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Benedict Baur

Elliptic Boundary Value Problems and Construction of L^p-Strong Feller Processes with Singular Drift and Reflection

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Mit einem Geleitwort von
Professor Dr. Martin Grothaus

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Preface

The present dissertation of Benedict Baur characterizes a milestone in the the theory of Dirichlet forms. In the last decades the theory of Dirichlet forms emerged to be a very useful concept for the construction and analysis of solutions to stochastic differential equations (SDEs). In particular, this theory was and is of great success in the construction of solutions to equations with singular coefficients, such as they are showing up in many applications from Physics. Moreover, the concepts of Dirichlet forms allow to treat equations in bounded domains with various boundary conditions. Classical existence results from the theory of SDEs in the presence of boundary conditions often are rather limited or, for certain boundary conditions, even not available up to now.

But the general theory of Dirichlet forms has a disadvantage. One can treat many equations, but in general it is not clear for which initial conditions. More precisely, one has a solution for only almost all starting points. Well, the notion "almost all" can even be refined, but in worst case one can not specify even a single starting point for which a solution exists. This disadvantage can be overcome by a combination of Dirichlet from techniques with strong Feller properties. This idea, for example, has been worked out by Masatoshi Fukushima, one of the giants and founders of the theory of Dirichlet forms. Then, approximately 10 years ago, these ideas were refined by Michael Röckner, a further giant of the theory of Dirichlet forms, to be applicable to much more general classes of equations. At that time those concepts were applied to an interesting system of SDEs from Statistical Physics. Later on these concepts were generalized to treat more and more examples.

The impressive contribution of Benedict Baur is the development of a general concept out of the above ideas. In his dissertation he invented a collection of functional analytic conditions. These imply the existence of a solution to a given SDE. The construction of the solution is via Dirichlet form techniques and, nevertheless, the solution process can be started in an explicitly known set of initial points. That these analytic conditions are of practical use, he illustrated by providing several challenging and interesting examples.

In the cases with reflecting boundary conditions, even the corresponding Skorokhod decomposition is provided. Furthermore, as a byproduct, elliptic regularity results up to the boundary were derived.

It is desirable that the present dissertation will serve as a standard reference for constructing solutions to SDEs via Dirichlet forms for an explicitly known set of initial conditions.

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Kaiserslautern Benedict Baur

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Contents

1 Introduction

1.1 Introduction

This thesis consists of three main parts: First, the construction of $\mathcal{L}^p$-strong Feller processes from sub-Markovian strongly continuous contraction semigroups on L^p-spaces that are associated with symmetric regular Dirichlet forms, see Chapter 2.

Second, providing a regularity result for weak solutions to elliptic equations under local assumptions on the coefficients, see Chapter 3 and Section 4.2.

Third, construction of elliptic diffusions with singular drift and reflecting boundary behavior and providing a Skorokhod representation (or semimartingale decomposition). This representation holds for every starting point that is not in the singularity set of the drift term and is either in the interior of the domain or at a C^2-smooth boundary part. See Chapter 4 and Chapter 6 for details.

All results are applied to construct stochastic dynamics for finite particle systems with singular interaction in continuum and for Ginzburg-Landau interface models, see Chapter 5, Section 6.4 and Section 6.5.

Let us now describe the results in more detail.

Construction of $\mathcal{L}^p$-strong Feller processes

We start with the first part, i.e., Chapter 2. In this chapter we provide a general construction scheme for $\mathcal{L}^p$-strong Feller processes that give solutions to a martingale problem for starting points from a known set. With $\mathcal{L}^p$-strong Feller we mean that for some $1 \leq p < \infty$ the semigroup of the process $(P_t)_{t\geq 0}$ maps $\mathcal{L}^p$ (w.r.t. to a specified measure) into $C^0(E_1)$, the space of continuous functions on a given set E_1.

The motivation is the following: Dirichlet form methods allow to construct stochastic processes in a very general setting, see [FOT11] and [MR92]. In particular, the construction of diffusions with very singular drift and general boundary behavior are possible.

However, these methods yield processes that solve the associated martingale problem for the corresponding L^2-generator for starting points outside

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an exceptional set only. In general, this set cannot be explicitly specified and in particular need not to be empty.

