamma \in \Gamma\}$ is the following:[6]

$$\mathbb{R}^\Gamma_f = \left\{ x \in \mathbb{R}^\Gamma \;\middle|\; x(\gamma) = 0 \ \text{ except for finitely many } \gamma \in \Gamma \right\},$$

which is a dense linear subspace of $\mathbb{R}^\Gamma$. The subspace $\mathbb{R}^\mathbb{N}_f$ of $\boldsymbol{s} = \mathbb{R}^\mathbb{N}$ is also denoted by $\boldsymbol{s}_f$, which is the space of finite sequences (with the product topology). When $\operatorname{card}\Gamma = \aleph_0$, the space $\mathbb{R}^\Gamma$ is linearly homeomorphic to the space of sequences $\boldsymbol{s} = \mathbb{R}^\mathbb{N}$, i.e., there exists a linear homeomorphism between $\mathbb{R}^\Gamma$ and $\boldsymbol{s}$, where the linear subspace $\mathbb{R}^\Gamma_f$ is linearly homeomorphic to $\boldsymbol{s}_f$ by the same homeomorphism. The following fact can easily be observed:

Fact. *The following are equivalent:*

(a) $\mathbb{R}^\Gamma$ *is metrizable;*
(b) $\mathbb{R}^\Gamma_f$ *is metrizable;*
(c) $\mathbb{R}^\Gamma_f$ *is first countable;*
(d) $\operatorname{card}\Gamma \le \aleph_0$.

The implication (c) $\Rightarrow$ (d) is shown as follows: Let $\{U_i \mid i \in \mathbb{N}\}$ be a neighborhood basis of $\mathbf{0}$ in $\mathbb{R}^\Gamma_f$. Then, each $\Gamma_i = \{\gamma \in \Gamma \mid \mathbb{R}\mathbf{e}_\gamma \not\subset U_i\}$ is finite. If Γ is uncountable, then $\Gamma \setminus \bigcup_{i\in\mathbb{N}} \Gamma_i \neq \emptyset$, i.e., $\mathbb{R}\mathbf{e}_\gamma \subset \bigcap_{i\in\mathbb{N}} U_i$ for some $\gamma \in \Gamma$. In this case, $U_i \not\subset \mathrm{pr}_\gamma^{-1}((-1,1))$ for every $i \in \mathbb{N}$, which is a contradiction.

Thus, every linear subspace L of $\mathbb{R}^\Gamma$ containing $\mathbb{R}^\Gamma_f$ is non-metrizable if Γ is uncountable, and it is metrizable if Γ is countable. On the other hand, due to the following proposition, every linear subspaces L of $\mathbb{R}^\Gamma$ containing $\mathbb{R}^\Gamma_f$ is non-normable if Γ is infinite.

Proposition 1.2.1. *Let Γ be an infinite set. Any norm on $\mathbb{R}^\Gamma_f$ does not induce the topology inherited from the product topology of $\mathbb{R}^\Gamma$.*

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[5]For topological linear spaces, refer Sect. 3.4.

[6]The linear subspace generated by a set B is called the **linear span** of B.

Proof. Assume that the topology of $\mathbb{R}^\Gamma_f$ is induced by a norm $\|\cdot\|$. Because $U = \{x \in \mathbb{R}^\Gamma_f \mid \|x\| < 1\}$ is an open neighborhood of $\mathbf{0}$ in $\mathbb{R}^\Gamma_f$, we have a finite set $F \subset \Gamma$ and neighborhoods V_γ of $0 \in \mathbb{R}$, $\gamma \in F$, such that $\mathbb{R}^\Gamma_f \cap \bigcap_{\gamma \in F} \mathrm{pr}_\gamma^{-1}(V_\gamma) \subset U$. Take $\gamma_0 \in \Gamma \setminus F$. As $\mathbb{R}\mathbf{e}_{\gamma_0} \subset U$, we have $\|\mathbf{e}_{\gamma_0}\|^{-1}\mathbf{e}_{\gamma_0} \in U$ but $\big\| \|\mathbf{e}_{\gamma_0}\|^{-1}\mathbf{e}_{\gamma_0} \big\| = \|\mathbf{e}_{\gamma_0}\|^{-1}\|\mathbf{e}_{\gamma_0}\| = 1$, which is a contradiction. □

The Banach space $\ell_\infty(\Gamma)$ and its closed linear subspaces $\boldsymbol{c}(\Gamma) \supset \boldsymbol{c}_0(\Gamma)$ are defined as follows:

