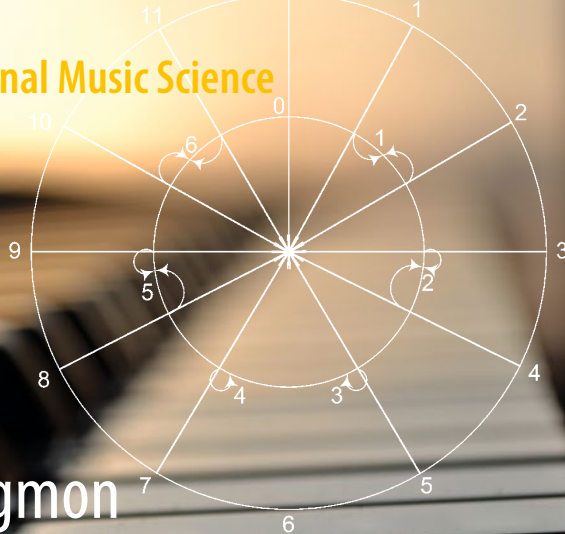


Computational Music Science



Eytan Agmon

# The Languages of Western Tonality

 Springer

# Computational Music Science

## **Series Editors**

Guerino Mazzola  
Moreno Andreatta

For further volumes:  
<http://www.springer.com/series/8349>



Eytan Agmon

# The Languages of Western Tonality

 Springer

Eytan Agmon  
Department of Music  
Bar-Ilan University  
Ramat-Gan  
Israel

ISSN 1868-0305

ISSN 1868-0313 (electronic)

Computational Music Science

ISBN 978-3-642-39586-4

ISBN 978-3-642-39587-1 (eBook)

DOI 10.1007/978-3-642-39587-1

Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013951826

© Springer-Verlag Berlin Heidelberg 2013

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media ([www.springer.com](http://www.springer.com))

# Preface

... Each of these four notes governs, as its subjects, a pair of tropes... Thus every melody... is necessarily led back to one of these same four [notes]. Therefore they are called “finals,” because anything that is sung finds its ending (*finem*) in [one of] them.

—Hucbald of Saint-Amand, late Ninth Century<sup>1</sup>

As an object of inquiry, “tonal music” is far from homogenous. The music of the ninth century with which Hucbald was familiar was very different, we may assume, from (say) the music of the seventeenth century. Nonetheless, there are striking points of contact. Most notably, a “background system” of exactly seven notes, orderable within the octave as a cyclic permutation of the sequence T-T-S-T-T-S of tones and semitones (alternatively, as a sequence of perfect fifths—the fifth being the most privileged interval following the perfect octave), is “governed” by one of its members—the final (and “co-governed,” one might add, by another—the cofinal, the perfect fifth above the final). How can one account for the remarkable stability of such basic features of “Tonality,” and, at the same time, do justice to the equally remarkable variety of styles—nay, *languages*—that the history of Western tonal music has taught us exist? This book is an attempt to answer these questions.

The book is divided into two main parts. Part I, *Proto-tonality*, studies the background system of notes prior to the selection of a final. The “proto-tonal system” ultimately posited is *harmonic* and contains a “harmonic message.” However, the harmonic message may be empty, in which case the system reduces to its *diatonic* component. In other words, a harmonic system is diatonic, but not vice versa (a diatonic system is oblivious of such constructs as “chord,” “chord progression,” and “voice leading”). An important component of every diatonic system is its “core”: a length-7 segment of the “line of fifths,” for example, F, C, . . . , B.

After some preliminaries that concern consonance and chromaticism, Part II, *The Languages of Western Tonality* (also the title of the book as a whole), begins with the notion “mode.” A mode is assumed to contain a “nucleus.” A nucleus is a subset of the core *that is consonant* while containing a maximal number of

---

<sup>1</sup> Trans. from Cohen (2002), p. 322.

elements. The mode's final is the unique nucleus element that is *a root* relative to every other nucleus element. For example, the final would be the lower member (rather than the upper) of a perfect-fifth dyadic subset of the nucleus.

Crucially, the notion of "consonance" is itself dependent on the proto-tonal system's status as diatonic or harmonic. In particular, in the harmonic case the major and minor third and sixth are deemed consonant in addition to the diatonic perfect octave, fifth, and fourth. As a result, the modal nucleus is a perfect-fifth dyad in the diatonic case and a triad (major or minor) in the harmonic case. Early in Part II, in other words, a distinction is established between two types of mode, *dyadic* and *triadic*. Moreover, within each type, the degree to which the nucleus (or a privileged subset thereof) is consonant *in relation to the non-nucleus core elements* defines "semi-key" as a special type of mode and "key" as a special type of semi-key. On the basis of these distinctions, seventeenth-century music (for example) presents itself as a (tonal) language of triadic semi-keys; ninth-century music, by contrast, is a (tonal) language of dyadic modes.

Thus, in a nutshell, the theory accounts for tonal variety. The question of tonal stability is addressed mainly in the proto-tonal Part I, though it continues to inform ideas put forth in Part II as well. In a nutshell again, tonality is seen as a highly successful "communication system." Communication, indeed, is the most important high-level principle that guides the theory offered in this book.

## Theory; History; Cognition

From the preceding remarks it is clear that the book draws upon three distinct fields of study, namely music theory, music history, and music cognition. Like the three edges of an equilateral triangle, the contribution of each field to the project as a whole is inconceivable without the other two.

Music theory is the oldest and most established of the three. Music theory has not only handed us, early in the nineteenth century, a valuable though elusive concept—Tonality—but is on record for centuries if not millennia for attempting to demystify that extraordinary gift of mankind to itself: music. The story of music theory is fascinating in its own right, replete as it is with turns and twists, progressions and regressions. Be that as it may, the present project is unthinkable in the absence of the rich and complex heritage of ideas that constitute the music-theoretic endeavor.

It is all too easy to absorb oneself in the familiar and the readily accessible, forgetting not only that the past may have been different from the present, but also that the present may be very different elsewhere. Historical musicology and ethnomusicology have taught us to respect the chronological and cultural Other.

