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Preface

The evolution of economic agglomeration in cities is observed historically. The most likely and most accepted scenario is the evolution from an evenly spread population of farmers with evenly spread economic activities en route to agglomeration of economic activities in a few urban regions. Urbanization is indeed the cradle of economic development and prosperity. Nowadays, megalopolises prosper worldwide: Tokyo, Jakarta, New York, Seoul, Manila, Mumbai, São Paulo, Mexico City, Delhi, and Shanghai, to name a few. A question to be answered is "*How* and *where* is spatial agglomeration self-organized?"

The problem of self-organization involves several aspects of human activities and, accordingly, is of interest in various fields of studies. The self-organization of hexagonal agglomeration patterns of industrial regions was first predicted by central place theory in economic geography based on an empirical investigation of southern German cities. Self-organization of such distributions in two-dimensional economic agglomeration was envisaged by Krugman, who developed a new economic geographical model for spatial agglomeration. This model incorporated microeconomic mechanisms, including the following: monopolistic competition model of Dixit–Stiglitz, increasing returns at the level of firms, iceberg transport costs, factor mobility, and so on. Krugman noticed the vital role of bifurcation in the evolution of economic agglomeration. This motivated a thorough study of the mathematical mechanism of bifurcation engendering economic agglomeration presented in this book.

A two-dimensional space for economic activities is modeled, in this book, by a hexagonal lattice with periodic boundaries. Places (cities) for economic activities, such as consumption and production, are located on the nodes of this lattice and are connected by roads forming a regular-triangular mesh. By virtue of periodic boundaries, every place on the lattice enjoys equal competition. Microeconomic interactions among the places are expressed by core–periphery models in new economic geography. Population distribution on this lattice is obtained as a solution to the governing equation of these models. Manufactured goods are transported along these roads at a certain transport cost. When the transport cost is high, the uniform state, in which each place has the same population and is in the same economic state, is stable. Yet, when the cost is reduced to a certain level in association with the progress of technology, the uniform state is destabilized by bifurcations to produce prospering places with increasing population and decaying ones with decreasing population. Then certain patterns, the hexagonal ones being of particular interest, are self-organized.

In this book, after a brief introduction of central place theory and new economic geography, the missing link between them is discovered by elucidating the mechanism of the evolution of bifurcating hexagonal patterns. Pattern formation by such bifurcation is a well-studied topic in nonlinear mathematics, and group-theoretic bifurcation analysis is a well-developed theoretical tool to investigate possible bifurcating patterns. A finite hexagonal lattice is used in this book to express uniformly distributed places, and the symmetry of this lattice is expressed by a finite group. Several mathematical methodologies indispensable for tackling the present problem are gathered in a self-contained manner. The existence of hexagonal distributions is verified by group-theoretic bifurcation analysis, first by applying the so-called equivariant branching lemma and next by solving the bifurcation equation. This book consequently offers a complete guide for the application of group-theoretic bifurcation analysis to economic agglomeration on the hexagonal lattice.

As a main technical contribution of this book, a complete analysis of bifurcating solutions for hexagonal distributions from critical points of multiplicity 12 is conducted. Mathematically, the analysis of hexagonal distributions is carried out in a streamlined manner by means of fundamental theoretical tools for integer matrices, such as the Smith normal form and determinantal divisors. In particular, a solvability criterion for a system of linear equations in integer unknown variables that refers to determinantal divisors plays a significant role. Duality nature between the two methods, one by the equivariant branching lemma and the other by bifurcation equations, is made clear. The equivalence of the results obtained by these two methods is established through a theorem for integrality of solutions, the so-called integer analogue of the Farkas lemma.

Numerical bifurcation analysis of an economy on the hexagonal lattice with periodic boundaries is conducted to demonstrate the emergence of hexagonal distributions envisaged in central place theory, led by Christaller and Lösch. Moreover, as a step toward a connection with the real world, self-organization of central places is demonstrated for a domain with the shape of southern Germany without periodic boundaries. This is the birthplace of central place theory, to which Christaller's theory was first applied, and nowadays cities of several sizes are scattered on its relatively flat land to exhibit a hierarchy of central places. Unlike the hexagonal lattice with symmetry, no bifurcation for agglomeration would occur on this domain as it has no symmetry, but it has turned out that bifurcation serves as an underlying mechanism for the progress of the agglomeration. The hexagonal patterns, observed also for the hexagonal lattice, have appeared in the middle of the domain, just as regularly-arrayed hexagonal cells appear in the experiment of Bénard convection in fluid dynamics. This indicates the generality of the emergence of hexagonal patterns in spatial agglomeration, regardless of the shape and the boundary conditions of the domain. Thus the agglomeration behaviors on the hexagonal lattice capture much desired realism at the expense of idealization of periodic boundaries. This suffices to show the role and the importance of the theoretical and computational analysis on the hexagonal lattice presented in this book.

