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László Lovász · Imre Z. Ruzsa · Vera T. Sós
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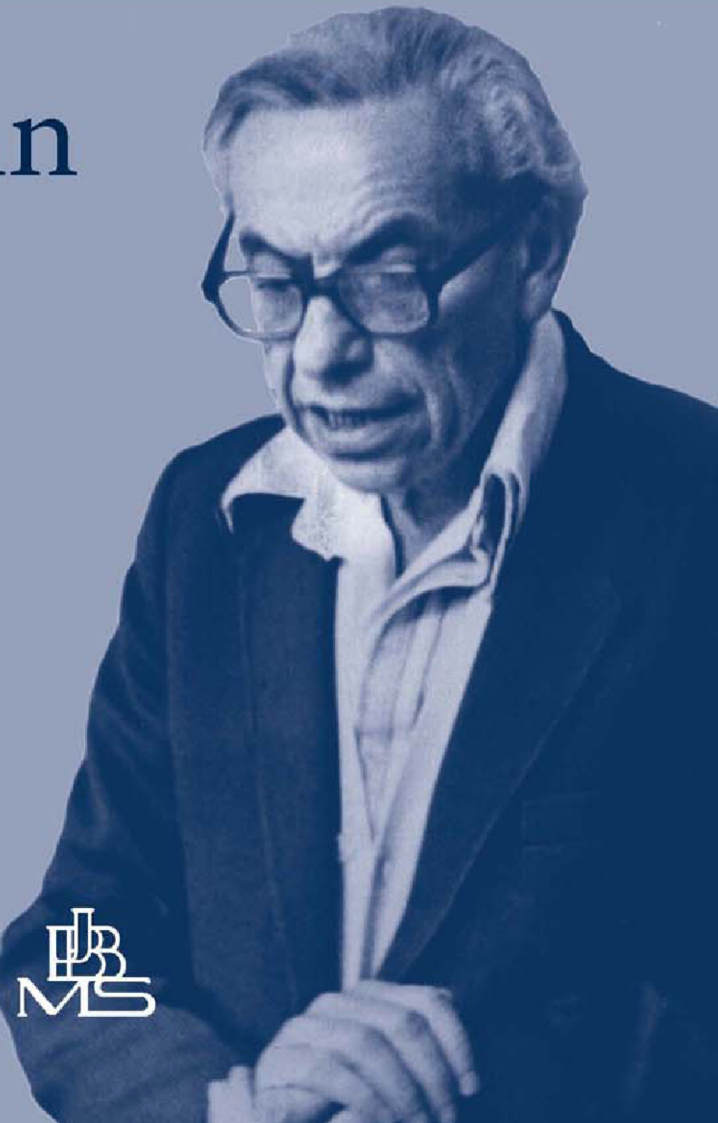

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Paul Erdős 1913–1996

Paul Erdös

Paul Erdős 1913–1996

Preface

Paul Erdős was one of the most influential mathematicians of the twentieth century. His work in number theory, combinatorics, set theory, and other branches of mathematics has determined the development in large areas of these fields. His name is forever attached to combinatorial and additive number theory, combinatorial geometry, extremal graph and hypergraph theory, random graphs, and the probabilistic method. His contributions to set theory, the theory of primes, analysis, probability, and other classical areas in mathematics are also fundamental.

Paul Erdős passed away in 1996. Three years later, a conference was organized in Budapest to survey his work, his contributions to mathematics, and the far-reaching impact of his work on many branches of mathematics. A 2-volume collection of papers, "Paul Erdős and his Mathematics" (János Bolyai Mathematical Society and Springer-Verlag 2002), was also published, which contained papers about his life, surveys of areas which he initiated or contributed to, and personal reminiscences by his friends and collaborators.

We feel that in 2013, on the 100th anniversary of his birth, it was time to have another look on the long-term impact of his work. We are organizing another conference devoted to his mathematics. This volume (which is not the Proceedings of this conference, but of course having the similar goals) undertakes the almost impossible task to describe the ways in which problems raised by him and topics initiated by him (indeed, whole branches of mathematics) continue to flourish.

Written by outstanding researchers in these areas, the papers in this volume include extensive surveys of classical results as well as of new developments. It would be even more hopeless to be comprehensive than in 1999, but we hope that this volume, as well as the lectures at the conference, will give a glimpse into how his mind was working, and a feeling for his tremendous influence on modern mathematics.

The interested reader should also consult the home page of the conference (`http://www.renyi.hu/erdos100`), which contains more material, including the program and abstracts of posters submitted to the conference. We plan that recordings of plenary talks will also be made available. The Paul Erdős page (`http://www.renyi.hu/~p_erdos`) contains scanned copies of most Erdős papers, along with many photos and a lot of other material.

Our thanks are due to Dömötör Pálvölgyi for his very careful and efficient work as managing editor of this volume, to Dezső Miklós for organizing the production, and to Ildikó Miklós for the expert production of the LaTeX files.

Budapest, May 2013

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Paul Erdős and Probabilistic Reasoning

NOGA ALON*

One of the major contributions of Paul Erdős is the development of the Probabilistic Method and its applications in Combinatorics, Graph Theory, Additive Number Theory and Combinatorial Geometry. This short paper describes some of the beautiful applications of the method, focusing on the long-term impact of the work, questions and results of Erdős. This is mostly a survey, but it contains a few novel results as well.

1. The Probabilistic Method

The Probabilistic Method is one of the most significant contributions of Paul Erdős, and part of his greatness is the fact that applications of the probabilistic method and of random graphs have become so common that it is now possible to use those without explicitly mentioning him. The method is a powerful tool with numerous applications in Combinatorics, Graph theory, Additive Number Theory and Geometry and had an immense impact on the development of theoretical Computer Science as well. The results and tools are far too numerous to cover in a short survey, even if the focus is only on those influenced directly by the work and problems of Erdős, and thus this paper is mainly a selection of topics that illustrate the method, and is not meant to be a comprehensive treatment of the whole area. Several books that contain more material on the subject are [13], [18], [54], [60].