As should be clear from the Preface's opening remarks, this study takes seriously the historical challenge, offering a theory that, while not explicitly diachronic, nonetheless renders conceivable a historical process of the sort that seems to have taken place in Western culture, namely from dyadic to triadic tonality, and more or

less concurrently, from modes, through semi-keys, to keys. Self-consciously focusing on the West, the study is obviously less responsive to the cultural challenge. Nonetheless, a reference to the Javanese *pélog* scale may be found in Part I, Sect. 8.1; a reference to north Indian music may be found in Part II, Sect. 14.1.

Finally, music cognition has made us keenly aware that music is a reflection of the human mind. In current music-cognitive discourse much emphasis is placed on *perception*. The present book, by contrast, places equal emphasis on *conception*. The difference reflects the book's communicative bias, coupled with the observation that communication takes place where perception meets conception. In many ways, we shall see, the languages of Western tonality reflect the logical and cognitive constraints that make musical communication possible.

Even for the professional music theorist, the book is no easy reading. This is especially true of Part I, *Proto-tonality*, where abstraction and formal rigor reign supreme. Formal rigor in the book, however, is no ornament. As explained in Chap. 2, the book strives for the highest possible standard of scientific acceptability, namely explanatory adequacy. To this end, it was necessary to strip tonality down to its barest elements, generalizing parameters whenever possible. That this approach pays off becomes apparent already in Chap. 8, where an alternative theory is compared to the proposed theory precisely in terms of explanatory adequacy.

A number of strategies may help the interested reader overcome the difficulties of Part I. The tried-and-true strategy of nonsequential reading may not only help combat fatigue but may offer a larger (if not completely coherent) perspective from which the intricacies of a given phase in the theory may be easier to digest. In particular, Chap. 4, "The Conventional Nomenclatures for Notes and Intervals," is a relatively accessible exercise that may be fruitfully studied before or concurrently with Chap. 3. In general, examples of formal definitions are presented *after* the definitions themselves. Easier access into an abstruse definition may be gained by skipping ahead to an accompanying example.

The reader may feel overwhelmed by the sheer number of definitions and notations introduced. Partial help in this matter is provided by three indices found in the back of the book: a General Index, where formal definitions are easily identified by the corresponding page number's formatting in bold; a List of Definitions; and a List of Notation. Two mathematical appendices provide the basic mathematical background.

The origins of this book go back to my 1986 PhD dissertation, "Diatonicism, Chromaticism, and Enharmonicism: A Study in Cognition and Perception." The book has thus acquired many debts over the years.

Carl Schachter, the dissertation's supervisor, with whom I spent a couple of years as a graduate student at the City University of New York, has had a profound influence on my identity as a music theorist. Schachter's immense knowledge and deep understanding of music and music theory, his astounding eloquence in verbalizing his ideas and insights, his generosity towards students and colleagues alike, and his down-to-earth, unpretentious human warmth have enriched and nourished me for life.



I was very lucky to have had Joel Lester as a reader of my dissertation. Lester showed interest in my work early on and has always been extremely generous in lending support and guidance. Lester provided important feedback on early versions of Chaps. 12 and 13.

John Rink and William Rothstein commented insightfully on an early version of Chap. 15. Section 15.2 in fact developed from a brilliant suggestion of Rothstein's.

A number of individuals provided invaluable help with the mathematical aspects of the book.

I feel very lucky to have two renowned mathematicians in my immediate family: my father, Shmuel Agmon, and my brother, Ehud de-Shalit. Both have been involved in my work since the dissertation. Without their patient guidance over the years, this book could not have come into being.

In 2010–2011 Nori Jacoby and I gave a course at Bar-Ilan University, “Music, Mathematics, and Cognition,” based on Part I of the book. Nori contributed numerous improvements to the mathematics, substantive as well as stylistic. Thanks must also go to Thomas Noll, who read Part I and offered valuable insights and suggestions, to Avinoam Braverman for commenting insightfully on Chap. 3, and to Reuven Naveh for commenting on a very early version of Chap. 4.

Although it has become a cliché to thank one's spouse and children, in the present case my wife, Lea, and two wonderful daughters, Einat and Orly, are true partners in the endeavor. My wife, an accomplished musician in her own right, has been part of this project from the very start, lending her ear, heart, and mind with uncompromising patience and devotion at every turn. My daughters have grown accustomed to music-theoretic discussions at family dinners. They have found early on that the best way to get Daddy genuinely upset is to argue with him that there is no difference between G# and A<sub>b</sub>.

I dedicate this book to Lea, Einat, and Orly, with love.

## Reference

- Cohen, D. (2002). Notes, scales, and modes in the earlier middle ages. In T. Christensen (Ed.), *The Cambridge history of Western music theory* (pp. 307–363). Cambridge: Cambridge University Press.

# Contents

<b>1 Proto-tonal Theory: Tapping into Ninth-Century Insights . . . . .</b>	<b>1</b>
References . . . . .	14
<b>Part I Proto-tonality</b>	
<b>2 Preliminaries . . . . .</b>	<b>19</b>
2.1 Descriptive and Explanatory Proto-tonal Adequacy: A Lesson from Linguistics . . . . .	20
2.2 The Communication Principle . . . . .	24
2.3 Three Additional Guiding Ideas . . . . .	26
2.3.1 The Economical Principle . . . . .	26
2.3.2 The Categorical Principle . . . . .	26
2.3.3 The Maximalist Principle . . . . .	27
2.4 Event Sequences . . . . .	28
References . . . . .	28
<b>3 Communicating Pitches and Transmitting Notes . . . . .</b>	<b>31</b>
3.1 Octave-Endowed Note Systems . . . . .	31
3.2 Bases of the Interval Space . . . . .	35
3.3 Pitch-Communication Systems . . . . .	36
3.4 Absolute, Relative, and Reflexive Pitch Communication . . . . .	45
3.4.1 Two Postscripts . . . . .	48
3.5 Composite Tone Systems . . . . .	50
References . . . . .	57
<b>4 The Conventional Nomenclatures for Notes and Intervals . . . . .</b>	<b>59</b>
4.1 The Conventional Nomenclatures for Notes and Intervals . . . . .	60
4.2 Staff Notation and Its Idiosyncrasies . . . . .	63