Hexagonal agglomeration in economic geography investigated in this book is a topic of interdisciplinary study of various fields, encompassing central place theory in economic geography, core–periphery models in new economic geography, and bifurcation theory in nonlinear mathematics. Naturally, several prerequisites in preparation for this study are contained in this book. The book can be read profitably by those who study applied mathematics as it presents related backgrounds of central place theory and geographical models in a self-contained manner. The book can also be addressed to professional researchers of economic geography and new economic geography. A complete guide of group-theoretic bifurcation analysis is provided encompassing introductory fundamental issues and an application to the economy on the hexagonal lattice. A complete classification of hexagonal distributions that can appear on the hexagonal lattice is obtained so as to assist the understanding of agglomeration behavior. The present methodology is endowed with extendibility to other problems with other symmetry groups. In addition, numerically obtained agglomeration patterns would contribute to gaining an intuitive understanding of two-dimensional agglomeration. In particular, agglomeration patterns of southern Germany points to a promising direction of a further study. Ample references are introduced to assist readers who are interested in further study.

The book comprises two parts. Part I is devoted to the preparation of fundamental issues, whereas the hexagonal agglomeration in economic geography is revealed in light of bifurcation in Part II.

Part I is organized as follows. Chapter 1 introduces several fundamental and introductory issues. The hexagonal market areas of Christaller and Lösch's hexagons studied in central place theory are introduced. As a step toward a connection with the real world, self-organization of a hexagonal distribution called Christaller's $k = 3$ system is demonstrated for a domain with the shape of southern Germany with a microeconomic mechanism of Krugman's core–periphery model in new economic geography. Such a distribution is observed much clearer for an economy on the hexagonal lattice with periodic boundaries. This shows the suitability of such hexagonal lattice as a spatial platform for agglomeration in economic geography. The equilibrium equation of Krugman's core–periphery model is formulated by assembling several relations on economic concepts, and its stability is described. The history of the study of self-organization of cities is reviewed, encompassing works in economic geography, new economic geography, and physics. Chapter 2 presents fundamentals of group-theoretic bifurcation theory. Agglomeration of population in a two-place economy is advanced in order to demonstrate the predominant role of bifurcation in economic agglomeration. The mechanism of this bifurcation is elucidated by group-theoretic bifurcation analysis of a system with dihedral-group symmetry. Bifurcation equation, equivariant branching lemma, and block-diagonalization are introduced as mathematical tools used to tackle bifurcation of a symmetric system. Chapter 3 serves as a bridge to the study in Part II. The spatial agglomeration in a racetrack economy is investigated as an application of group-theoretic bifurcation analysis to a problem with a simple group, the dihedral group. Theoretically possible agglomeration (bifurcation) patterns of this economy are predicted by this analysis and the existence of these patterns is demonstrated by numerical bifurcation analysis. Such prediction and demonstration are conducted in Part II, in a more general setting, for a larger group expressing the symmetry of economy on a hexagonal lattice to clarify the existence of the hexagonal patterns of Christaller and Lösch.

In Part II, we would like to tackle the objective of this book: investigation of the mechanism of the hexagonal agglomeration in economic geography on a hexagonal lattice in light of bifurcation. Hexagonal population distributions of several sizes are shown to be self-organized from a uniformly inhabited state, which is modeled by a system of places (cities) on a hexagonal lattice. Microeconomic interactions among the places are expressed by core–periphery models in new economic geography. We search for hexagonal distributions of Christaller and Lösch using group-theoretic bifurcation theory. The symmetries of possible bifurcating solutions can be determined from the algebraic structure of the group that describes the symmetry of the system. Hence, the first step of the bifurcation analysis is to identify the underlying group and its algebraic structure. After an introduction of the hexagonal lattice as a two-dimensional spatial platform of economic agglomeration, the symmetry group of this lattice is presented (Chap. 5). In comparison with the dihedral group, which describes the symmetry of the racetrack economy (Chap. 3 in Part I), this symmetry group has a more complicated structure, thereby, entailing a far more complicated bifurcation mechanism. Such complexity, however, is untangled by group-theoretic (equivariant) bifurcation analysis in Chaps. 8 and 9.

Part II is organized as follows. Chapter 4, serving as a prelude of Part II, presents the equilibrium equation of core–periphery models and gives the proof of the equivariance of this equation. Theoretical results of Chaps. 5–9 that elucidate the mechanism of these bifurcations are previewed. Bifurcations on the hexagonal lattice engendering hexagonal distributions of interest are demonstrated by numerical bifurcation analysis. Chapter 5 introduces a hexagonal lattice as a two-dimensional discretized uniform space for economic agglomeration. Hexagonal distributions on this lattice, corresponding to those envisaged by Christaller and Lösch in central place theory (Sect. 1.2), are explained, parameterized, and classified. The symmetry group of the hexagonal lattice is presented. Chapter 6 gives a derivation of irreducible representations of this group according to a standard procedure known as the method of little groups in group representation theory. Chapter 7 presents matrix representations of the group for the hexagonal lattice. Among the irreducible representations of this group, those which are relevant to this lattice are identified. Chapters 8 and 9 present group-theoretic bifurcation analysis, respectively, by using the equivariant branching lemma and by solving the bifurcation equation.

Irreducible representations and the sizes of the hexagonal lattice that can engender hexagonal patterns of interest are set forth and classified. Asymptotic forms of bifurcating equilibrium paths and the directions of these paths are presented.