- $\ell_\infty(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \sup_{\gamma \in \Gamma} |x(\gamma)| < \infty\}$ with the sup-norm
$$\|x\|_\infty = \sup_{\gamma \in \Gamma} |x(\gamma)|;$$
- $\boldsymbol{c}(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \exists t \in \mathbb{R}$ such that $\forall \varepsilon > 0,\ |x(\gamma) - t| < \varepsilon$ except for finitely many $\gamma \in \Gamma\}$;
- $\boldsymbol{c}_0(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \forall \varepsilon > 0,\ |x(\gamma)| < \varepsilon$ except for finitely many $\gamma \in \Gamma\}$.

These are linear subspaces of $\mathbb{R}^\Gamma$, but are not topological subspace according to Proposition 1.2.1. The space $\boldsymbol{c}(\Gamma)$ is linearly homeomorphic to $\boldsymbol{c}_0(\Gamma) \times \mathbb{R}$ by the correspondence

$$\boldsymbol{c}_0(\Gamma) \times \mathbb{R} \ni (x, t) \mapsto (x(\gamma) + t)_{\gamma \in \Gamma} \in \boldsymbol{c}(\Gamma).$$

This correspondence and its inverse are Lipschitz with respect to the norm $\|(x,t)\| = \max\{\|x\|_\infty, |t|\}$. Indeed, let $y = (x(\gamma)+t)_{\gamma \in \Gamma}$. Then, $\|y\|_\infty \le \|x\|_\infty + |t| \le 2\|(x,t)\|$. Because $x \in \boldsymbol{c}_0(\Gamma)$ and $|t| \le |y(\gamma)| + |x(\gamma)| \le \|y\|_\infty + |x(\gamma)|$ for every $\gamma \in \Gamma$, it follows that $|t| \le \|y\|_\infty$. Moreover, $|x(\gamma)| \le |y(\gamma)| + |t| \le 2\|y\|_\infty$ for every $\gamma \in \Gamma$. Hence, $\|x\|_\infty \le 2\|y\|_\infty$, and thus we have $\|(x,t)\| \le 2\|y\|_\infty$.

Furthermore, we denote $\mathbb{R}^\Gamma_f$ with this norm as $\ell^f_\infty(\Gamma)$. We then have the inclusions:

$$\ell^f_\infty(\Gamma) \subset \boldsymbol{c}_0(\Gamma) \subset \boldsymbol{c}(\Gamma) \subset \ell_\infty(\Gamma).$$

The topology of $\ell^f_\infty(\Gamma)$ is different from the topology inherited from the product topology. Indeed, $\{\mathbf{e}_\gamma \mid \gamma \in \Gamma\}$ is discrete in $\ell^f_\infty(\Gamma)$, but $\mathbf{0}$ is a cluster point of this set with respect to the product topology.

We must pay attention to the following fact:

Proposition 1.2.2. *For an arbitrary infinite set Γ,*

$$w(\ell_\infty(\Gamma)) = 2^{\operatorname{card}\Gamma} \quad \textit{but} \quad w(\boldsymbol{c}(\Gamma)) = w(\boldsymbol{c}_0(\Gamma)) = w(\ell^f_\infty(\Gamma)) = \operatorname{card}\Gamma.$$

Proof. The characteristic map $\chi_\Lambda : \Gamma \to \{0, 1\} \subset \mathbb{R}$ of $\Lambda \subset \Gamma$ belongs to $\ell_\infty(\Gamma)$ ($\chi_\emptyset = \mathbf{0} \in \ell_\infty(\Gamma)$), where $\|\chi_\Lambda - \chi_{\Lambda'}\|_\infty = 1$ if $\Lambda \ne \Lambda' \subset \Gamma$. It follows that $w(\ell_\infty(\Gamma)) = c(\ell_\infty(\Gamma)) \ge 2^{\operatorname{card}\Gamma}$. Moreover, $\mathbb{Q}^\Gamma \cap \ell_\infty(\Gamma)$ is dense in $\ell_\infty(\Gamma)$, and hence we have

$$w(\ell_\infty(\Gamma)) = \operatorname{dens} \ell_\infty(\Gamma) \le \operatorname{card} \mathbb{Q}^\Gamma = \aleph_0^{\operatorname{card}\Gamma} = 2^{\operatorname{card}\Gamma}.$$