<b>5</b>	<b>Communicating the Primary Intervals</b> . . . . .	67
5.1	Efficient Tone Systems . . . . .	68
5.2	Coherent Tone Systems . . . . .	74
5.3	Categorical Equal Temperament . . . . .	79
	References . . . . .	81
<b>6</b>	<b>Receiving Notes</b> . . . . .	83
6.1	Note Reception: A Lesson by Bartók . . . . .	84
6.2	Note-Reception Systems . . . . .	86
6.3	Proto-diatonic Systems . . . . .	90
6.4	Diatonic Systems . . . . .	93
6.5	Properties of Diatonic Systems . . . . .	98
	References . . . . .	102
<b>7</b>	<b>Harmonic Systems</b> . . . . .	103
7.1	The Grammatical Basis of Harmonic Communication . . . . .	103
7.2	Generic Klang Systems . . . . .	107
7.3	Functional Klangs and Klang Classes . . . . .	110
7.4	Harmonic Systems, Voice-Leading Enabled . . . . .	114
7.5	Efficient Harmonic Systems . . . . .	118
	References . . . . .	125
<b>8</b>	<b>Proto-tonality</b> . . . . .	127
8.1	Proto-tonal Systems . . . . .	127
8.2	Categorical ET: Theory Lagging (Far) Behind Practice? . . . . .	130
8.3	A Possible Alternative to the Theory of Proto-tonal Systems . . . . .	134
	References . . . . .	143
<b>Part II The Languages of Western Tonality</b>		
<b>9</b>	<b>Tonal Preliminaries</b> . . . . .	147
9.1	Dyadic and Triadic Consonance and Stability . . . . .	147
9.2	The Chromatic Content of the Cluster . . . . .	151
	References . . . . .	155
<b>10</b>	<b>Modal Communication</b> . . . . .	157
10.1	Modes, Semi-keys, and Keys . . . . .	157
10.2	Modal Communication Systems . . . . .	162
10.3	Scale Degrees . . . . .	168
10.4	Robust and Semi-robust Communication Systems . . . . .	173
10.5	Congruent and Standard Modes . . . . .	177
	References . . . . .	178
<b>11</b>	<b>Topics in Dyadic and Triadic Theory</b> . . . . .	179
11.1	Glarean, Lippius, and Modal Theory . . . . .	179
11.2	Aspects of Triadic Consonance and Stability . . . . .	182
11.3	Relative and Parallel Triadic Keys . . . . .	183
11.4	Robust Triadic Keys and Schenker's "Mixture" . . . . .	184
	References . . . . .	187

<b>12</b>	<b>Modes, Semi-keys, and Keys: A Reality Check</b> . . . . .	189
12.1	The Octenary Doctrine and the “Reality of Mode” . . . . .	189
12.2	The Seventeenth-Century “Church Keys” as Triadic Semi-keys . . . . .	198
12.3	On the Reality of Triadic Keys . . . . .	204
	References . . . . .	214
<b>13</b>	<b>A Neo-Riepelian Key-Distance Theory</b> . . . . .	217
13.1	Key-Distance Theories of the Eighteenth and Nineteenth Centuries . . . . .	218
13.2	The Krumhansl/Kessler Torus and Its Relation to Weber’s . . . .	224
13.3	A Neo-Riepelian Key-Distance Theory . . . . .	230
	References . . . . .	235
<b>14</b>	<b>Tonal Communication</b> . . . . .	237
14.1	Dyadic and Triadic Heptads . . . . .	238
14.2	Scales and Tonalities . . . . .	244
	References . . . . .	249
<b>15</b>	<b>The Tonal Game</b> . . . . .	251
15.1	The Tonal Game . . . . .	251
15.2	Chopin’s Mazurka, Op. 24, No. 2, and Fétis’s “Tonal Perfection” . . . . .	258
	References . . . . .	263
	<b>Appendix A: Mathematical Preliminaries</b> . . . . .	265
	<b>Appendix B: <math>\mathbb{Z}</math> Modules and Their Homomorphisms</b> . . . . .	269
	<b>Index</b> . . . . .	273



# List of Figures

Fig. 1.1	(a) A segment of the conceptually infinite series of disjunct T-S-T tetrachords that forms the <i>enchiriadis</i> tone system ( <i>Musica enchiriadis</i> ). Copyright © 1981 Bayerische Akademie der Wissenschaften. Redrawn from H. Schmid, <i>Musica et scolica enchiriadis</i> , by permission. (b) The line of fifths. The “note-class content” of the <i>enchiriadis</i> tone system is shown in <i>square brackets</i> .....	7
Fig. 1.2	Semitonal displacement (E to E <sub>b</sub> ) in the C-G-C pentachord ( <i>Scolica enchiriadis</i> ). Copyright © 1981 Bayerische Akademie der Wissenschaften. Redrawn from H. Schmid, <i>Musica et scolica enchiriadis</i> , by permission .....	9
Fig. 1.3	A triple of “block diagrams” from a late tenth- or early-eleventh-century copy of <i>Scolica enchiriadis</i> . Stadt Köln, Historisches Archiv. HASTK, Best. 331, fol. 219. Reproduced by permission .....	10
Fig. 1.4	Notes as integer pairs. After <i>Scolica enchiriadis</i> .....	11
Fig. 1.5	The octave as a combined fifth and fourth ( <i>Scolica enchiriadis</i> ). Copyright © 1981 Bayerische Akademie der Wissenschaften. Redrawn from H. Schmid, <i>Musica et scolica enchiriadis</i> , by permission .....	12
Fig. 2.1	A (compound) augmented seventh in the first movement (m. 116) of J. S. Bach’s Brandenburg Concerto No. 3 .....	21
Fig. 2.2	Shannon’s General Communication System .....	25
Fig. 2.3	The note-communication system .....	25
Fig. 2.4	The prototype $p_i$ and the category $C_i$ on the perceptual continuum $P$ . $L_i$ and $R_i$ are the category’s left and right boundaries, respectively .....	27
Fig. 2.5	Prototypicality as a function of perception relative to an arbitrary category $C_i = (L_i, R_i)$ with prototype $p_i$ .....	27
Fig. 3.1	A pitch-communication system .....	37
Fig. 3.2	The main objects and relationships studied in Theorem 3.27 .....	43