In recent years it turned out that additional techniques allow to refine these results to get a process that can be started from every point in a specified set of admissible starting points. Now the process yields solutions to the martingale problem for starting points from this set and has continuous paths in this set. In applications this specified set is naturally related to coefficients in equations describing the process, like the formulation of the martingale problem.

Albeverio, Kondratiev and Röckner ([AKR03]) construct distorted Brownian motion on $\mathbb{R}^d$, $d \in \mathbb{N}$, with strongly singular drift (see also Theorem 2.4.2). The process can be started from those points where the drift is not singular. Fattler and Grothaus ([FG07] and [Fat08]) generalize these methods to construct Brownian motion with singular drift in the interior and reflecting boundary behavior on domains with certain smoothness assumptions. There one has to exclude all points with singular drift and all non-smooth boundary points. In both cases drifts with very strong (repulsive) singularities are allowed, in particular potentials of Lennard-Jones type can be treated.

Both works make use of an elliptic regularity of Bogachev, Krylov and Röckner, see [BKR97] and [BKR01], and path-regularity techniques of Dohmann, see [Doh05].

The construction method in [FG07] is quite similar to the one of [AKR03]. We generalize this method to an abstract setting in the following way: We start with a regular symmetric strongly local Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, \mu)$ with E being a locally compact separable metric space. Well-known theorems (see Theorem 7.2.3) yield that there exists an associated sub-Markovian strongly continuous contraction semigroup $(T_t^p)_{t\geq 0}$ and resolvent $(G_\lambda^p)_{\lambda>0}$ on $L^p(E, \mu)$ for $1 \leq p < \infty$ with associated generator $(L_p, D(L_p))$. We assume that there exists a Borel set $E_1 \subset E$ complemented by an exceptional set such that for some $1 < p < \infty$ we have

- $D(L_p) \hookrightarrow C^0(E_1)$ and the embedding is locally continuous.
- $D(L_p)$ is point separating on E_1 in the sense of Condition 2.1.2(ii).

From this we construct a semigroup of $\mathcal{L}^p$-strong Feller transition kernels $(P_t)_{t\geq 0}$ and resolvents of $\mathcal{L}^p$-strong Feller kernels $(R_\lambda)_{\lambda>0}$. Both give regularized version of the corresponding L^p-semigroup $(T_t^p)_{t\geq 0}$ and L^p-resolvent $(G_\lambda^p)_{\lambda>0}$, i.e., for $u \in L^p(E, \mu)$

$$P_t u\,(x) = \widetilde{T_t^p u}\,(x) \quad \text{and} \quad R_\lambda u\,(x) = \widetilde{G_\lambda^p u}\,(x) \quad \text{for } t > 0, \lambda > 0 \text{ and } x \in E_1.$$

Here $\widetilde{T_t^p u}$ ($\widetilde{G_\lambda^p u}$) denotes the continuous version of $T_t^p u$ ($G_\lambda^p u$) which exists due to the regularity assumption on $D(L_p)$ and the mapping properties of the semigroup and resolvent.

These kernels give rise to an associated process, solving the martingale problem (for functions in a certain space) for starting points in E_1. With techniques of [Doh05] and [AKR03] we get continuity of the paths in $[0, \infty)$. For the right-continuity at $t = 0$ it is crucial to have point separating functions in $D(L_p)$.

So altogether, we obtain a general construction result for processes from symmetric regular Dirichlet forms that can be started from every point in a known set. The generality of the construction scheme is comparable to the construction of classical Feller processes but works under local assumptions. In Section 2.4 we provide concrete examples for the application of the construction scheme.

Elliptic regularity up to the boundary

We aim to apply this general scheme for construction of reflected elliptic diffusions on sets $\overline{\Omega}$ with open interior $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$. Therefore, we have to provide an elliptic regularity result which gives regularity of weak solutions both in the interior Ω and at boundary parts. This is the main part of Chapter 3.