It is convenient to classify the applications of probabilistic techniques in Discrete Mathematics into three groups. The first one deals with the study of random combinatorial objects, like random graphs or random matrices. The results here are essentially results in Probability Theory,

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although many of them are motivated by problems in Combinatorics. The second group consists of probabilistic constructions. These are applications of probabilistic arguments in order to prove the existence of combinatorial structures which satisfy a list of prescribed properties. Existence proofs of this type often supply extremal examples to various questions in Discrete Mathematics. The third group, which contains some of the most striking examples, focuses on the application of probabilistic reasoning in the proofs of deterministic statements whose formulation does not give any indication that randomness may be helpful in their study.

Random graphs are covered in another chapter of this volume. The present chapter contains a brief description of several results in each of the other two groups, as well as a very brief discussion of some of the applications of the probabilistic method in theoretical Computer Science. The influence of the work and questions of Paul Erdős in all these has been crucial.

This is mostly a survey paper, but it contains several new results, presented in subsections 3.2 and 3.5, as well.

2. Probabilistic constructions

The applications of probabilistic constructions have yielded numerous results in Combinatorics, Graph Theory, Combinatorial Geometry and Additive Number Theory. Below is a selection of several representative examples.

2.1. Ramsey Numbers

Let $H_1, H_2, \dots, H_k$ be k finite, undirected, simple graphs. The (multicolor) *Ramsey number*

$$r(H_1, H_2, \dots, H_k)$$

is the minimum integer r such that in every edge coloring of the complete graph on r vertices by k colors, there is a monochromatic copy of H_i in color i for some $1 \leq i \leq k$. By a (special case of) a well known theorem of Ramsey (c.f., e.g., [49]), this number is finite for every sequence of graphs H_i.

The determination or estimation of these numbers is usually a very difficult problem. When each graph H_i is a complete graph with more than two vertices, the only values that are known precisely are those of $r(K_3, K_m)$ for $m \leq 9$, $r(K_4, K_4)$, $r(K_4, K_5)$ and $r(K_3, K_3, K_3)$. Even the determination of the asymptotic behavior of Ramsey numbers up to a constant factor is a hard problem, and despite a lot of efforts by various researchers (see, e.g., [49], [22] and their references), there are only a few infinite families of graphs for which this behavior is known.

In one of the first applications of the probabilistic method in Combinatorics, Erdős [26] proved that if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ then $R(K_k, K_k) > n$, that is, there exists a 2-coloring of the edges of the complete graph on n vertices containing no monochromatic clique of size k. This implies that $R(K_k, K_k) > 2^{k/2}$ for all $k \geq 3$. The proof is extremely short: the probability that a random two-edge coloring of K_n contains a monochromatic copy of K_k is at most $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, and hence there is a coloring with the required property.

It is worth noting that although this argument seems almost trivial today, it was far from being obvious when published in 1947. In fact, several prominent researchers believed, before the publication of this short paper, that $R(K_k, K_k)$ may well be bounded by a polynomial in k. In particular, Paul Turán writes in [67] that he had conjectured for a while that $R(K_k, K_k)$ is roughly k^2, and that Erdős's result showed that this quantity behaves very differently than expected.

A particularly interesting example of an infinite family for which the asymptotic behavior of the Ramsey number is known, is the following result of Kim and of Ajtai, Komlós and Szemerédi.

Theorem 2.1 ([56], [3]). *There are two absolute positive constants c_1, c_2 such that*

$$c_1 m^2/\log m \leq r(K_3, K_m) \leq c_2 m^2/\log m$$

for all $m > 1$.

The upper bound, proved in [3], is probabilistic, and applies a certain random greedy algorithm. There are several subsequent proofs, all are based on probabilistic arguments. The lower bound is proved by a "semi-random" construction and proceeds in stages. The detailed analysis is subtle, and is based on certain large deviation inequalities. An alternative analysis of this probabilistic construction, inspired by the differential equation method of Wormald [71], is given by Bohman in [17]. It is worth noting that the question of obtaining a super-linear lower bound for $r(K_3, K_m)$ is mentioned already in [26], and Erdős has established in [28], by an appropriate probabilistic construction, an $\Omega(m^2/\log^2 m)$ lower bound. More on this appears in another chapter of this volume.

Even less is known about the asymptotic behavior of multicolor Ramsey numbers, that is, Ramsey numbers with at least 3 colors. The asymptotic behavior of $r(K_3, K_3, K_m)$, for example, has been very poorly understood for quite some time, and Erdős and Sós conjectured in 1979 (c.f., e.g., [22]) that

$$\lim_{m \mapsto \infty} \frac{r(K_3, K_3, K_m)}{r(K_3, K_m)} = \infty.$$

This has been proved in [12], where it is shown that in fact $r(K_3, K_3, K_m)$ is equal, up to logarithmic factors, to m^3. A more complicated, related result proved in [12], that supplies the asymptotic behavior of infinitely many families of Ramsey numbers up to a constant factor is the following.

Theorem 2.2. *For every $t > 1$ and $s \geq (t-1)! + 1$ there are two positive constants c_1, c_2 such that for every $m > 1$*

$$c_1 \frac{m^t}{\log^t m} \leq r(K_{t,s}, K_{t,s}, K_{t,s}, K_m) \leq c_2 \frac{m^t}{\log^t m},$$

where $K_{t,s}$ is the complete bipartite graph with t vertices in one color class and s vertices in the other.