Fig. 3.3	Harmonic templates generated by <b>(a)</b> a broadband noise stimulus and <b>(b)</b> a random click train with random widths. Reprinted with permission from Shamma and Klein, <i>Journal of the Acoustical Society of America</i> 107(5), p. 2635 (2000). Copyright © 2000, Acoustical Society of America . . . . .	49
Fig. 3.4	<b>(a)</b> <i>Tonnetz</i> by Euler (1739). <b>(b)</b> <i>Tonnetz</i> by Ellis. Copyright © 1954 Dover Publications. Reproduced from H. Helmholtz, <i>On the sensations of tone</i> , by permission . . . . .	54
Fig. 4.1	The notes $(0, -1)$ and $(-1, 0)$ in conventional staff notation . . .	65
Fig. 4.2	A quadruply diminished ascending third and a sextuply diminished descending fourth in conventional staff notation . . .	65
Fig. 5.1	A usual or anti-usual efficient tone system is coherent . . . . .	76
Fig. 5.2	Pitch-interval prototypicality relative to three arbitrary note intervals $(u-1, v'')$ , $(u, v)$ , and $(u+1, v')$ . . . . .	80
Fig. 6.1	Bartók, <i>Music for String Instruments, Percussion, and Celesta</i> , I, beginning <b>(a)</b> and end <b>(b)</b> . Copyright © 1937 Universal Edition A.G., Wien/UE34129 . . . . .	85
Fig. 6.2	Schubert, String Quintet in C major, D. 956, mm. 1–6 . . . . .	93
Fig. 6.3	A proto-diatonic system with a diatonic note $(w, t_i)$ . . . . .	94
Fig. 6.4	The scalar ordering of the core relative to its quintic ordering. <b>(a)</b> $(a, b) = (13, 7)$ , $(c, d) = (2, 1)$ ; <b>(b)</b> $(a, b) = (12, 7)$ , $(c, d) = (7, 4)$ . . . . .	100
Fig. 7.1	Beethoven, String Quartet in F minor, Op. 95, III. <b>(a)</b> Score of first phrase; <b>(b)</b> melodic content of bracketed segment; <b>(c)</b> harmonic content of bracketed segment . . . . .	105
Fig. 7.2	Six different “C-major triads” . . . . .	108
Fig. 7.3	The seven-dot circles <b>(a)</b> and <b>(b)</b> represent $N_7$ . The <i>solid dots</i> may be seen as a triad <b>(a)</b> and a tetrad <b>(b)</b> with root $r = 0$ and $r = 6$ , respectively, relative to $e = 2$ . (The <i>solid dots</i> may also be seen as a triad and a tetrad with root $r' = 4$ and $r' = 5$ , respectively, relative to $e' = 5$ ) . . . . .	109
Fig. 7.4	Relative to the usual or anti-usual generic klang system the number of notes ( <i>open note heads</i> ) that a triad <b>(a)</b> or a tetrad <b>(b)</b> does <i>not</i> share with a referential triad and tetrad, respectively, increases with the distance between the roots, measured in multiples of thirds (or sixths) . . . . .	111
Fig. 7.5	Two congruent chord progressions . . . . .	117
Fig. 7.6	<i>Lines</i> connecting elements of the fifth chord with elements of the sixth chord represent a voice leading between them . . . . .	118
Fig. 7.7	A voice leading between two congruent chords expressing a triad. A pair of notes connected by a <i>straight line</i> is a linear pair . . . . .	119
Fig. 7.8	All voice leadings in <b>(a)</b> are efficient; in <b>(b)</b> only the second is efficient . . . . .	120

Fig. 7.9	A usual harmonic system is efficient. There exist efficient voice leadings between chords expressing every pair of functional klangs. <b>(a)</b> Triad to triad; <b>(b)</b> tetrad to tetrad; <b>(c)</b> triad to tetrad; <b>(d)</b> tetrad to triad. After Agmon (1991), Figs. 2 and 6. Copyright © 1991 Universitätsverlag Brockmeyer, redrawn from <i>Musikometrika 3</i> , by permission . . . . .	122
Fig. 7.10	<b>(b)</b> is an efficient realization of the congruent progression <b>(a)</b> . . .	122
Fig. 8.1	String-trio intonation of selected intervals. Data from Shackford (1961, 1962a). For each interval, pitch-interval size in cents ( <i>horizontal axis</i> ) is plotted against percentage of occurrences ( <i>vertical axis</i> ). Also given are the sample size (N), as well as its mode and mean . . . . .	132
Fig. 8.2	Mean subjective purity ratings for intervals that comprised two simultaneous sinusoidal tones. From Vos (1986). Copyright © 1986 Regents of the University of California. Reproduced from <i>Music Perception 3</i> (3), by permission . . . . .	134
Fig. 9.1	The dyadic set of systemically privileged intervals as a proper subset of the triadic set . . . . .	148
Fig. 9.2	Two rooted consonances. Regardless of direction of measurement, in <b>(a)</b> the root is the lower note, whereas in <b>(b)</b> it is the upper . . . . .	150
Fig. 9.3	From <i>Ars contrapunctus secundum Philippum de Vitriaco</i> (de Vitry, 1864–1876) . . . . .	153
Fig. 10.1	Modes, semi-keys, and keys (dyadic or triadic), and the tribal classes to which they reduce . . . . .	161
Fig. 10.2	An absolute <b>(a)</b> and a relative <b>(b)</b> modal communication system with the same transmitted, C# Aeolian mode. The received modes are Mixolydian with finals D $\flat$ and C, respectively . . . . .	163
Fig. 10.3	Possible mappings of selected message elements into their received images in a modal communication system . . . . .	166
Fig. 10.4	The <i>Mu</i> -ian degree cluster. <b>(a)</b> $\mu = 1$ (Ionian); <b>(b)</b> $\mu = 4$ (Aeolian) . . . . .	169
Fig. 10.5	Mappings of message elements into their received images in <b>(a)</b> a robust and <b>(b)</b> a semi-robust relative communication system . . . . .	176
Fig. 10.6	The scores of two congruent triadic keys, C and D $\flat$ major . . . . .	178
Fig. 11.1	The three figured-bass positions of a major triad . . . . .	183
Fig. 11.2	Standard parallel major/minor keys . . . . .	185
Fig. 11.3	Two Beethoven excerpts as robust keys. <b>(a)</b> Piano Sonata in C major, Op. 53 (“Waldstein”), I, mm. 1–13; <b>(b)</b> Piano Sonata in C minor, Op. 27/1, II, 1–16 . . . . .	185
Fig. 11.4	Score elements of two robust relative keys are enharmonically related if, and only if, their scale-degree representations are vertically aligned or diagonally connected . . . . .	187
Fig. 11.5	Chopin, Scherzo, Op. 31. <b>(a)</b> Mm. 1–9; <b>(b)</b> mm. 49–58 . . . . .	187