It would be our great pleasure if this book contributes to a better understanding of self-organization of economic agglomeration.

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The authors met with the concept of self-organizing economy in the book:

P. Krugman, *The Self-organizing Economy*, Blackwell, Oxford, 1996.

This book opened the eyes of the authors to interdisciplinary studies encompassing social science and natural science.

The hexagonal patterns of economic agglomeration on the hexagonal lattice were found in May 2009 by a master course student of the first author. This finding triggered our serious group-theoretic analysis to sort out Christaller's hexagons of three kinds, which was reported in:

K. Ikeda, K. Murota, T. Akamatsu, T. Kono, Y. Takayama, G. Sobhaninejad, A. Shibasaki: Self-organizing hexagons in economic agglomeration: core-periphery models and central place theory, Technical Report METR 2010-28, Department of Mathematical Informatics, University of Tokyo, October 2010.

This report and a study thereafter that blossomed into this book were supported by researchers in various fields, encompassing economic geography, new economic geography, and nonlinear mathematics. The authors would like to acknowledge the contributions of the following: Professor Takashi Akamatsu provided us with knowledge on core–periphery models in new economic geography and Professor Tatsuhito Kono introduced us to fundamentals of economic geography. Yuki Takayama, Reza Sobhaninejad, Akira Shibasaki, and Naoki Kondo contributed to the numerical nonlinear bifurcation analysis of hexagonal patterns on the hexagonal lattice.

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Contents

Part I
Economic Agglomeration and Bifurcation: Introduction

Chapter 1
Hexagonal Distributions in Economic Geography and Krugman's Core–Periphery Model

Abstract At the beginning of this book, several fundamental concepts related to the study of hexagonal economic agglomerations are presented and importance of this study is demonstrated. Christaller's three hexagonal market areas associated with market, traffic, and administrative principles and Lösch's hexagons derived from geometrical consideration in central place theory are introduced. As a step toward a connection with the real world, self-organization of central places is demonstrated for a domain with the shape of southern Germany with a microeconomic mechanism of Krugman's core–periphery model in new economic geography. Such a distribution is observed much clearer for an economy on the hexagonal lattice with periodic boundaries to demonstrate the importance of the group-theoretic study on this lattice conducted in this book. Nonlinear equilibrium equations and the stability of Krugman's core–periphery model are introduced. History of the study of self-organization of cities is reviewed, encompassing works in economic geography, new economic geography, and physics.

Keywords Agglomeration of population • Bifurcation • Central place theory • Christaller's hexagonal market area • Core–periphery model • Economic agglomeration • Krugman model • Lösch's hexagons • Southern Germany • Spatial equilibrium • Stability

1.1 Introduction

Hierarchical urbanization of megalopolises, cities, towns, villages, and so on displays interesting scattering patterns that hint at the existence of an underlying mechanism. A first attempt to elucidate such a mechanism was conducted for southern Germany by Christaller, 1933 [8]. Figure 1.1 depicts a distribution of large cities in southern Germany, where cities of various sizes are distributed. In particular, Frankfurt, Stuttgart, Nuremberg, and Munich appear to be approximately equidistant. A question to be answered is "*How* and *where* are these cities

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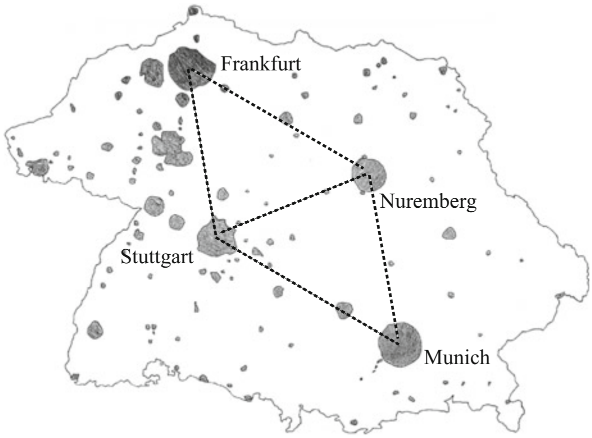

Fig. 1.1 Distribution of large cities in southern Germany

self-organized?" Although diversified studies have been conducted in social and natural sciences, to be described in Sect. 1.6, the elucidation of this mechanism remains difficult.

Successful simulation of self-organization is indeed a difficult task as it involves the modeling of various aspects: geometry of locations, microeconomic activities, interaction of places via transportation of goods, and so on. In addition, there are detailed factors, such as the location of houses and factories and the shipment of goods by trucks and trains. In this book, to avoid excessive complexity, we specifically examine the most likely and most generally accepted scenario: an evolution from an evenly spread population of farmers to an agglomeration of economic activities in a few urban regions. In accordance with this scenario, we introduce the modeling of several issues:

- For the modeling of locations, we refer to *central place theory*, which describes geometrically possible spatial patterns of urbanization that are self-organized from a uniform economic space, as explained in Sect. 1.2.
- For the modeling of economic activities, we utilize the *core–periphery model* in new economic geography that incorporates several microeconomic mechanisms, such as interactions occurring among production with increasing returns, transport costs, and factor mobility, as expounded in Sect. 1.5.