We provide in Chapter 3 an Sobolev space regularity result for weak solutions of elliptic equations. This result is a (partial) generalization of a result of Morrey, see [Mor66, Theo. 5.5.4']. However, therein only a short sketch of the proof is given. Our proof is based on techniques of Shaposhnikov, see [Sha06]. There a detailed proof of an a-priori estimate of Morrey, [Mor66, Theo. 5.5.5'], is given. See also Chapter 3 for a further discussion.

We can prove local regularity at all points where the coefficient matrix is continuous and strictly elliptic, and that are either interior points or located at a C^1-smooth boundary part. Since we do not assume any global assumptions on the matrix, we can handle gradient Dirichlet forms with density, having a non-trivial zero set.

Let us now describe the proof of the regularity result: For interior points we represent a weak solution in terms of potentials containing Green's function. With this representation we can conclude iteratively higher regularity, starting from local $H^{1,2}$-regularity. For boundary points we use a reflection method to reduce this case to the interior point case.

Construction of elliptic diffusions with reflection at the boundary

Combining the results of Chapter 2 and Chapter 3 we construct elliptic diffusions in Chapter 4. They are constructed as $\mathcal{L}^p$-strong Feller processes associated with gradient Dirichlet forms. So let A be a matrix-valued mapping of symmetric and strictly elliptic matrices and ϱ a density on a set $\overline{\Omega} \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with open interior Ω. Our Dirichlet form is constructed as the closure of the pre-Dirichlet form

$$\mathcal{E}(u,v) = \int_\Omega (A\nabla u, \nabla v)\, d\mu,$$
$$u,v \in \mathcal{D} := \big\{ u \in C_c(\overline{\Omega}) \,|\, u \in H^{1,1}_{\mathrm{loc}}(\Omega), \mathcal{E}(u,u) < \infty \big\},$$

see (4.1). For the construction of the process we assume that the matrix coefficient A is C^1-smooth, the boundary of Ω is C^2-smooth boundary (except for a set of capacity zero) and certain weak differentiability conditions on the density, see Condition 4.1.1 and Condition 4.1.6. These stronger assumptions on the boundary and matrix are imposed to construct point separating functions in the domain of the L^p-generator. Nevertheless, our assumptions on the density are so weak that very singular drift terms can be handled. In particular, interacting particle systems with Lennard-Jones type potentials can be treated. For the construction we fix a boundary part $\Gamma_2 \subset \partial\Omega$, open in $\partial\Omega$, and complemented in $\partial\Omega$ by a set of capacity zero.

The set of all admissible starting points E_1 consists of all points where the density is non-zero and that are either in the interior or at the smooth boundary part Γ_2, i.e., $E_1 = (\Omega \cup \Gamma_2) \cap \{\varrho > 0\}$.

We can show that the domain of the L^p-generator contains a subspace $\mathcal{D}_{\mathrm{Neu}}$ of C^2-functions with compact support in E_1 and Neumann-type boundary on Γ_2, see (4.4). On this set the generator has the form of an elliptic differential operator of second order with singular drift, denoted by $\hat{L}$. So for $u \in \mathcal{D}_{\mathrm{Neu}}$ it holds

$$L_p u = \hat{L} u := \sum_{i,j=1}^d a_{ij}\partial_i\partial_j u + \sum_{j=1}^d \left(\sum_{i=1}^d \partial_i a_{ij} + \sum_{i=1}^d \frac{1}{\varrho} a_{ji}\partial_i \varrho \right) \partial_j u,$$

see (4.5). Using the general construction scheme from the first part together with the elliptic regularity result from the second part we obtain an $\mathcal{L}^p$-strong Feller diffusion

$$\mathbf{M} = (\boldsymbol{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\mathbf{X}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E\cup\{\Delta\}}),$$

see Chapter 2 or Section 7.3 for the notion. Then the process solves the martingale problem for $u \in D(L_p)$, in particular for $u \in \mathcal{D}_{\mathrm{Neu}}$. So we have that

$$M_t^{[u]} := \widetilde{u}(\mathbf{X}_t) - \widetilde{u}(\mathbf{X}_0) - \int_0^t L_p u(\mathbf{X}_s)\, ds,\ t \geq 0,$$

is an $(\mathcal{F}_t)$-martingale under $\mathbb{P}_x$ for all $x \in E_1$ and $u \in D(L_p)$. Here $\widetilde{u}$ denotes the continuous version of u on E_1 provided by the elliptic regularity result.