Fig. 12.1	The nested hierarchy Nucleus-Core-Cluster .....	205
Fig. 12.2	Percentages of probe-tone rating, major. (a) Separately for each degree; (b) grouped by category. Data from Table 12.2 ....	207
Fig. 12.3	Percentages of probe-tone rating, minor. (a) Separately for each degree; (b) grouped by category. Data from Table 12.2 ....	207
Fig. 12.4	Percentages of degree occurrence, major. (a) Separately for each degree; (b) grouped by category. Data from Table 12.3 ....	209
Fig. 12.5	Percentages of degree occurrence, minor. (a) Separately for each degree; (b) grouped by category. Data from Table 12.3 ....	209
Fig. 12.6	Percentages of degree occurrence in Brahms, Op. 24 (B $\flat$ major). (a) Separately for each degree; (b) grouped by category .....	212
Fig. 12.7	Percentages of degree occurrence in Brahms, Op. 98, IV (E minor). (a) Separately for each degree; (b) grouped by category .....	212
Fig. 12.8	First- vs. second-order chromatic degrees in Brahms, Op. 24 (a), and Op. 98, IV (b). ** = highly significant at $p < 0.01$ .....	213
Fig. 12.9	“Applied dominants” to all non-tonic major and minor diatonic triads in C major (a) and C minor (b). <i>Black note heads</i> indicate second-order chromatic degrees .....	213
Fig. 13.1	(a) Heinichen’s (1728) “musical circle.” (b) Heinichen’s circle as a connected graph .....	219
Fig. 13.2	(a) Kellner’s circular arrangement of keys (1737; 1st ed., 1732). (b) Kellner’s circle as a connected graph .....	220
Fig. 13.3	Weber’s “table of key relationships” .....	221
Fig. 13.4	Weber’s table as a connected graph .....	222
Fig. 13.5	Ratner’s “solar” arrangement of keys. (a) From Ratner (1980). (b) An alternative representation that includes C minor .....	223
Fig. 13.6	The Krumhansl/Kessler torus. (a) From Krumhansl (1990a). (b) From Krumhansl and Kessler (1982). Copyright © 1982 American Psychological Association. Reproduced with permission .....	226
Fig. 13.7	Key distance derived from “harmonic hierarchies.” From Krumhansl (1990a), Fig. 7.4 on p. 187. By permission of Oxford University Press .....	229
Fig. 13.8	Two Riepel Spaces. (a) The C-major space. (b) The C-minor space .....	231
Fig. 13.9	Schubert, String Quintet in C major, D. 956, II, mm. 91–94 ...	233
Fig. 13.10	The diatonic subspace of a Riepel Space forms a metric space ...	234
Fig. 14.1	Two type-equivalent dyadic heptads. <i>Arrows</i> represent transposition by major sixth .....	241
Fig. 14.2	Liszt, Piano Sonata in B minor, mm. 1–7, 453–459, and 750–760 .....	243
Fig. 14.3	Aligned pairs of major/minor degrees are line-of-fifths inverses .....	243

Fig. 15.1	The three main default levels of the Tonal Game .....	252
Fig. 15.2	Chopin, Mazurka, Op. 24/2, mm. 1–4 .....	255
Fig. 15.3	Tonal analysis of Chopin, Mazurka Op. 24/2 .....	256
Fig. 15.4	The modulatory path in Chopin's Mazurka, Op. 24/2, plotted on the C-major Riepel Space. <i>Parenthetical numbers in bold</i> represent <i>relative</i> key distances, to the extent that they exist ...	259
Fig. 15.5	A progression illustrating the <i>ordre omnitonique</i> from Fétis's <i>Traité</i> . Note the Tristan Chord in the first measure .....	263



# List of Tables

Table 4.1	Examples of the bijection NAME (first two columns); the third column provides the conventional equivalent .....	62
Table 4.2	Examples of the bijections NAME <sup>±</sup> (first two columns); the third column provides the conventional equivalent .....	64
Table 4.3	Musical intuitions as algebraic expressions .....	64
Table 6.1	The A-centric note collection of Bartók’s fugue subject .....	85
Table 8.1	The main assumptions of proposed proto-tonality (left-hand column, with definition number in parentheses) and the principles that they express (top row) .....	136
Table 9.1	Chromatic cluster elements relative to the natural core (left-hand column), displayed in terms of Definition 6.2 .....	153
Table 10.1	Modes, semi-keys, and keys (dyadic or triadic), and the tribal classes to which they reduce .....	161
Table 10.2	Diatonic and first-order chromatic degrees relative to tribal class .....	172
Table 11.1	The 24 standard major/minor key classes .....	184
Table 12.1	Modes, semi-keys, and keys (dyadic or triadic), and the tribal classes to which they reduce .....	199
Table 12.2	The “probe-tone ratings” of Krumhansl and Kessler (Redrawn from Krumhansl (1990), Table 2.1 on p. 30, by permission of Oxford University Press) .....	206
Table 12.3	Degree distribution in selected triadic compositions (After Krumhansl (1990), Table 3.3 on p. 67). Redrawn by permission of Oxford University Press. ....	208
Table 13.1	The correlations between keys from which the Krumhansl/ Kessler torus was derived (From Krumhansl (1990a), Table 2.4 on p. 38. By permission of Oxford University Press) .....	229