The proof of the lower bound forms yet another example of a probabilistic construction, where each of the first three color classes is a randomly shifted copy of an appropriate $K_{t,s}$-free graph that contains a relatively small number of large independent sets, as shown by combining some spectral techniques with character sum estimates.

2.2. Combinatorial Geometry

There are several striking examples where a probabilistic construction supplies rather easily counter-examples to well studied conjectures in Combinatorial Geometry. The following result of Erdős and Füredi illustrates this point.

Theorem 2.3 ([34]). *For every $d \geq 1$ there is a set of at least $\left\lfloor \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^d \right\rfloor$ points in the d-dimensional Euclidean space R^d, such that all angles determined by three points from the set are strictly less than $\pi/2$.*

The proof is obtained by considering a random set of binary vectors in R^d. We omit the details but mention that this disproves an old conjecture of Danzer and Grünbaum [23] which suggests that the maximum cardinality of such a set is at most $2d-1$. The authors of [23] did prove, motivated by a question of Erdős and Klee, that the maximum cardinality of a set of points in R^d in which all angles are at most $\pi/2$ is 2^d.

A *range space* S is a pair (X, R), where X is a (finite or infinite) set and R is a (finite or infinite) family of subsets of X. The members of X are called *points* and those of R are called *ranges.* If A is a subset of X then $P_R(A) = \{r \cap A : r \in R\}$ is the *projection* of R on A. In case this projection contains all subsets of A we say that A is *shattered.* The *Vapnik-Chervonenkis* dimension (or VC-dimension) of S, denoted by $VC(S)$, is the maximum cardinality of a shattered subset of X. If there are arbitrarily large shattered subsets then $VC(S) = \infty$.

A subset $N \subset A$ is an *ε-net* for A if any range $r \in R$ satisfying $|r \cap A| \geq \varepsilon|A|$ contains at least one point of N.

A well known result of Haussler and Welzl [52], following earlier work of Vapnik and Chervonenkis [68], asserts that for any n and $\varepsilon > 0$, any set of size n in a range space of VC-dimension d contains an ε-net of size at most $O\big(\frac{d}{\varepsilon}\log(1/\varepsilon)\big)$.

The authors of [61] asked in 1990 whether or not in all natural geometric scenarios of bounded VC-dimension, there always exists an ε-net of size $O(1/\varepsilon)$. This problem received a considerable amount of attention over the years, until it has finally been answered negatively in [5] and in [62], by constructions that have essential probabilistic ingredients. The following, however, is still open.

Problem 2.4. Are there sets X_n of points in the plane and a sequence $\varepsilon_n > 0$ tending to zero so that the minimum size of an ε_n-net for X_n with respect to line ranges is $\Omega(\frac{1}{\varepsilon_n}\log(\frac{1}{\varepsilon_n}))$?

2.3. Additive Number Theory

Erdős and Turán [41] asked if for any asymptotic basis of order 2 of the positive integers (that is, a set A of positive integers so that each sufficiently large integer has a representation as a sum of two elements of A), there must be, for any constant t, integers that have more than t such representations.

Erdős has used in [27] a probabilistic construction to prove the existence of a set A of integers such that every n is represented as $n = x + y$ with $x, y \in A$ at least once but at most $O(\ln n)$ times. This settles a problem posed by Sidon and shows that in the Erdős-Turán question mentioned above one cannot expect to necessarily have too many representations of an integer n, although the question, as posed, is still wide open.

A somewhat similar question is considered by Canfield and Wilf in [21] and by Ljujić and Nathanson in [59]. For two sets A and M of positive integers and for a positive integer n, let $p(n, A, M)$ denote the number of partitions of n with parts in A and multiplicities in M, that is, the number of representations of n in the form $n = \sum_{a \in A} m_a a$ where $m_a \in M \cup \{0\}$ for all a, and all numbers m_a but finitely many are 0. There are simple examples of M and A in which M is finite so that $p(n, A, M) = 1$ for all n, but it seems more difficult to find infinite sets A and M for which $p(n, A, M)$ has a polynomial growth in n. For the specific cases of $A = \{k!\}_{k=1}^{\infty}$, $A = \{k^k\}_{k=1}^{\infty}$ (and many other cases), the existence of such an infinite M is proved in

[6] using a probabilistic construction and answering questions raised in [21] and [59]. These constructions are tailored to fit the growth of the given sequence A, and are general enough to ensure that the same sequence M can work simultaneously for several sequences A. The analysis is based on some large deviation inequalities.

Erdős and Newman studied in [39] another problem dealing with bases for sets of integers. They studied bases for m-element subsets A of $\{1, \ldots, n\}$, where a set B is a basis for A if $A \subset B + B = \{b_1 + b_2 : b_1, b_2 \in B\}$. Since $\{0\} \cup A$ is a basis for A, and there is a set X with at most $c\sqrt{n}$ elements such that $X + X \supset \{1, \ldots, n\}$ it follows that for any m-element subset of $\{1, \ldots, n\}$ there is always a basis of size $\min(c\sqrt{n}, m + 1)$. Erdős and Newman showed by a simple probabilistic construction that if m is somewhat smaller than $\sqrt{n}$, say $m = O(n^{1/2-\varepsilon})$, then almost no m-element set has a basis of size $o(m)$. Similarly, if m is at least $n^{1/2+\varepsilon}$ then almost all m-element sets require a basis of size at least $c\sqrt{n}$. For the borderline case when m is of the order $\sqrt{n}$ their counting argument only yields existence of sets that need a basis of size $c\sqrt{n} \log\log n/ \log n$, and they asked if every m-set of size $m = \sqrt{n}$ has a basis with $o(m)$ elements. This is established in [7], where it is shown that in fact any such set has a basis of size $O(\sqrt{n} \log\log n/ \log n)$. The argument is probabilistic.