Next we aim to investigate the boundary behavior of the constructed diffusion process. We construct the local time of the process on the boundary part $\Gamma_2 \cap \{\varrho > 0\}$. We show that the process solves a martingale problem even for C^2-functions with compact support in E_1 that do not have the Neumann boundary condition. More precisely,

$$M_t^{[u]} := u(\mathbf{X}_t) - u(\mathbf{X}_0) - \int_0^t \hat{L}u(\mathbf{X}_s)\, ds + \int_0^t (A\nabla u, \eta)\, \varrho\,(\mathbf{X}_s)\, d\ell_s,\ t \geq 0,$$

is an $(\mathcal{F}_t)$-martingale under $\mathbb{P}_x$ for all $x \in E_1$ and $u \in C_c^2(E_1)$. Here η denotes the outward unit normal at Γ_2.

We can characterize the quadratic variation process of the martingale $(M_t^{[u]})_{t\geq 0}$ in terms of the matrix coefficient. Altogether, we get a semimartingale decomposition for $(u(\mathbf{X}_t) - u(\mathbf{X}_0))_{t\geq 0}$. Using a localization technique we get such a decomposition (or Skorokhod representation) for the process itself. Denote by $(b_i)_{1\leq i\leq d}$ the first-order coefficients of $\hat{L}$. Then we have

$$\mathbf{X}_{t\wedge\mathcal{X}}^{(i)} - \mathbf{X}_0^{(i)} = \int_0^{t\wedge\mathcal{X}} b_i(\mathbf{X}_s)\, ds - \int_0^{t\wedge\mathcal{X}} (e_i, A\eta)\varrho\,(\mathbf{X}_s)\, d\ell_s + M_{t\wedge\mathcal{X}}^{(i)},\ t \geq 0,$$

$\mathbb{P}_x$-a.s. for $x \in E_1$ and $1 \leq i \leq d$. The $(M_t^{(i)})_{t\geq 0}$, $1 \leq i \leq d$, are continuous local martingales (up to the lifetime $\mathcal{X}$) with quadratic variation process (up to $\mathcal{X}$)

$$\langle M^{(i)}, M^{(j)}\rangle_{\cdot\wedge\mathcal{X}} = 2\,(a_{ij}\cdot t)_{\cdot\wedge\mathcal{X}} \quad \text{for } 1 \leq i,j \leq d.$$

Let us emphasize that these decompositions hold under the path measures $\mathbb{P}_x$ for every $x \in E_1$, i.e., we have again a pointwise statement. For conservative processes we can further conclude existence of weak solutions.

The construction of the boundary local time and the semimartingale decomposition for $u \in C_c^2(E_1)$ is based on [FOT11, Ch. 5]. To apply these results to our setting we need the absolute continuity of the semigroup

$(P_t)_{t\geq 0}$ and certain regularity of potentials of the surface measure at compact boundary parts. We first have to refine the construction theorem in [FOT11, Theo. 5.1.6] to our setting since the semigroup $(P_t)_{t\geq 0}$ is in our case absolutely continuous on E_1 only. Then we apply our regularity result from Chapter 3 to potentials of the surface measure at compact boundary parts to conclude the regularity needed to apply the construction of additive functionals. Our regularity result implies even continuity properties of the potentials, but for the construction boundedness properties are already sufficient. Additional care has to be taken in our setting due to the singularity of the drifts.

We apply our results to concrete models in Mathematical Physics. We construct stochastic dynamics for finite particle systems with singular interaction in continuum and reflection at the boundary of the state space. Our approach allows very singular interaction potentials of Lennard-Jones type.

Furthermore, we construct stochastic dynamics for Ginzburg-Landau interface models with reflection (also called: entropic repulsion) at a hard wall. There we can also handle general potentials. These dynamics describe the random evolution of an interface, e.g., the surface of a liquid that is conditioned to stay above a hard wall.