On the other hand, $\mathbf{e}_\gamma \in \ell_\infty^f(\Gamma)$ for each $\gamma \in \Gamma$ and $\|\mathbf{e}_\gamma - \mathbf{e}_{\gamma'}\|_\infty = 1$ if $\gamma \neq \gamma'$. Since $\ell_\infty^f(\Gamma) \subset \boldsymbol{c}_0(\Gamma)$, it follows that

$$w(\boldsymbol{c}_0(\Gamma)) \ge w(\ell_\infty^f(\Gamma)) = c(\ell_\infty^f(\Gamma)) \ge \operatorname{card}\Gamma.$$

Moreover, $\boldsymbol{c}_0(\Gamma)$ has the following dense subset:

$$\mathbb{Q}_f^\Gamma = \{x \in \mathbb{Q}^\Gamma \mid x(\gamma) = 0 \text{ except for finitely many } \gamma \in \Gamma\},$$

and so it follows that

$$w(\boldsymbol{c}_0(\Gamma)) = \operatorname{dens} \boldsymbol{c}_0(\Gamma) \le \operatorname{card} \mathbb{Q}_f^\Gamma \le \aleph_0 \operatorname{card} \operatorname{Fin}(\Gamma) = \operatorname{card}\Gamma.$$

Thus, we have $w(\boldsymbol{c}_0(\Gamma)) = w(\ell_\infty^f(\Gamma)) = \operatorname{card}\Gamma$. As already observed, $\boldsymbol{c}(\Gamma) \approx \boldsymbol{c}_0(\Gamma) \times \mathbb{R}$, hence $w(\boldsymbol{c}(\Gamma)) = w(\boldsymbol{c}_0(\Gamma))$. □

When $\Gamma = \mathbb{N}$, we write

- $\ell_\infty(\mathbb{N}) = \ell_\infty$ — the space of bounded sequences,
- $\boldsymbol{c}(\mathbb{N}) = \boldsymbol{c}$ — the space of convergent sequences,
- $\boldsymbol{c}_0(\mathbb{N}) = \boldsymbol{c}_0$ — the space of sequences convergent to 0, and
- $\ell_\infty^f(\mathbb{N}) = \ell_\infty^f$ — the space of finite sequences with the sup-norm,

where $\ell_\infty^f \neq \boldsymbol{s}_f$ as (topological) spaces. According to Proposition 1.2.2, $\boldsymbol{c}$ and $\boldsymbol{c}_0$ are separable, but ℓ_∞ is non-separable. When $\operatorname{card}\Gamma = \aleph_0$, the spaces $\ell_\infty(\Gamma)$, $\boldsymbol{c}(\Gamma)$, and $\boldsymbol{c}_0(\Gamma)$ are linearly isometric to these spaces ℓ_∞, $\boldsymbol{c}$ and $\boldsymbol{c}_0$, respectively.

Here, we regard $\operatorname{Fin}(\Gamma)$ as a directed set by $\subset$. For $x \in \mathbb{R}^\Gamma$, we say that $\sum_{\gamma\in\Gamma} x(\gamma)$ is **convergent** if $\left(\sum_{\gamma\in F} x(\gamma)\right)_{F\in\operatorname{Fin}(\Gamma)}$ is convergent, and define

$$\sum_{\gamma\in\Gamma} x(\gamma) = \lim_{F\in\operatorname{Fin}(\Gamma)} \sum_{\gamma\in F} x(\gamma).$$

In the case that $x(\gamma) \ge 0$ for all $\gamma \in \Gamma$, $\sum_{\gamma\in\Gamma} x(\gamma)$ is convergent if and only if $\left(\sum_{\gamma\in F} x(\gamma)\right)_{F\in\operatorname{Fin}(\Gamma)}$ is upper bounded, and then

$$\sum_{\gamma\in\Gamma} x(\gamma) = \sup_{F\in\operatorname{Fin}(\Gamma)} \sum_{\gamma\in F} x(\gamma).$$

By this reason, $\sum_{\gamma\in\Gamma} x(\gamma) < \infty$ means that $\sum_{\gamma\in\Gamma} x(\gamma)$ is convergent.

For $x \in \mathbb{R}^\mathbb{N}$, we should distinguish $\sum_{i\in\mathbb{N}} x(i)$ from $\sum_{i=1}^\infty x(i)$. When the sequence $\left(\sum_{i=1}^n x(i)\right)_{n\in\mathbb{N}}$ is convergent, we say that $\sum_{i=1}^\infty x(i)$ is **convergent**, and define