By virtue of this modeling, a nonlinear equilibrium equation can be formulated, in which the population migrating among places serves as an independent variable and the transport cost serves as a bifurcation parameter. As a solution to this equation, we can obtain loci of equilibria expressing the progress of agglomeration parameterized by the transport cost. The analysis of the equation is, however, complicated due to the multiplicity of solutions by bifurcations.

The book, accordingly, has become an interdisciplinary study of hexagonal patterns in central place theory in economic geography, core–periphery models for economic agglomeration in new economic geography, and group-theoretic bifurcation theory. At the beginning of this book, fundamentals of central place theory and core–periphery models are presented, whereas the history of economic geography and new economic geography is given in Sect. 1.6 and the group-theoretic bifurcation theory is shown in detail in Chap. 2.

This book offers a group-theoretic methodology to elucidate the mechanism of self-organization of hexagonal patterns in economic agglomeration. In this chapter, the occurrence of self-organization is demonstrated and explained on the basis of computational simulations of economic agglomeration in southern Germany and in a hexagonal lattice.

This chapter is organized as follows. Hexagonal market areas in central place theory in economic geography are introduced in Sect. 1.2. Economic agglomeration in a domain of the shape of southern Germany with regular-triangular meshes is simulated to demonstrate the emergence of hexagonal patterns in Sect. 1.3. Emergence of hexagonal patterns on a hexagonal lattice with regular-triangular meshes and periodic boundaries is demonstrated numerically in Sect. 1.4. Krugman's core–periphery model is presented in Sect. 1.5. History of the study of self-organization of cities is reviewed in Sect. 1.6.

1.2 Christaller's Hexagonal Market Areas and Lösch's Hexagons

Self-organization of hexagonal distributions has been studied by Christaller and Lösch in central place theory.[1] The concept of *flat earth* is introduced on the basis of several simplifying assumptions, such as

- The land surface is completely flat and homogeneous in every aspect. It is, in technical terms, an *isotropic plain*.
- Movement can occur in all directions with equal ease and there is only one type of transportation.
- The plain is limitless or unbounded, so that complications that tend to occur at boundaries do not need to be dealt with.
- The population is spread evenly over the plain.

1.2.1 Christaller's Hexagonal Distributions

Christaller, 1933 [8] considered a hierarchical structure of industries with different sizes of demand and showed that a nested set of *hexagonal market areas* of

[1]For central place theory, see, for example, Lösch, 1940 [23]; Lloyd and Dicken, 1972 [22]; Isard, 1975 [20]; and Beavon, 1977 [4].

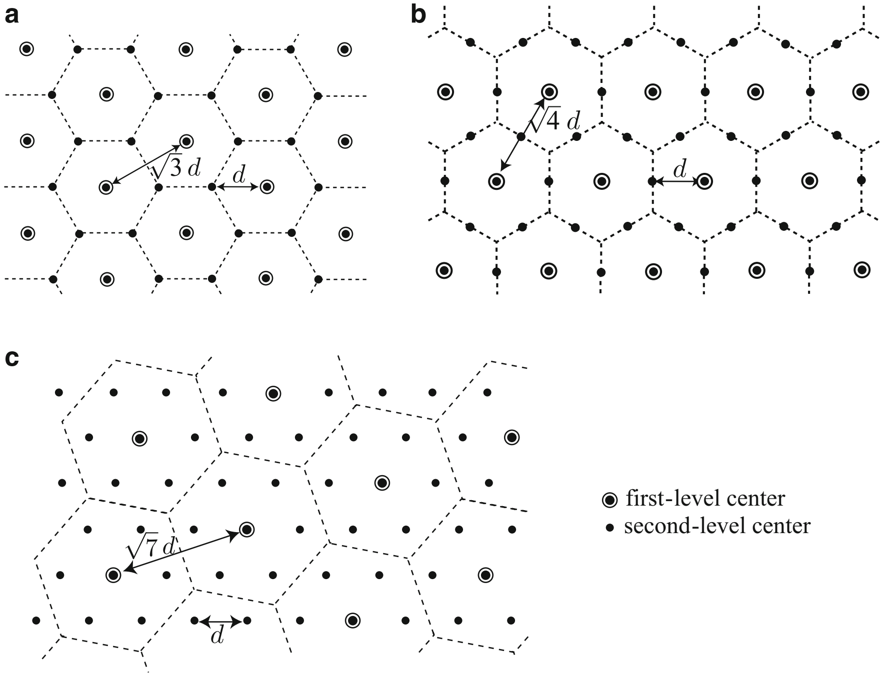

Fig. 1.2 Three systems predicted by Christaller. (**a**) Christaller's $k = 3$ system. (**b**) Christaller's $k = 4$ system. (**c**) Christaller's $k = 7$ system. The *dashed lines* denote hexagonal market areas

places, such as cities, towns, and villages, emerges. A hierarchy of places with different levels exists in each market area governed by the highest-level (first-level) center with the largest population, the second-level center with the second largest population, and so on.[2] Self-organization of hexagonal market areas of three kinds shown in Fig. 1.2 was advanced as a key concept.