Table 13.2	The data from which Fig. 13.7 was derived (From Krumhansl (1990a), Table 7.10 on p. 183. By permission of Oxford University Press) .....	230
Table 14.1	The 32 dyadic Ionian heptads .....	241
Table 14.2	The 32 major and minor heptads .....	245

# List of Definitions

Definition 2.1	Sequence; Length (of a sequence); Empty Sequence; Position (of an element in a sequence); Sequence Membership .....	28
Definition 2.2	Sequence Exchange .....	28
Definition 3.1	Note, Note Interval .....	31
Definition 3.2	Note-Interval Function .....	32
Definition 3.3	Note-Transposition Function .....	32
Definition 3.5	Interval Space .....	33
Definition 3.6	Note System .....	33
Definition 3.7	Cognitive Octave; Octave Relation .....	33
Definition 3.10	Octave-Related Notes .....	34
Definition 3.11	Octave-Endowed Note System; Usual Note System .....	34
Definition 3.12	Basis of the Interval Space .....	35
Definition 3.16	Pitch; Pitch Interval .....	37
Definition 3.17	Pitch-Interval Function; Pitch-Transposition Function .....	37
Definition 3.18	Pitch-Interval Space .....	38
Definition 3.19	Pitch System .....	38
Definition 3.20	Pitch-Communication System (PC system) .....	39
Definition 3.22	Standard; Anti-standard; Faithful PC System .....	39
Definition 3.23	Frequency .....	40
Definition 3.24	Periodic Wave Class (relative to a fundamental frequency) .....	41
Definition 3.25	Pitch-Transmission Function; Transmittable Pitch Range .....	41
Definition 3.29	Overtone Pair (of order $n$ ) .....	45
Definition 3.30	Privileged Pitch Interval; Psychoacoustical Octave .....	45
Definition 3.31	$\psi$ -Standard PC System .....	45
Definition 3.32	Pitch Height; Pitch-Interval Direction; Pitch Register .....	46
Definition 3.33	Reflexive PC System .....	46
Definition 3.34	Absolute PC System .....	47

Definition 3.35	$\varphi$ -Centered PC System; Relative $\varphi$ -Centered PC System .....	47
Definition 3.36	Note and Note-Interval Transmission Functions .....	50
Definition 3.42	Composite Tone System (with message <i>MSG</i> ) .....	54
Definition 3.43	Note Height; Note-Interval Direction; Note Register .....	55
Definition 3.45	Oriented Type- <i>t</i> System; Naturally Oriented System; Usual Composite Tone System .....	56
Definition 4.1	Natural Core .....	60
Definition 4.4	Usual Primary Intervals .....	61
Definition 4.8	Staff-Note Height; Staff Note-Interval Direction; Staff-Note Register .....	64
Definition 5.1	Primary Intervals; Quintic Element .....	68
Definition 5.4	Semi-efficient Tone System (of order <i>M</i> ) .....	69
Definition 5.6	Efficient Tone System .....	70
Definition 5.11	Standard Efficient Tone System .....	73
Definition 5.13	Usual/Anti-usual Efficient Tone System .....	74
Definition 5.14	Coherent Tone System .....	75
Definition 5.19	Quintic Line; Quintic Segment; Line of Fifths .....	78
Definition 5.21	Categorical ET Functions; Prototypicality .....	79
Definition 5.22	(updating Definition 3.42) Composite Tone System .....	80
Definition 6.2	Image (of a message element); Empty Image .....	87
Definition 6.5	Note-Reception System (with message <i>MSG</i> ); Core; Received Message; Usual/Anti-usual Note-Reception System .....	88
Definition 6.7	Cluster (of a note-reception system) .....	89
Definition 6.10	Enharmonic Notes .....	90
Definition 6.11	Proto-diatonic System .....	90
Definition 6.13	Stable Proto-diatonic System .....	91
Definition 6.18	Diatonic Note .....	94
Definition 6.21	Diatonic System .....	96
Definition 6.25	Standard Diatonic System .....	98
Definition 6.26	Usual/Anti-usual Diatonic System .....	98
Definition 6.29	Interval Content of the Core .....	99
Definition 7.1	Generic Note Class; Generic Note-Class Interval .....	107
Definition 7.2	Generic Interval Function; Generic Note-Transposition Function .....	107
Definition 7.3	Generic Note System .....	108
Definition 7.4	Klang ( <i>m</i> -ad) with Root <i>r</i> .....	108
Definition 7.7	Klang Class (of order $\omega$ ) .....	109
Definition 7.9	Generic Klang System; Dual of a Generic Klang System .....	110
Definition 7.11	Usual/Anti-usual Generic Klang System .....	110
Definition 7.12	Functional Klang; Functional Klang Class .....	111
Definition 7.19	Chord (expressing a klang); Functional Chord .....	114