Let us mention other works concerning the construction of reflected diffusions, see also the beginning of Chapter 6 for a more detailed comparison with other works. Strong solutions are constructed by Lions and Sznitman, see [LS84]. Strook and Varadhan construct reflected diffusions as solutions to the sub-martingale problem, see [SV71]. Moreover, there are several works on reflected diffusions and Dirichlet forms: Let us mention [FT95] and [FT96] where classical Feller processes associated with gradient Dirichlet forms with uniformly elliptic coefficient matrix, but without density are constructed. Using the results of [FOT94] a semimartingale decomposition is given. Note, however, that their setting does not cover the case of diffusions with singular drift term. Trutnau (see [Tru03]) constructs diffusions with reflection and singular drift using generalized non-symmetric Dirichlet forms, admitting a more general class of drift terms. The derived semimartingale decomposition, however, holds for quasi-every starting point only.

Furthermore, one can use operator semigroups with Feller(-type) regularizing properties also in the infinite dimensional setting for constructing martingale (or even weak) solutions to stochastic partial differential equations, see e.g. [PR02] and [RS06].

Finally, let us summarize the core results and progress achieved by this work:

- We obtain a general construction result for $\mathcal{L}^p$-strong Feller processes from analytic assumptions, see Theorem 2.1.3.

- We prove a local Sobolev space regularity result up to the boundary for elliptic equations under local assumptions on the coefficients and boundary, see Theorem 3.1.1.
- We construct $\mathcal{L}^p$-strong Feller elliptic diffusions with singular drift and reflection at the boundary, see Theorem 4.1.14.
- We provide a pointwise Skorokhod decomposition for the constructed diffusions, see Theorem 6.2.9 and Theorem 6.3.2, and obtain weak solutions, see Theorem 6.3.5.

In the Appendix we provide several auxiliary results. Most of them are well-known but we needed from time to time modified versions which apply for our specific settings.

1.2 Notation

The notation we use in this work is quite standard. By $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ we denote the set of all natural, rational, real and complex numbers, respectively. Notions like *positive* or *increasing* are meant in the non-strict sense. By $|\cdot|$ we denote the euclidean norm on $\mathbb{R}^d$, $d \in \mathbb{N}$, which yields in one dimension just the modulus. By $(\cdot,\cdot)$ we always denote the Euclidean scalar product on $\mathbb{R}^d$, $d \in \mathbb{N}$. All other scalar products will be explicitly distinguished. For real numbers a and b we denote by $a \wedge b$ and $a \vee b$ the minimum and maximum, respectively.

For a topological space (E, τ) we denote by $\mathcal{B}(E)$ the Borel σ-algebra generated by the open sets. By $\mathcal{B}_b(E)$, respectively $\mathcal{B}^+(E)$, we denote the set of Borel-measurable real-valued bounded, respectively positive, functions. For a subset $A \subset E$ we denote by $\overset{\circ}{A}$, $\overline{A}$ and ∂A the interior, closure and boundary of A, respectively. For $A \subset B$ we say A is open (closed) in B if A is open (closed) w.r.t. the trace topology of E on B. For a matrix A we denote by $A^\top$ the transpose of A.