Christaller introduced the so-called *k value* as an important index to characterize hexagonal market areas, as stated by Dicken and Lloyd, 1990 (p. 28) [11] as

> Christaller's model, then, implies a fixed relationship between each level in the hierarchy. This relationship is known as a k value (k meaning a constant) and indicates that each center dominates a discrete number of lower-order centers and market areas in addition to its own.

The k value has a geometrical implication in that it is proportional to the size (area) of the hexagonal market area. Its square root $\sqrt{k}$ is proportional to the *spatial period* L, the shortest distance between the first-level centers, which represents the radius of hexagons. The three smallest values, $k = 3$, 4, and 7, are associated with Christaller's systems (Fig. 1.2).

[2]Such a hierarchy is also called metropolis, city, town, village, hamlet; or A-level center, B-level center, C-level center, and so on.

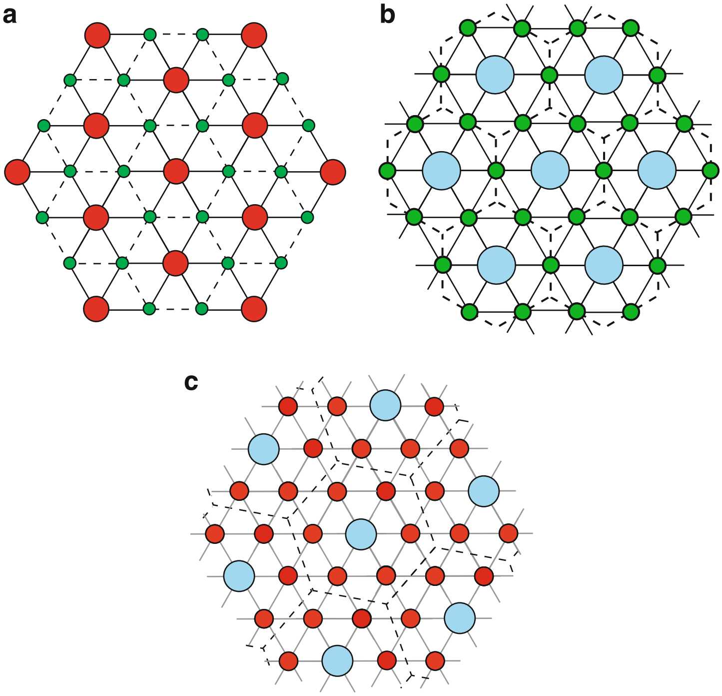

Fig. 1.3 Hexagonal distributions of Christaller on the hexagonal lattice. (**a**) Christaller's $k = 3$ system. (**b**) Christaller's $k = 4$ system. (**c**) Christaller's $k = 7$ system. The *larger circles* represent the first-level centers and the *smaller ones* represent the second-level centers; the *dashed lines* denote hexagonal market areas

Christaller's three systems are explained by market, traffic, and administrative principles, respectively (Christaller, 1933 [8]; Dicken and Lloyd, 1990, Chap. 1 [11]). These principles are explained below with reference to Fig. 1.3, which displays Christaller's hexagonal distributions on a hexagonal lattice.

- In the $k = 3$ *system* in Fig. 1.3a, neighboring first-level centers (larger circles) are connected by two kinked roads, each of which passes a second-level center (smaller circle) at the kink. This system is explained by Christaller's *market principle* of supplying the maximum number of evenly distributed consumers from the minimum number of central places.
- In the $k = 4$ *system* in Fig. 1.3b, neighboring first-level centers are connected by a straight road that passes a second-level center. This system is explained by Christaller's *traffic principle* of achieving efficient transportation. Christaller wrote: "The traffic principle states that the distribution of central places is most favorable when as many important places as possible lie on one traffic route between two important towns, the route being as straightly and as cheaply as possible."

- The distribution in the $k = 7$ *system* in Fig. 1.3c agrees with Christaller's *administrative principle* of avoiding the sharing of a satellite place by two first-level centers to prevent administrative conflict. Christaller stated: "The ideal of such a spatial community has the nucleus as the capital (a central place of a higher rank), around it, a wreath of satellite places of lesser importance, and toward the edge of the region a thinning population density—and even uninhabited areas."

The number N_j of the jth level centers dominated by the first-level center is given by the recurrence formula[3]

$$N_1 = 1, \quad N_j = k^{j-1} - k^{j-2}, \quad j = 2, 3, \ldots; \; k = 3, 4, 7. \tag{1.1}$$

For example, we have

$$N_1 : N_2 : N_3 : \cdots = \begin{cases} 1 : 2 : 6 : 18 : 54 : 162 : \cdots & \text{for } k = 3 \text{ system,} \\ 1 : 3 : 12 : 48 : 192 : \cdots & \text{for } k = 4 \text{ system.} \end{cases}$$