Estimating the size of the smallest possible basis for explicitly given sets is often far harder. Erdős and Newman showed that any basis for the set of squares $\{t^2 : t = 1, \ldots, n\}$ (which is a subset of $\{1, 2, \ldots, n^2\}$) is of size at least $n^{2/3-o(1)}$ for large values of n, which is an improvement over the trivial lower bound of $n^{1/2}$. They constructed a small basis for the squares, of size only $O\big(\frac{n}{\log^M n}\big)$ for any M. Wooley asked about powers other than the squares. Whereas it is likely that any basis for the set of d-th powers $\{t^d : t = 1, \ldots, n\}$ is of size $\Omega(n^{1-\varepsilon})$ for every $\varepsilon > 0$ and $d \geq 2$, only a modest improvement of the $n^{2/3-o(1)}$ lower bound of Erdős and Newman for large values of d is proved in [7], where it is shown that the set $\{t^d : t = 1, \ldots, n\}$ does not have a basis of size $O\Big(n^{3/4-\frac{1}{2\sqrt{d}}-\frac{1}{2(d-1)}-\varepsilon}\Big)$ for any $\varepsilon > 0$.

3. Deterministic Theorems

3.1. Sum-free subsets

A subset A of an abelian group is called sum-free if there is no solution to the equation $x+y=z$ with $x,y,z \in A$. Erdős [31] showed that any set of n positive integers contains a sum-free subset of size at least $n/3$. The proof is a simple yet intriguing application of the probabilistic method, and proceeds as follows. Let A be a set of n positive integers, choose a real x uniformly between 0 and 1 and let $B=B_x$ be the set of all $a \in A$ so that $ax \bmod 1 \in (1/3, 2/3)$. It is not difficult to check that B is always sum-free, and that the expected value of the size $|B_x|$ of B is $n/3$. Therefore, there is a fixed x so that the size of B_x is at least $n/3$, providing the required result.

In [8] the authors showed that a similar proof gives a lower bound of $(n+1)/3$. Bourgain [20] has further improved this estimate to $(n+2)/3$. It seems possible that the constant $1/3$ cannot be replaced by a larger constant, but this is an open problem. The best known upper bound is $11/28$, proved by Lewko [58], improving earlier estimates of $3/7$ in [31] and $12/29$ in [8]. In subsection 3.2 we present a further (modest) improvement. It is worth noting that for general abelian groups there is a similar result proved in [8]: any set of n nonzero elements in any abelian group contains a sum-free subset with more than $2n/7$ elements. The constant $2/7$ is best possible.

3.2. The sum-free subset constant

For a set B of nonzero integers, let $s(B)$ denote the maximum cardinality of a sum-free subset of B. The infimum value of the ratio $\frac{s(B)}{|B|}$ as B ranges over all nonempty sets of nonzero integers is called the sum-free subset constant, and is denoted by δ. As mentioned in the previous subsection Erdős proved that $\delta \geq 1/3$ and observed that $\delta \leq 3/7$. The upper bound has been improved in [8] and further improved in [58]. All these upper bounds are established by exhibiting a set B and by computing $s(B)$. The next statement shows that for any given example B it is possible to construct another one which gives a (slightly) better upper bound for δ.

Proposition 3.1. *Let B be a finite set of b nonzero integers and define $s=s(B)$. Put*

$$p=[b(b-1)+1](b-s+1), \quad q=\lceil p!(e-e^{-1}+3)/2\rceil - p+2$$

and

$$m=\left\lceil \frac{q}{b(b-1)+1} \right\rceil b.$$

Then there is a set C of at most m elements so that

$$\frac{s(C)}{|C|} \leq \frac{s(B)}{|B|} - \frac{1}{|C|}.$$

The result of [58] is proved by exhibiting an explicit set B of 28 nonzero integers for which $s(B) = 11$. Therefore $\delta \leq 11/28$. By the proposition above this can be improved to $11/28 - \varepsilon$ for some ε which is roughly $10^{-50,000}$. It is possible to get a slightly bigger value of ε, but as this is certainly far from giving a tight bound, we make no serious attempt to optimize this value here. Note that the proposition above implies that δ is an infimum, and not a minimum, that is, there is no finite set B so that $\delta = \frac{s(B)}{|B|}$.

Proof. Put $|B| = b,\ s = s(B)$. Let n be a large integer, to be chosen later, and let G be the graph whose set of vertices is $\{1, 2, \ldots, n\}$, where i and j are adjacent iff the two sets iB and jB intersect (and $i \neq j$). It is clear that the maximum degree of this graph is at most $b(b-1)$ and hence, by the Hajnal-Szemerédi Theorem [51], it has a proper coloring f with $k = b(b-1)+1$ colors and nearly equal color classes. This coloring provides a partition of $[n] = \{1, 2, \ldots, n\}$ into k sets I_j, so that each of the set $B_j = \cup_{i \in I_j} iB$ is a set of exactly $|I_j|b$ nonzero integers.

Claim: If n is sufficiently large then at least one of these sets B_j does not contain a sum-free subset containing s elements from each of the sets iB for all $i \in I_j$.

Indeed, assuming this is not the case, fix a sum-free subset A_j in each B_j so that $|A_j \cap iB| = s$ for all $i \in I_j$. Using the sets A_j, define a coloring g of I_j by $b - s + 1$ colors as follows. Let $x_1 < x_2 < \ldots < x_b$ be the members of B and suppose $i \in I_j$. By assumption A_j contains at least one of the elements ix_q for some $q \in \{1, 2, \ldots, b - s + 1\}$. Let q be the smallest index for which this holds and define $g(i) = q$. The ordered pair $(f(i), g(i))$ defines a coloring of the integers in $[n]$ by $k(b - s + 1) = [b(b-1)+1](b - s + 1)$ colors.