Definition 7.21	Congruent Chords .....	115
Definition 7.23	Chord Progression (expressing a klang progression) . . . .	115
Definition 7.25	Harmonic System; Empty Harmonic System .....	116
Definition 7.26	Usual/Anti-usual Harmonic System .....	116
Definition 7.28	Congruent Chord Progressions; Congruent Harmonic Systems .....	117
Definition 7.29	Voice Leading; Voice-Leading Enabled Harmonic System .....	117
Definition 7.31	Density (of VL); Span (of <i>LP</i> ); Activity (of VL) .....	118
Definition 7.33	Efficient Voice Leading .....	119
Definition 7.36	Efficient Harmonic Systems .....	121
Definition 8.1	Tertial Element .....	127
Definition 8.2	Proto-tonal System .....	128
Definition 8.7	Composite Maximally Even Set .....	137
Definition 8.9	Generic Maximally Even Set .....	137
Definition 8.15	Proposed Proto-tonality (reduced) .....	141
Definition 8.16	ME Proto-tonality (reduced) .....	141
Definition 9.2	Dyadic/Triadic Consonance (Dissonance) .....	148
Definition 9.3	Root (of a consonance) .....	149
Definition 9.4	Consonant Set .....	150
Definition 9.6	Root (of a consonant set) .....	150
Definition 9.8	Accidental; Natural, Sharp, Flat .....	151
Definition 9.9	Chromatic/Non-chromatic Note .....	151
Definition 10.1	Nucleus .....	158
Definition 10.4	Dyadic/Triadic Mode; Score; Final, Cofinal .....	158
Definition 10.5	Diatonic/Chromatic Score Element; Accidental Index .....	159
Definition 10.7	Mode Class .....	159
Definition 10.8	Tribal Class .....	160
Definition 10.9	Sub-nucleus .....	160
Definition 10.10	Key; Semi-key .....	160
Definition 10.12	Major/Minor Mode .....	162
Definition 10.14	(Context-Dependent) Modal Communication System; Transmitted/Received Mode; (Modal) Message, Received (Modal) Message .....	162
Definition 10.21	Diatonic/Chromatic Scale Degree; Degree Cluster .....	168
Definition 10.25	First-Order/Second-Order Chromatic Degree .....	169
Definition 10.30	(Caret notation for diatonic degrees) .....	171
Definition 10.32	(Caret notation for first-order chromatic degrees) .....	172
Definition 10.33	(Caret notation for second-order chromatic degrees) . . . .	172
Definition 10.34	Robust Mode .....	173
Definition 10.39	(Semi-)robust Communication System .....	175
Definition 10.41	Congruent Modes .....	177
Definition 10.43	Standard Mode .....	177



Definition 11.1	Perfect/Imperfect Triadic Consonance .....	182
Definition 11.4	Relative Major/Minor Keys .....	183
Definition 11.6	Parallel Major/Minor Keys .....	184
Definition 14.1	Heptad .....	238
Definition 14.4	Depth (of a heptad) .....	239
Definition 14.6	Degree Interval .....	239
Definition 14.7	Degree-Interval Set .....	239
Definition 14.9	Saturation (of a heptad) .....	239
Definition 14.10	Signature (of a heptad) .....	240
Definition 14.13	Heptad Type Equivalence; Heptad Type Class .....	240
Definition 14.18	Harmonic Degree .....	244
Definition 14.20	Quality of a Harmonic Degree; Standard Quality; Major/Minor Quality .....	246
Definition 14.21	Tendency Degree (Melodic) .....	246
Definition 14.22	Tonal Heptads .....	246
Definition 14.24	Major/Minor Scale; Complete Scale; Major/Minor Tonality .....	247
Definition 14.27	Tonal Communication System .....	249
Definition 15.1	Cue-Cell; Minimal Cue-Cell .....	253
Definition A1	Cartesian Product .....	265
Definition A3	Relation .....	265
Definition A5	Function .....	265
Definition A8	Surjection .....	266
Definition A9	Injection (One-to-One Function) .....	266
Definition A11	Bijection .....	266
Definition A13	Induced Function .....	267
Definition A15	Semigroup .....	267
Definition A16	Group; Abelian Group .....	267
Definition A18	Homomorphism .....	268
Definition B1	Ring .....	269
Definition B3	Module .....	269
Definition B5	Module Homomorphism .....	270

# List of Notation

$X(a) = b$ .....	28
$N$ ; $NVL$ .....	31
$NI$ ; $NT$ .....	32
$NVL$ .....	33
$NS$ .....	33
$(a, b)$ ; $\equiv (a, b)$ ; $\neq (a, b)$ .....	33
$\overset{(a, b)}{ONS}$ ; $ONS$ .....	34
$((g_1, h_1), (g_2, h_2))$ .....	35
$P$ ; $PVL$ .....	37
$PI$ ; $PT$ .....	38
$PVL$ .....	38
$PS$ .....	38
$M_\beta^\alpha$ , $PCS_\beta^\alpha$ .....	39
$f$ .....	40
$\omega(f)$ .....	41
$\Omega$ ; $P(\Omega)$ ; $T^f$ .....	41
$R_\sigma^\rho$ .....	43
$\psi$ .....	45
$REG(p)$ .....	46
$PCS_\varphi^\psi$ .....	47
$T = \{T_1, T_2\}$ .....	51
$MSG$ ; $\overset{(a, b)}{CTS}_\varphi^\psi$ .....	55
$REG(s, t)$ .....	55
$CTS_\varphi$ .....	56
$RVL$ .....	61
$(c, d)$ ; $\overset{(c, d)}{RVL}$ ; $(y_k, z_k)$ ; $(y, z)_k$ .....	68
$\overset{(a, b; c, d)}{FTS}_\varphi^\psi$ .....	69

$(c, d)$ $RVL / oct$ .....	69
$(a, b; c, d)$ $(a, b)$ $FTS_\varphi$ ; $CTS_\varphi$ .....	73
$FTS_\varphi$ .....	74
$Q_j$ ; $Q_j^i$ .....	78
$CET = \{CET_1, CET_2\}$ .....	79
$PCS_\varphi$ .....	80
$(w, x)$ $IMG (s, t)$ .....	87
$CORE$ ; $MSG/R$ ; $(a, b; c, d)$ $NRS_\varphi$ .....	88
$NRS_\varphi$ .....	88
$CLUST$ .....	89
$(a, b; c, d)$ $(a, b; c, d)$ $NRS_\varphi^i$ ; $DTS_\varphi$ .....	90
$CORE^K$ ; $(a, b; c, d)$ $DTS_\varphi^K$ .....	98
$DTS_\varphi^K$ .....	98
$I/CORE^K$ .....	99
$N_\beta$ ; $NVL_\beta$ .....	107
$GI$ ; $GT$ .....	107
$GNS_\beta$ .....	108
$K(r, m)$ .....	108
$KC(\omega, m)$ .....	109
$U^e$ ; $GKS_\beta^e$ .....	110
$GKS$ ; $GKS'$ .....	110
$C(K)$ .....	114
$KP$ ; $CP$ .....	115
$(a, b; c, d; e)$ $HS_\varphi^K$ .....	116
$HS_\varphi^K$ .....	116
$LP$ ; $VL$ .....	118
$(a, b; i; j)$ $CME$ .....	137
$(b, m; k)$ $GME$ .....	137
$3S_\varphi^K$ ; $2S_\varphi^K$ .....	147