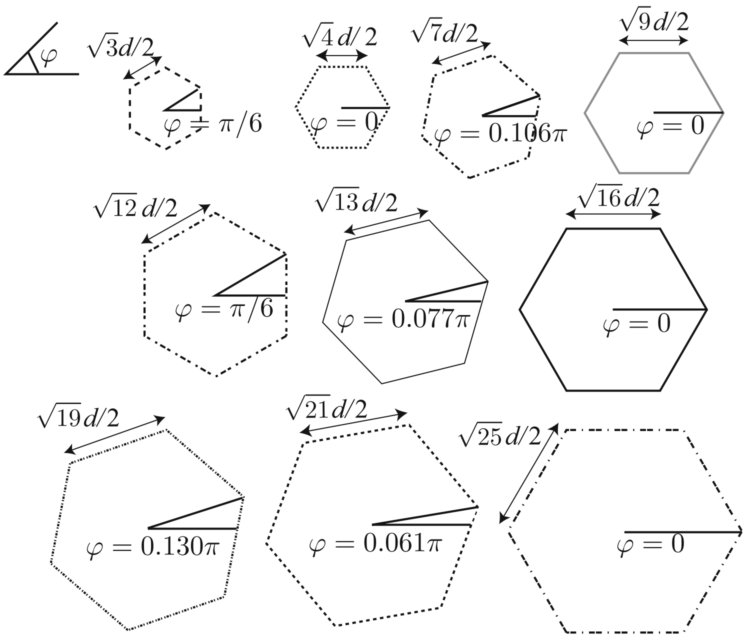

Fig. 1.4 Lösch's ten smallest hexagons

[3]See Christaller, 1933 (1966, p. 67) [8] for the case of $k = 3$ and Dicken and Lloyd, 1990, Chap. 1 [11] for the case of $k = 4$.

1.2.2 Lösch's Hexagons

Lösch, 1940 [23] demonstrated, for a single industry, that market areas must be hexagonal in order to minimize transport costs for a given density of central places. The *ten smallest hexagons* shown in Fig. 1.4 were presented as fundamental sizes of market areas. Based on the geometry of the hexagonal lattice, the *normalized spatial period* L/d was shown to take some specific values, such as

$$\frac{L}{d} = \sqrt{D} = 1, \sqrt{3}, 2, \sqrt{7}, 3, \sqrt{12}, \sqrt{13}, 4, \sqrt{19}, \sqrt{21}, 5, \ldots, \tag{1.2}$$

where d is the distance between two neighboring places and D is an important parameter for the characterization of Lösch's hexagons.

It should be emphasized that central place theory relies on a normative and geometrical approach and does not reveal the microeconomic or mathematical mechanism of self-organization of hexagonal patterns. This book underpins this theory by mathematical study of the geometry of the hexagonal lattice (Chap. 5) and elucidates the mechanism of self-organization in light of bifurcation theory (Chaps. 6–9).

1.3 Agglomeration in Southern Germany: Realistic Spatial Platform

As a step toward a connection with the real world, self-organization of central places is demonstrated here by economic agglomeration analysis[4] of the domain in Fig. 1.5. The shape of this domain was chosen to mimic the shape of southern Germany.[5]

In comparison with the hexagonal lattice to be studied in Sect. 1.4, this domain has the following two characteristics.

- An irregular shape without any symmetry.
- Nonperiodic boundaries.

No bifurcation for agglomeration would occur on this domain as it has no symmetry, but it will turn out that bifurcation serves as an underlying mechanism for the

[4]This domain comprises 404 places connected by a set of triangular meshes and the nominal mesh size $d = 1/32$. For microeconomic modeling, the core–periphery model with (4.6) in Sect. 4.5 was used with the parameter values of $(\sigma, \mu, \theta) = (5.0, 0.4, 10000)$. See also Sect. 1.5 for fundamental issues of this model.

[5]Southern Germany is the birthplace of central place theory to which Christaller's theory was first applied (Christaller, 1933 [8]), and nowadays cities of several sizes are scattered on its relatively flat land to exhibit a hierarchy of central places.

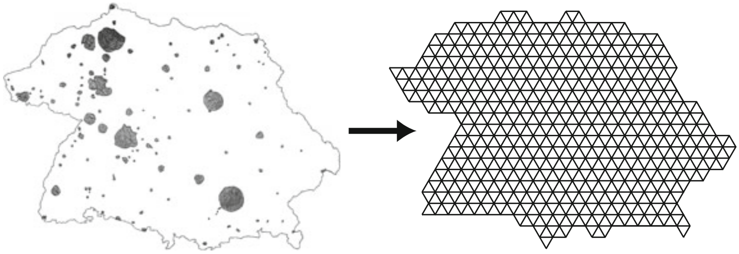

Fig. 1.5 Grid model of southern Germany. Places are located on the nodes of the grid; a regular-triangular mesh of roads is assumed to exist even at the location where the grid is absent