Note that there is no monochromatic Schur triple in this coloring, that is, there are no $i, j, t \in [n]$ so that $i + j = t$ and $(f(i), g(i)) = (f(j), g(j)) = (f(t), g(t))$. This is because if there is such a triple then for $(f', g') = (f(i), g(i))$ we have $iB \cup jB \cup tB \subset B_{f'}$, and for $x_{g'} \in B \ \ ix_{g'}, jx_{g'}, tx_{g'}$ all lie in $A_{f'}$. This contradicts the fact that $A_{f'}$ is sum-free, as $ix_{g'} + jx_{g'} = tx_{g'}$. Thus there are indeed no monochromatic Schur triples.

An old Theorem of Schur (c.f., e.g., [49]) asserts that if n is sufficiently large as a function of the number of colors used then there must be a monochromatic Schur triple, contradiction. This contradiction proves the assertion of the claim.

Returning to the proof of the proposition, note that the number of colors in the construction above is $p = [b(b-1)+1](b-s+1)$. By [70] if n is at least $q = \lceil p!(e - e^{-1}+3)/2\rceil - p + 2$ then there is a monochromatic Schur triple. This implies that if indeed n is at least that large, then at least one of the sets B_j cannot contain a sum-free subset that consists of s elements from each iB for $i \in I_j$. Hence $s(B_j) \leq |I_j|s - 1$ and as the size of each set I_j is at most $\lceil \frac{q}{b(b-1)+1} \rceil$ the set $C = B_j$ completes the proof of the proposition. ■

3.3. List coloring and Euclidean Ramsey Theory

The *list chromatic number* (or *choice number*) $\chi_\ell(G)$ of a graph $G = (V, E)$ is the minimum integer s such that for every assignment of a list L_v of s colors to each vertex v of G, there is a proper vertex coloring of G in which the color of each vertex is in its list. This notion was introduced independently by Vizing in [69] and by Erdős, Rubin and Taylor in [40]. In both papers the authors realized that this is a variant of usual coloring that exhibits several new interesting properties, and that in general $\chi_\ell(G)$, which is always at least as large as the chromatic number of G, may be arbitrarily large even for graphs G of chromatic number 2.

It is natural to extend the notion of list coloring to hypergraphs. The list chromatic number $\chi_\ell(H)$ of a hypergraph H is the minimum integer s such that for every assignment of a list of s colors to each vertex of H, there is a vertex coloring of H assigning to each vertex a color from its list, with no monochromatic edges.

An intriguing property of list coloring of graphs, which is not shared by ordinary vertex coloring, is the fact that the list chromatic number of any (simple) graph with a large average degree is large. Indeed, it is shown in [4] that the list chromatic number of any graph with average degree d is at least $(\frac{1}{2} - o(1))\log_2 d$, where the $o(1)$-term tends to zero as d tends to infinity. For $r \geq 3$, simple examples show that there is no nontrivial lower bound on the list chromatic number of an r-graph in terms of its average degree. However, such a result does hold for simple hypergraphs. Recall that a hypergraph is *simple* if every two of its distinct edges share at most one vertex. The following result is proved in [10].

Theorem 3.2. *For every fixed $r \geq 2$ and $s \geq 2$, there is a $d = d(r, s)$, such that the list chromatic number of any simple r-graph with n vertices and nd edges is greater than s.*

A similar result for the special case of d-regular 3-uniform simple hypergraphs has been obtained independently in [53]. A subsequent proof with a better upper estimate for $d(r, s)$ appears in a recent paper of Saxton and Thomason [66].

The proof of the theorem is probabilistic and proceeds by induction on r. For simplicity we only outline the idea for the case of graphs with a large minimum degree. Let $G = (V, E)$ be a graph with n vertices and minimum degree d. Choose a random set B of about $n/\sqrt{d}$ vertices and assign a random list of size s out of a set S of $2s - 1$ colors to each vertex of B. A simple computation shows that if, say, $d > 10^s$, then with positive (and in fact high) probability many of the vertices v not in B have every subset of size s of S assigned to at least one of their B-neighbors. Fix such a choice of the set B and lists of colors to its vertices. Note now that for each fixed choice of a coloring f of the vertices of B from their lists, at least s distinct colors appear on the B-neighbors of any vertex v of the type mentioned above. If we now assign a random list to such a vertex v, then with probability at least $\binom{2s-1}{s}^{-1} > 4^{-s}$ it will be a forbidden list, that is, it will consist only of colors assigned by f to its neighbors, showing that the coloring f of the B vertices cannot be extended to a proper list coloring of the whole graph. There are only $s^{|B|}$ possible colorings of the vertices of B from their lists, and the probability that no vertex v gets a forbidden list is small enough to ensure that this will not happen for any of these colorings. This argument suffices to show that the list chromatic number of G exceeds s. The hypergraph case is more complicated, and we do not include it here.

The argument above suggests an interesting algorithmic question: given a graph $G = (V, E)$ with minimum degree $d > 10^s$, can we find, deterministically and efficiently, lists of size s for each $v \in V$ so that there is no proper coloring of G assigning to each vertex a color from its list? This problem is open, as is the simpler NP version of it, that is, that of finding sets S_v and providing a certificate that there is no proper coloring using the lists. Here the sets do not have to be found efficiently, and we only require that one will be able to check the certificate efficiently.

The last theorem has an interesting application in Euclidean Ramsey Theory – yet another subject initiated by Erdős and his collaborators. A well known problem of Hadwiger and Nelson is that of determining the minimum number of colors required to color the points of the Euclidean plane so that no two points at distance 1 have the same color. Hadwiger showed already in 1945 that 7 colors suffice, and Moser and Moser noted in 1961 that 3 colors do not suffice. These bounds have not been improved, despite a considerable amount of effort by various researchers, see [55, pp. 150–152] and the references therein for more on the history of the problem.