$V_n$ .....	148
$\sqrt{S}$ .....	150
$ALT((w, x), \#)$ .....	151
$S_\alpha; nNU$ .....	158
$SCORE; nM_\alpha^\mu$ .....	158
$nMC_\alpha^\mu$ .....	159
$nTC^\mu$ .....	160
$^\circ(s, t); ^\circ CLUST^\mu$ .....	168
$^\circ CLUST^\mu/d; ^\circ CLUST^\mu/c$ .....	168
$IQ_j^i$ .....	168
$+K_{\alpha^+}; -K_{\alpha^-}$ .....	183
$HK$ .....	230
$nH^\mu$ .....	238
$\Sigma(H)$ .....	240
$+H; -H$ .....	243
$HD$ .....	246
$QC^+; QC^-$ .....	253

# Chapter 1

## Proto-tonal Theory: Tapping into Ninth-Century Insights

**Abstract** The ninth-century treatise *Scolica enchiriadis* (SE) offers two notions of “interval,” namely ratio (proportion) and step distance. The latter notion entails a “generic” distance (cf. “fifth”); however, suggestive diagrams clarify that a “specific” distance is assumed as well (cf. “perfect fifth”). SE raises the question, how to pair step distances such as perfect octave (*diapason*), perfect fifth (*diapente*), and perfect fourth (*diatessaron*), with ratios such as 2:1, 3:2, and 4:3, respectively. In answer, SE departs from the Boethian tradition whereby the distinction between say, duple (2:1) and *diapason*, is merely terminological. Moreover, SE points out that multiplication of ratios corresponds to *addition* of step distances, in a manner to which a modern-day mathematician would apply the term *homomorphism*. Even though the “daseian” tone system proposed in SE (and the “sister” treatise *Musica enchiriadis*) was discarded already in the middle ages, the SE insights into “proto-tonal” theory, the background system of tones prior to the selection of a central tone or “final,” are still relevant.

Well into the third part of *Scolica enchiriadis*, the second of a well-known pair of Carolingian treatises dated by some scholars as early as 850 CE, the disciple asks the master the following question:

Although it has been sufficiently shown that the principle of commensurability joins musical pitches (*voces*) to one another, how nevertheless can one know to which proportion any symphony must be assigned? For how is it known that the diapason must be assigned to the duple relationship, the diapente to the sescupule, the diatessaron to the epitritus, the diapente-plus-diapason to the triple, [and] the disdiapason to the quadruple?<sup>1</sup>

---

This chapter is reprinted by permission from *Music Theory Spectrum* 35(1), pp. 103–110, with minor adaptations. Copyright © 2013 Society for Music Theory.

<sup>1</sup>Erickson (1995), p. 79. All excerpts from *Musica enchiriadis* (ME) and *Scolica enchiriadis* (SE) cited henceforth are in Erickson’s translation. Unless otherwise noted, all excerpts from Boethius’s *De institutione musica* (IM) are in Bower’s translation (Boethius 1989). All excerpts

The question, which appears under the title “How it may be known by what proportion any symphony is formed” and to which I shall refer henceforth as Q, follows a detailed exposé (based on Boethius’s *De institutione arithmetica*) of the various types of numerical proportions, including the “commensurable” and “connumerable,” that is, the multiple proportion (e.g., 2:1, 3:1) and the superparticular (3:2, 4:3). The purpose of this mathematical discussion is to answer a question concerning the “symphonies,” that is, the perfect octave, fifth, and fourth (diapason, diapente, and diatessaron), and their compounds, a question posed about halfway through the second part of the treatise (p. 64): “why at some intervals the voices are consonant, whereas in others they are either discordant or not as agreeable?”<sup>2</sup>

It would seem that by rationalizing the commensurable and connumerable proportions the question of consonance has been settled. As the master notes (p. 68), “. . . the symphonies at the diapason and disdiapason are more perfect than those at the diatessaron and diapente, because the former are of multiple inequality, the latter of superparticular inequality. For multiple inequality is more perfect than superparticular inequality.” However, as Q makes clear, for the disciple in SE this explanation is at best incomplete. This is because the master has given no reason for *pairing* the symphonies with the privileged proportions, in particular, the perfect octave with 2:1 (duple), the perfect fifth with 3:2 (sescuple or sesquialter), and the perfect fourth with 4:3 (epitritus or sesquitercian). Indeed, in answering the question “what is a symphony” at the beginning of the treatise’s second part, “Concerning the Symphonies” (p. 53), *the master makes no mention whatsoever of numerical proportions*, a topic introduced later in the second part, but treated in detail only in the third. Rather, a symphony (“an agreeable combination of certain pitches”) is characterized by *the number of steps* it contains: the diapason eight, the diapente five, and the diatessaron four (counting both extremes in each case). As the master explains,

*diapason* is Greek and in Latin is translated “through all,” because the ancient kithara contained only eight strings. . . *diapente* means “through five,” because it comprehends five pitches. *Diatessaron* is translated “through four,” because it encloses four pitches.

Henceforth in this chapter the terms “type-1 interval” and “type-2 interval” shall be used, respectively, to refer to these two senses of “musical interval” that SE provides, namely, “step distance between two tones” and numerical proportion. The strictly “generic” characterization of type-1 intervals, to borrow Clough and Myerson’s (1985) useful term, is misleading. Certain SE diagrams to be discussed shortly (though some appear early on in the treatise) strongly suggest that a type-1 interval is in fact *a pair* of distances, one generic and the other “semitonal” or,

---

from Ptolemy’s *Harmonics* are in Solomon’s (2000) translation. Special thanks to Oliver Wiener for granting permission to use his Dasia font.

<sup>2</sup>In the course of the mathematical discussion the author presents a diagram that represents the musically most important proportions in terms of the integers 6, 8, 9, 12, 16, 18, and 24. For an interesting interpretation of this diagram see Carey and Clampitt (1996).