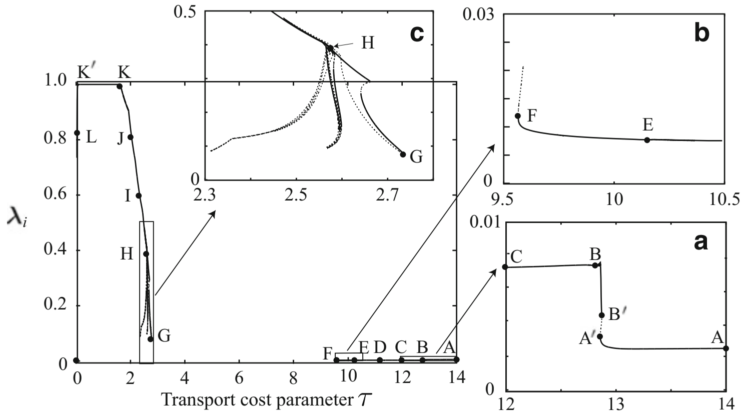

Fig. 1.6 Change of the population size λ_i at a place in the middle of the domain plotted against the transport cost parameter τ. *Solid curves* mean stable equilibria; *dotted curves*, unstable equilibria

progress of the agglomeration. The major objective of this book is to develop a theoretical framework to explain the mechanism of this agglomeration behavior.

The population λ_i at a place in the middle is plotted[6] against the transport cost parameter τ in Fig. 1.6 ($0 < \tau < 14.0$) with enlarged views (a)–(c), and in Fig. 1.7 population distributions are shown by expressing population size by the area of *black circle*. The curves in Fig. 1.6c are very complicated forming several *loops* that are a mixture of stable and unstable equilibria (see Sect. 1.5.4 for the definition of stability). Several characteristic states of the progress of agglomeration as τ decreases (in association with the development of technology) were observed as explained below.

[6]The population versus the transport cost parameter curve in Fig. 1.6 was obtained only in the ranges of $\tau > 9.57$ and $\tau < 2.73$ because the curve formed a plethora of loops and was too complicated in the range of $2.73 < \tau < 9.57$ between points G and F.

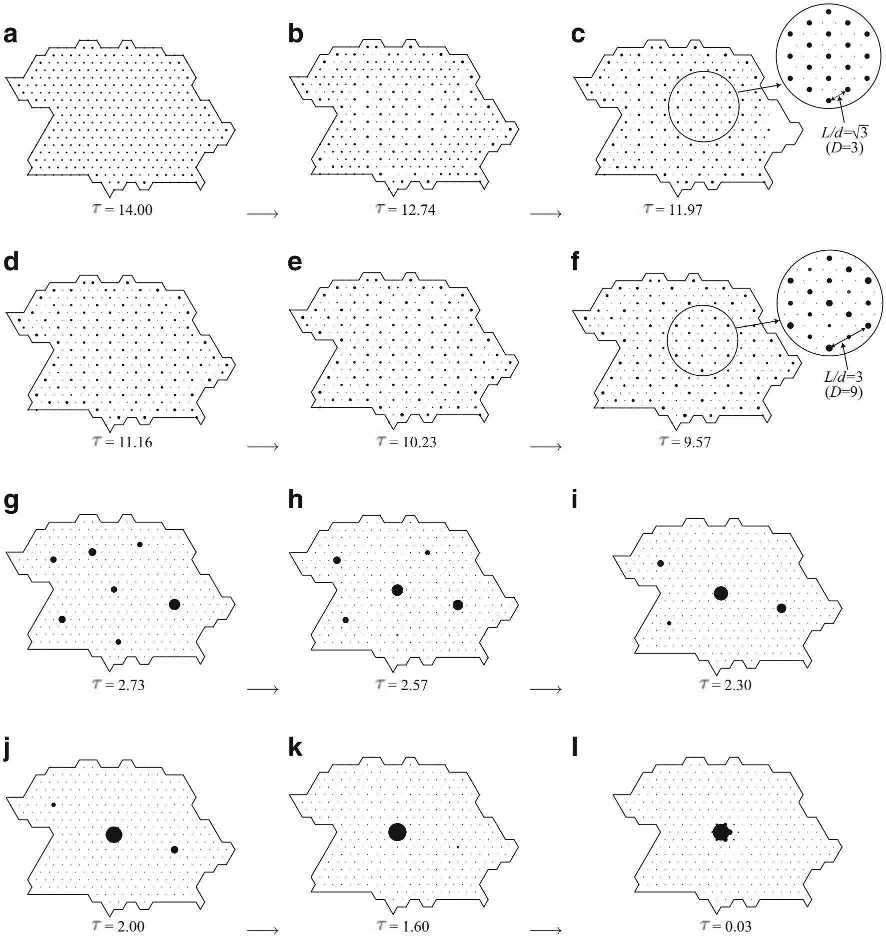

Fig. 1.7 Progress of agglomeration in association with the decrease of transport cost parameter τ observed at points A–L in Fig. 1.6. The area of *black circle* indicates population size. (**a**) Point A. (**b**) Point B. (**c**) Point C. (**d**) Point D. (**e**) Point E. (**f**) Point F. (**g**) Point G. (**h**) Point H. (**i**) Point I. (**j**) Point J. (**k**) Point K. (**l**) Point L

Emergence of Christaller's $k = 3$ System

In an early state ($14.0 > \tau > 12.86$), the population curve in Fig. 1.6 remained almost flat and the population of each place changed slowly, and the population was distributed almost uniformly (see Fig. 1.7a for $\tau = 14.0$).