A more general problem is considered in [35], [36], [37], where the main question is the investigation of finite point sets K in the Euclidean space for which any coloring of an Euclidean space of dimension d by r colors must contain a monochromatic copy of K. There are lots of intriguing conjectures that appear in these papers. One of them asserts that for any set K of 3 points which do not form an equilateral triangle the minimum number of colors required for coloring the plane with no monochromatic isometric copy of K is 3. The situation is very different for list coloring. A simple Corollary of the theorem above is the following.

Theorem 3.3 ([10]). *For any finite set X in the Euclidean plane and for any positive integer s, there is an assignment of a list of size s to every point of the plane, such that whenever we color the points of the plane from their lists, there is a monochromatic isometric copy of X.*

3.4. Turán numbers and Dependent random choice

For a graph H and an integer n, the Turán number $ex(n, H)$ is the maximum possible number of edges in a simple graph on n vertices that contains no copy of H. The asymptotic behavior of these numbers for graphs H of chromatic number at least 3 is well known, and is determined by the Erdős-Stone-Simonovits Theorem. For bipartite graphs H, however, the situation is considerably more complicated, and there are relatively few nontrivial such graphs H for which the order of magnitude of $ex(n, H)$ is known. A rather general result with a relatively simple proof, described in [11], asserts that for every fixed bipartite graph H in which the degrees of all vertices in one color class are at most r, there is a constant $c = c(H)$ so that $ex(n, H) \leq cn^{2-1/r}$. This is tight for all values of r, as it is known that for every r and $t > (r-1)!$, there is a simple graph with n vertices and at least $c_{r,t}n^{2-1/r}$ edges, containing no copy of the complete bipartite graph $K_{r,t}$.

The basic tool in the proof is a simple and yet surprisingly powerful method, whose probabilistic proof may be called "dependent random choice", as it involves a random selection of a set of vertices, where the choices are dependent in a way that increases the probability that r-tuples of the selected vertices will have many common neighbors. An early version of this lemma has first been proven in [50] and [57], and many variants and extension have been obtained afterwards. See [44] for a survey containing lots of applications in Extremal Graph Theory and in Additive Number Theory.

One of the basic versions of the lemma is the following.

Lemma 3.4 ([11]). *Let a, b, n, r be positive integers. Let $G = (V, E)$ be a graph on $|V| = n$ vertices with average degree $d = 2|E|/n$. If*

$$\frac{d^r}{n^{r-1}} - \binom{n}{r}\left(\frac{b-1}{n}\right)^r > a - 1\,, \tag{1}$$

then G contains a subset A_0 of at least a vertices so that every r vertices of A_0 have at least b common neighbors.

The proof proceeds by considering a (multi)-set T of r random vertices of G, chosen uniformly with repetitions. Let A be the set of all vertices of G which are neighbors of all members of T. The crucial fact is that the expected value of $|A|$ is large, by linearity of expectation and convexity, whereas the expected number of r-tuples of vertices of A with a small number of common neighbors is small, as it is not likely that all vertices of T fall into such a small set of common neighbors. The set A_0 can thus be obtained from A by deleting a vertex from each such undesirable r-tuple.

The lemma above easily implies the following result, that can also be derived from an earlier result of Füredi [47] proved by a different method, in response to a question of Erdős.

Theorem 3.5. *Let H be a bipartite graph with maximum degree r on one side. Then there exists a constant $c = c(H) > 0$ such that*

$$ex(n, H) < cn^{2-\frac{1}{r}}.$$

The method yields several related results, but does not suffice to settle the following problem, suggested by Erdős.

Problem 3.6 ([33]). A graph is *r-degenerate* if every subgraph of it contains a vertex of degree at most r. Is it true that for every fixed r-degenerate bipartite graph H, $ex(n, H) \le O(n^{2-1/r})$?

As shown in [11], the method of dependent random choice with some twists does imply that for each such H on h vertices, $ex(n, H) \le h^{1/2r} n^{2-\frac{1}{4r}}$.

3.5. Hypergraph coloring

Erdős realized already in the 60s that probabilistic methods are powerful in the study of hypergraph coloring problems. Several examples appear in [29], [30], [38]. A k-uniform hypergraph is two-colorable if it has a vertex coloring by two colors so that no edge is monochromatic. In [29], [30] Erdős applies probabilistic arguments to prove that the minimum possible number of edges in a k-hypergraph that is not two-colorable is at least 2^{k-1} and at most $O(k^2 2^k)$. The lower bound has been improved several times, and all the improved proofs apply the probabilistic method. The current record is $\Omega\Big(\sqrt{\frac{k}{\log k}}2^k\Big)$, due to Radhakrishnan and Srinivasan [64]. See also [63] for a weaker $\Omega(k^{1/4}2^k)$ bound, with a beautiful short (probabilistic) proof.

One of the main motivations for proving the Lovász Local Lemma in [38] has also been the study of the minimum possible number of edges of a *simple* k-uniform hypergraph which is not two-colorable.

A recent result of Blais, Weinsein and Yoshida [16] deals with a new intriguing variant of hypergraph coloring. In the rest of this section we describe this notion and present some new results about it.

A hypergraph $\mathcal{F}$ is t-intersecting if the intersection of any two of its edges is of size at least t. A vertex coloring of $\mathcal{F}$ is c-strong if any edge F contains vertices of at least $\min\{|F|, c\}$ colors. Let $\chi(t, c)$ denote the minimum f so that any t-intersecting hypergraph admits a c-strong coloring with at most f colors, (∞ if there is no such f).