By $C^0(E)$, $E \subset \mathbb{R}^d$, we denote the space of all continuous functions on E, by $C^{0,\alpha}(E)$, $0 < \alpha < 1$, the space of all Hölder continuous functions of order α. By $C^{m,\alpha}(E)$, $E \subset \mathbb{R}^d$, $d, m \in \mathbb{N}$, $0 < \alpha < 1$, we denote the set of all m-times continuously differentiable functions in $\overset{\circ}{E}$ such that the derivatives up to order $m-1$ admit a continuous continuation to $\partial E \cap E$ (possibly empty, e.g. if E is open) and the m-th derivatives admit a Hölder continuous extension of order α. The subindex c marks that the functions are supposed to have compact support in E. The subindex b marks that the function and its derivatives up to order m are bounded on E. By *support of a function* u we denote the closure of all points where u is not zero, denoted by supp$[u]$. Denote by $C^\infty(E)$ the intersection of all $C^m(E)$, $m \in \mathbb{N}$. With *cutoff* for A in B we mean a $C_c^\infty(E)$-function that is constantly equal 1 on A and has compact support in B. We call an open subset of $\mathbb{R}^d$ sometimes also a *domain*. For a differentiable function u on an open set $\Omega \subset \mathbb{R}$ we denote by ∇u the gradient, seen as a column vector, and by $\partial_i u$, $1 \le i \le d$, the partial derivative in direction e_i, e_i the i-th unit vector. The expression $(\nabla u)(x-y)$ means evaluating the gradient of u at $x-y$ rather than differentiating the function $x \mapsto u(x-y)$ or $y \mapsto u(x-y)$. By dx we denote the Lebesgue measure, by ε_x the point measure in a point x. Let $(E, \mathcal{B})$ be a measurable space with a topology τ. For a measure μ we denote by *topological support of* μ all points for which an open neighborhood with positive μ-measure exists. We denote by $\mathcal{L}^p(E,\mu)$, $1 \le p \le \infty$, the space of p-integrable functions and by $L^p(E,\mu)$ the corresponding equivalence classes. For a function space $\mathcal{L}$ we denote by $\sigma(\mathcal{L})$ the σ-algebra generated by $\mathcal{L}$, i.e., the smallest σ-algebra for which all functions in $\mathcal{L}$ are measurable. By $H^{m,p}(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, $m, p \in \mathbb{N}$, $p \ge 1$, we denote the *Sobolev space* of m-times weakly differentiable L^p-functions with $L^p(\Omega, dx)$-integrable derivatives. The corresponding local spaces are marked by the subindex loc, they are introduced in Section 7.5. In a metric space $(E, \mathbf{d})$ we denote by $B_r(x)$ the ball with radius $r > 0$ around $x \in E$. By dist(x, A), $x \in E$, $A \subset E$, we denote the *distance* of x to A.

2 Construction of $\mathcal{L}^p$-Strong Feller Processes

In this chapter we provide a general construction scheme for $\mathcal{L}^p$-strong Feller processes on locally compact separable metric spaces. The construction result yields that starting from certain regularity conditions on the semigroup associated with a symmetric Dirichlet form, one obtains a diffusion process which solves the corresponding martingale problem for every starting point from an explicitly known set. In Theorem 2.3.10 we mention further useful properties of the process, formulated also as pointwise statements. In Section 2.4 we provide concrete examples. Our results and their proofs are based on [AKR03] and [Doh05]. We got also many ideas from [FG07], [FG08] and [Sti10]. For the construction of classical Feller processes from strongly continuous contraction semigroups on spaces of continuous functions vanishing at infinity, see e.g. [BG68, Ch. I, Theo 9.4]. There are also results on the construction of Hunt processes from resolvents of kernels, see [Sto83] and Remark 2.3.1 below.

We have published the results of this chapter already in [BGS13].

2.1 A General Construction Scheme

For readers, who are unfamiliar with the concepts of Dirichlet forms or $\mathcal{L}^p$-strong Feller processes, it might help to have a look at the examples provided in Section 2.4 first.

Throughout Section 2.2 and 2.3 we fix a metric space $(E, \mathbf{d})$, a measure μ on the Borel σ-algebra $\mathcal{B}(E)$ and a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, \mu)$.

We assume the following conditions.

Condition 2.1.1.
(i) $(E, \mathbf{d})$ is a locally compact separable metric space.
(ii) μ is a locally finite Borel measure with full topological support.
(iii) $(\mathcal{E}, D(\mathcal{E}))$ is symmetric, regular and strongly local.

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Except for the locality assumption these are the standard assumptions under which Dirichlet forms and stochastic processes are considered in [FOT11]. With locally finite Borel measure we mean that μ is defined on the Borel σ-algebra and is finite on compact sets. With full topological support we mean that for every $x \in E$ there exists a neighborhood U of x such that $\mu(U) > 0$. By the Beurling-Deny theorem there exists an associated strongly continuous contraction semigroup on $L^r(E,\mu)$ (L^r-s.c.c.s) $(T_t^r)_{t>0}$ with generator $(L_r, D(L_r))$ for every $1 \leq r < \infty$, see Theorem 7.2.3. If $r > 1$ then $(T_t^r)_{t>0}$ is the restriction of an analytic semigroup. Here associated means that for $f \in L^1(E,\mu) \cap L^\infty(E,\mu)$ it holds $T_t^2 f = T_t^r f$ for every $t \geq 0$ where $(T_t^2)_{t\geq 0}$ is the unique L^2-s.c.c.s associated with $(\mathcal{E}, D(\mathcal{E}))$.