A rapid increase of population λ_i started at $\tau = 12.86$ (point A′ in the enlarged view in Fig. 1.6a). There was a short unstable curve A′B′ subject to a snap back (shown by the dotted curve). The stability was lost at the minimal point A′ of τ and recovered at the maximal point B′ of τ. In association with monotonic reduction of the value of the transport cost parameter τ, we encounter the stable path AA′, a *dynamical shift* between A′ and B (bypassing B′), and another stable path BC in this sequence.

Along the curve AA′B′B, some places lost their population, other places gained population to grow into first-level centers, and, in turn, to self-organize a spatial pattern. At point B with $\tau = 12.74$ in Fig. 1.6, a zone containing sparsely and regularly distributed central places appeared in the middle of the domain (Fig. 1.7b). Since the distance between these places is equal to $L/d = \sqrt{D} = \sqrt{3}$, this demonstrates a self-organization of Christaller's $k = 3$ system with $D = 3$. This zone grew stably thereafter (Figs. 1.7c–e). At $\tau = 10.23$ at point E in Fig. 1.6, the sparsely- and regularly-distributed zone covered the domain away from the boundary (Fig. 1.7e). The curve AA′B′BC had a step-like shape with a snap back followed by a plateau. Such a shape of an equilibrium path is characteristic in the emergence of hexagonal patterns, as we will see in Sect. 1.4.

Transition via the Hexagon with $D = 9$

After a relatively stable and calm era of the dominance of the $k = 3$ system with the spatial period $L/d = \sqrt{D} = \sqrt{3}$ during $12.74 > \tau > 10.23$, there appeared a transient state in Fig. 1.7f at $\tau = 9.57$, in which the spatial period among growing places elongated approximately to $L/d = \sqrt{D} = \sqrt{9} = 3$. Thus we encountered a cascade of bifurcations, in which the spatial period became $\sqrt{3}$ times repeatedly as

$$\frac{L}{d} = \sqrt{D} : \ 1 \to \sqrt{3} \to 3. \tag{1.3}$$

Such elongation progressed gradually without undergoing bifurcation and the initiation of the elongation was not clear.[7]

Agglomeration and Redispersion State

In the agglomeration and redispersion state ($2.73 > \tau > 0.0$), as shown in Fig. 1.6, the population λ_i in the middle of the domain grew rapidly between points G–K, hit the plateau between K–K′, and then decreased between K′–L. As shown in Fig. 1.7g, at point G with $\tau = 2.73$, the population is agglomerated at seven places denoted by *black circle*, which are approximately equidistant. Although these places do not have the same population and do not display a regular-hexagonal distribution due to the irregular shape of the domain, each place keeps sufficient and almost

[7] The cascade of spatial period elongation in (1.3) was observed more clearly for the 9×9 hexagonal lattice with periodic boundaries, for which this cascade was entailed by a cascade of bifurcations (Sects. 1.4.2 and 4.5.1).

the same distances from neighboring places to maintain its own market area. Such distribution agrees with Christaller's administrative principle of avoiding the sharing of a satellite place so as to prevent administrative conflict (Sect. 1.2).

The number of agglomerated places decreased in association with the decrease of τ (Figs. 1.7g–j), until the emergence of a megalopolis with a completely agglomerated population (Fig. 1.7k). Thereafter, the redispersion took place (Fig. 1.7l).

1.4 Hexagons on Hexagonal Lattice: Idealized Spatial Platform

As we have seen in Sect. 1.3, an agglomeration analysis for hexagonal patterns on the irregular shaped domain of southern Germany involved too complicated solution curves with a number of loops. In this book, we advance the hexagonal lattice as an idealized platform for the analysis of spatial agglomeration. The theoretical analysis and the numerical analysis of this lattice are previewed in this section. It is demonstrated that the agglomeration behavior on this lattice can capture essential characteristics of agglomeration in southern Germany at the expense of several idealizations.

1.4.1 Hexagonal Lattice and Possible Hexagonal Distributions

A completely homogeneous infinite two-dimensional land surface, the *flat earth* in central place theory (Sect. 1.2), needs to be expressed compatibly with the discretized analysis of core–periphery models. For this purpose, an $n \times n$ finite hexagonal lattice with periodic boundaries is used in this book.[8] Nodes on this lattice represent uniformly spread places of economic activities. These places are connected by roads of the same length d forming a regular-triangular mesh (see Fig. 1.8 for an example of $n = 4$), and goods are transported along these roads. In comparison with the domain with the shape of southern Germany studied in Sect. 1.3, the hexagonal lattice is endowed with uniformity, the same geometrical environment for every place, owing to periodic boundaries. Agglomeration would occur on this domain by way of bifurcations.

By a theoretical consideration, pertinent lattice sizes n that would engender hexagonal distributions of interest are given in Chap. 5. For example, Christaller's distributions (Fig. 1.3) are shown to be compatible with the lattice sizes

[8] The term of *hexagonal lattice* is commonly used in many fields of mathematical sciences, although it is also called *regular-triangular lattice*.