This notion is defined in [16] where the authors observe that $\chi(t, c)$ is infinite for all $t \leq c-2$, $\chi(c-1, c) \geq 2c-1$ and that $\chi(t, c) \geq 2c-2$ for all $t \geq c \geq 2$, and prove that $\chi(c, c) < \sqrt{c}e^c$ and that for all $t \geq 2c$, $\chi(t, c) \leq 2c^2$.

They raise several questions regarding the determination of this function, and in particular note that their method does not provide any sub-quadratic (in c) bound for $\chi(t, c)$ for any t, and ask whether or not for each fixed c the limit of $\chi(t, c)$ as t tends to infinity is $2c-2$.

The following theorem nearly settles this question.

Theorem 3.7. *For every fixed $c \geq 2$ there exists a $t_0 = t_0(c)$ $\big(\leq O(c^2)\big)$ so that for all $t > t_0$, $\chi(t, c) \leq 2c-1$.*

The proof follows the basic approach of [16], showing that a random coloring with $2c-1$ colors provides a c-strong coloring with positive probability bounded away from zero. We note that the example of all subsets of cardinality at least $(n+t)/2$ of an n-element set, where $n \gg t^2$, shows that for a random coloring $2c-2$ colors do not suffice, as with high probability the largest $c-1$ color classes will contain more than $(n+t)/2$ elements. A more careful analysis sketched at the end of this section shows that for random colorings with $2c-1$ colors, the $O(c^2)$ estimate for the intersection t is optimal as well.

We need a result about the biased measure of t-intersecting hypergraphs. A sharp version of this result was first proved in [2], and can be deduced from the main result of [1]. See also [14], [24], [46] for subsequent related statements. Here we give a much simpler, self-contained proof of a somewhat weaker estimate that suffices for our purpose.

For a hypergraph $\mathcal{F}$ and a real p, $0 \leq p \leq 1/2$, let $\mu_p(\mathcal{F})$ denote the p-measure of $\mathcal{F}$, that is, the probability that a random set of vertices of $\mathcal{F}$ obtained by selecting each vertex, randomly and independently, with probability p, forms an edge in $\mathcal{F}$. Thus $\mu_p(\mathcal{F}) = \sum_{F \in \mathcal{F}} \mu_p(F)$, where $\mu_p(F) = p^{|F|}(1-p)^{n-|F|}$, and n is the number of vertices of $\mathcal{F}$. It is convenient to formulate the results in terms of escape probabilities of random walks. A p-biased random walk of length n is a sequence of independent, identically distributed random variables $X_1, X_2, \ldots, X_n$ where each X_i is $+1$ with probability p and -1 with probability $1-p$. Put $S_i = \sum_{j=1}^{i} X_j$, let $W(p,t,i)$ be the probability that $S_i \geq t$ and let $W(p,t)$ denote the probability that there exists some i so that $S_i \geq t$.

Associate each subset F of $[n] = \{1, 2, \ldots, n\}$ with an assignment of values to the variables $X_1, X_2, \ldots, X_n$ by defining $X_i = 1$ if $i \in F$ and $X_i = -1$ otherwise. With this assignment, $\mu_p(F)$ is exactly the probability of the corresponding walk.

Let W_i denote the set of all walks for which $S_i \geq t$, and let F_i denote the corresponding family of subsets. It is easy to see that this family is t-intersecting. Indeed, if two sets in the family correspond to the walks $(X_1, X_2, \ldots X_n)$ and $(Y_1, Y_2, \ldots, Y_n)$, then $\sum_{j=1}^{i}(X_j + Y_j) \geq 2t$ and as each term $X_j + Y_j$ lies in $\{-2, 0, 2\}$, at least t of the terms are 2, providing the required intersection. Therefore, for every $i \leq n$ there is a t-intersecting family of subsets of $[n]$ of p-measure at least $W(p,t,i)$. It turns out that the maximum possible p-measure of such a family is exactly $\max_{i \leq n} W(p,t,i)$.

Lemma 3.8 ([2]). *For any t-intersecting hypergraph $\mathcal{F}$ on n vertices and any $p < 1/2$, $\mu_p(\mathcal{F}) \leq \max_{i \leq n} W(p,t,i)$.*

Here we give a simple proof of the following weaker estimate

Lemma 3.9. *For any (finite) t-intersecting hypergraph $\mathcal{F}$ and any $p < 1/2$, $\mu_p(\mathcal{F}) \leq W(p,t)$.*

Proof: We apply shifting, which is a common technique in the area, see, e.g., [45]. Let $[n]$ be the set of vertices of $\mathcal{F}$. For each $1 \leq i <$

$j \leq n$ define an operator S_{ij} on the edges of $\mathcal{F}$, where for each $F \in \mathcal{F}$, $S_{ij}(F) = F - \{j\} \cup \{i\}$ if $j \in F$, $i \notin F$ and $F - \{j\} \cup \{i\} \notin \mathcal{F}$, and $S_{ij}(F) = F$ otherwise. Put $S_{ij}(\mathcal{F}) = \{S_{ij}(F) : F \in \mathcal{F}\}$. Is is easy and well known that if $\mathcal{F}$ is t-intersecting so is $S_{ij}(\mathcal{F})$. It is also clear that $S_{ij}(\mathcal{F})$ has exactly the same p-measure as $\mathcal{F}$. Moreover, if $S_{ij}(\mathcal{F})$ differs from $\mathcal{F}$, then the sum of elements in all edges of $S_{ij}(\mathcal{F})$ is smaller than that of the elements in all edges of $\mathcal{F}$. We can thus keep applying the shift operators S_{ij} to our hypergraph until the process stabilizes, providing a left-shifted family of subsets, which, with a slight abuse of notation, we also denote by $\mathcal{F}$. By the comments above this is still t-intersecting and has the same measure as the original family. The important property of the shifted family is that if it contains an edge F, it also contains every set obtained from F by shifting elements to the left, that is, by replacing some elements of F by smaller elements not in F.