We assume the following stronger conditions that are needed to get refined pointwise results.

Condition 2.1.2.
There exists a Borel set $E_1 \subset E$ with $\mathrm{cap}_{\mathcal{E}}(E \setminus E_1) = 0$ and $1 < p < \infty$ such that

(i) $D(L_p) \hookrightarrow C^0(E_1)$ and the embedding is locally continuous, i.e., for $x \in E_1$ there exists an E_1-neighborhood U and a constant $C_1 = C_1(U) < \infty$ such that

$$\sup_{y\in U} |\widetilde{u}(y)| \leq C_1 \|u\|_{D(L_p)} \quad \text{for all}\, u \in D(L_p). \tag{2.1}$$

Here $\widetilde{u}$ denotes the continuous version of u.

(ii) For each point $x \in E_1$ there exists a sequence of functions $(u_n)_{n\in\mathbb{N}}$ in $D(L_p)$ such that

a) Either $\{u_n^2 \,|\, n \in \mathbb{N}\} \subset D(L_p)$ or $0 \leq u_n \leq 1$ and $u_n(x) = 1$ for all $n \in \mathbb{N}$.

b) The sequence $(u_n)_{n\in\mathbb{N}}$ is point separating in x.

Here $C^0(S)$ denotes the space of all continuous functions on a topological space S. By $\|\cdot\|_{D(L_p)}$ we denote the graph norm of $(L_p, D(L_p))$. Point separating in x means that for every $y \neq x$, $y \in E$, there exists u_n such that $u_n(y) = 0$ and $u_n(x) = 1$. We adjoin to E an extra point Δ which is not contained in E. We endow $E^\Delta := E \cup \{\Delta\}$ with the topology of the Alexandrov one-point compactification of E. The open neighborhoods of Δ are given by the complements of compact subsets of E.

Under Condition 2.1.1 and Condition 2.1.2 we obtain the following theorem.

Theorem 2.1.3. *There exists a diffusion process (i.e., a strong Markov process having continuous sample paths on the time interval* $[0,\infty)$*)* $\mathbf{M} = (\boldsymbol{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\mathbf{X}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E\cup\{\Delta\}})$ *with state space* E *and cemetery* Δ*, the Alexandrov point of* E*. The process leaves* $E_1 \cup \{\Delta\}$ $\mathbb{P}_x$*-a.s.,* $x \in E_1 \cup \{\Delta\}$*, invariant. The transition semigroup* $(P_t)_{t\geq 0}$ *is associated with* $(T_t^2)_{t\geq 0}$ *and is* $\mathcal{L}^p$*-strong Feller, i.e.,* $P_t\mathcal{L}^p(E,\mu) \subset C^0(E_1)$ *for* $t > 0$*. The process has continuous paths on* $[0,\infty)$ *and it solves the martingale problem associated with* $(L_p, D(L_p))$ *for starting points in* E_1*, i.e.,*

$$M_t^{[u]} := \widetilde{u}(\mathbf{X}_t) - \widetilde{u}(x) - \int_0^t L_p u(\mathbf{X}_s)\, ds,\ t \geq 0,$$

is an $(\mathcal{F}_t)$*-martingale under* $\mathbb{P}_x$ *for all* $u \in D(L_p)$ *and* $x \in E_1$*. As filtration* $(\mathcal{F}_t)_{t\geq 0}$ *we take the natural filtration, defined in* (7.8) *below.*

Here $(P_t)_{t\geq 0}$ being associated with $(T_t^2)_{t\geq 0}$ means that $P_t f$ is a μ-version of $T_t^2 f$ for $f \in \mathcal{L}^1(E,\mu) \cap \mathcal{B}_b(E)$ (the space of Borel-measurable bounded functions). By $\mathcal{L}^p(E,\mu)$ we denote the space of all p-integrable functions on (E,μ).

Remark 2.1.4. The continuity holds with respect to topology of the Alexandrov one-point compactification of E^Δ. This means that the process has continuous paths in E and reaches Δ only by leaving continuously every compact set of E. The integral in $M_t^{[u]}$ exists and is independent of the μ-version of $L_p u$. This will be seen in the proof below.

The theorem is proven in Section 2.3, see page 31 and Theorem 2.3.11 below. Further useful properties of the constructed process are proven in Theorem 2.3.10 below.

Under additional conditions, the corresponding resolvent of kernels $(R_\lambda)_{\lambda>0}$ are even *strong Feller*, i.e., $R_\lambda\mathcal{B}_b(E) \subset C^0(E_1)$. More precisely, we have the following theorem.

Theorem 2.1.5. *Assume the following conditions.*

(i) For every $x \in E_1$ *there exists a neighborhood* $U \subset E_1$ *such that for the closure in* E *it holds* $\overline{U} \subset E_1$ *and* $\overline{U}$ *is compact.*

(ii) For every sequence $(u_n)_{n\in\mathbb{N}}$ *in* $D(L_p)$ *such that* $((1-L_p)u_n)_{n\in\mathbb{N}}$ *is uniformly bounded in the* $\|\cdot\|_{L^\infty}$*-norm it holds that* $(u_n)_{n\in\mathbb{N}}$ *is equicontinuous.*

Then $(R_\lambda)_{\lambda>0}$ is strong Feller.

For the proof see Section 2.2 (page 21).

Remark 2.1.6. In [AKR03] it is shown that strong Feller property of $(R_\lambda)_{\lambda>0}$ and conservativity of $(\mathcal{E}, D(\mathcal{E}))$ imply that $(P_t)_{t>0}$ is strong Feller. The proof generalizes to the case considered here.

Having strong Feller properties of the resolvent family at hand, we can provide a conservativity criterion for the process $\mathbf{M}$.

Corollary 2.1.7. *Assume that $(\mathcal{E}, D(\mathcal{E}))$ is conservative and $(R_\lambda)_{\lambda>0}$ is strong Feller, then $\mathbf{M}$ from Theorem 2.1.3 is conservative for every starting point $x \in E_1$.*

See p. 34 for the proof.

For constructions in the later chapters it is convenient to consider the so-called restriction of the process from Theorem 2.1.3 to $E_1 \cup \{\Delta\}$. We obtain the following corollary.

Corollary 2.1.8. *Let $\mathbf{M} = (\mathbf{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\mathbf{X}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E\cup\{\Delta\}})$ be the diffusion process constructed in Theorem 2.1.3. Let the restricted process $\mathbf{M}^1 := (\mathbf{\Omega}^1, \mathcal{F}^1, (\mathcal{F}_t^1)_{t\geq 0}, (\mathbf{X}_t^1)_{t\geq 0}, (\mathbb{P}_x^1)_{x\in E_1\cup\{\Delta\}})$ be defined as in Definition 7.3.18 with $\widetilde{E}_1 := E_1 \cup \{\Delta\}$. Then $\mathbf{M}^1$ is a $\mathcal{L}^p$-strong Feller diffusion process with state space E_1 and cemetery Δ. The transition semigroup $(P_t^{E_1\cup\{\Delta\}})_{t\geq 0}$ is absolutely continuous on E_1.*

See p. 35 for the proof.

Remark 2.1.9. The filtration $(\mathcal{F}_t^1)_{t\geq 0}$ and $\mathcal{F}^1$ are important for the construction of additive functionals in Chapter 6. There some subsets of $\mathbf{\Omega}$ have full $\mathbb{P}_x$-measure for $x \in E_1 \cup \{\Delta\}$ only.

2.2 Construction of $\mathcal{L}^p$-strong Feller Kernels

We start with the construction of a semigroup of kernels $(P_t)_{t>0}$ and resolvent of kernels $(R_\lambda)_{\lambda>0}$ which yield a μ-version of $(T_t^p)_{t>0}$ and $(G_\lambda^p)_{\lambda>0}$. For this we assume Condition 2.1.1 and Condition 2.1.2 and fix a $1 < p < \infty$ as in Condition 2.1.2. We adapt the structure of [AKR03, Sec. 3] and modify the statements and proofs there in order to cover the abstract setting.