We claim that in the shifted family we cannot have a set corresponding to a walk whose partial sums are all at most $t-1$. This is because if we have such a set, we can show that it intersects some shifted copy of itself by less than t elements, contradiction. Indeed, let F be such a set. Using F, define another set G as follows. Consider the elements of F one by one, in order, starting with the smallest. The first (smallest) $t-1$ elements of F stay in G. Each subsequent element of F in its turn is replaced by the smallest element which is not in F and is also not one of the elements placed already in G. We claim that in this process, every element of F besides the first $t-1$ is replaced by a smaller element (which is not in F). Indeed, otherwise the first time in which the process fails to replace a member of F by a smaller member is some element f_{t-1+i} in F, where the elements of F are listed in increasing order, so that there are only $i-1$ non-elements of F smaller than it. But this means that the random walk corresponding to F has $t-1+i$ times $+1$ and only $i-1$ times -1 up to this point, meaning its value at this point is t, contradicting the assumption. Therefore G is obtained from F by left shifts, and as $\mathcal{F}$ is shifted, G belongs to $\mathcal{F}$ as well. But by construction G intersects F in only $t-1$ elements, contradicting the assumption that $\mathcal{F}$ is t-intersecting.

The claim about the measure follows, completing the proof. ■

We need the following standard estimate for Binomial distributions. See, e.g., [13], Theorem A.1.4.

Lemma 3.10. *Let $Y_i, 1 \leq i \leq n$ be independent identically distributed random variables where each Y_i is $+1$ with probability p and -1 with probability $1-p$, and put $Y = \sum_{i=1}^{n} Y_i$. Then the probability that $Y - E(Y) \geq b$ is at most $e^{-b^2/2n}$.*

Corollary 3.11. *Suppose $c \geq 2$, and put $p = \frac{c-1}{2c-1}$. Then:*

(i) For all t and i, $W(p,t,i) \leq e^{-t/c}$. In particular, if $t \geq 2c^2$ then $W(p,t,i) < e^{-2c}$.

(ii) For all $t \geq 8c^2$, $W(p,t) < e^{-2c}$.

Proof. Part (i) follows by substituting $n = i$, $E(Y) = -\frac{i}{2c-1}$ and $b = t + \frac{i}{2c-1}$ in Lemma 3.10. This gives

$$W(p,t,i) \leq e^{-b^2/(2i)} \leq e^{-4it/[2i(2c-1)]} = e^{-2t/(2c-1)} \leq e^{-t/c},$$

as needed. To prove part (ii) note that if for a random walk $X_1, X_2, X_3, \ldots$ no partial sum $S_{it} = \sum_{j \leq it} X_j$ satisfies

$$S_{it} \geq t/2 \tag{2}$$

then all partial sums S_i stay below t. We can thus bound $W(p,t)$ by the sum of probabilities of the events in (2), which we denote by E_i. By Lemma 3.10 the probability of E_i is at most

$$e^{-(\frac{it}{2c-1}+\frac{t}{2})^2/(2it)} \leq e^{-\frac{(i+c)^2 t}{8c^2 i}}.$$

The right hand side is at most $e^{-t/(2c)}$ for all i, since $(i+c)^2 \geq 4ic$, and it is also at most $e^{-it/(8c^2)}$ for all i. Therefore, for $t \geq 8c^2$, the sum over all $i \geq 1$ is smaller than

$$\sum_{i=1}^{8c^2} e^{-t/(2c)} + \sum_{i>8c^2} e^{-it/(8c^2)} < 8c^2 e^{-t/(2c)} + e^{-t} \tag{3}$$

where the last term is an upper estimate for the infinite geometric series $\sum_{i>8c^2} e^{-it/(8c^2)}$. For $t \geq 8c^2$ (and $c \geq 2$) the quantity in (3) is smaller than e^{-2c}, completing the proof. ∎

Proof of Theorem 3.7. Let $\mathcal{F}$ be a t-intersecting hypergraph, and let $[n]$ be its set of vertices. Add to the hypergraph any subset of $[n]$ that contains a member of $\mathcal{F}$ and note that the modified hypergraph is still t-intersecting and its p-measure $\mu_p(\mathcal{F})$ is precisely the probability that a random subset of $[n]$ obtained by picking each element independently with probability p contains an edge of the hypergraph. Put $p = \frac{c-1}{2c-1}$, and let ε be smaller than $\binom{2c-1}{c-1}^{-1}$. Choose t_0 so that $W(p,t) < \varepsilon$ for all $t > t_0$. Note that by Corollary 3.11, part (ii) $t_0 \leq O(c^2)$. Now color randomly by $2c-1$ colors. The probability there is a set that gets only $c-1$ colors is bounded by $\binom{2c-1}{c-1}\mu_p(\mathcal{F})$, implying the desired result. ∎

Remarks:

- The proof above together with Lemma 3.8 and Corollary 3.11, part (i) shows that the statement of Theorem 3.7 holds with $t_0 = 2c^2$ (with room to spare). Lemma 3.9 and Corollary 3.11, part (ii) provide a simple, self-contained proof that works with a somewhat larger value of t_0 (which is still $O(c^2)$).
- The above argument, with an appropriate choice of parameters, supplies a tradeoff between the number of colors used and the required size of the intersection. In particular it implies, for example, that $\chi(2c, c) \leq O(c)$.
- As mentioned above, if we apply random colorings, both the term $2c - 1$ and the $O(c^2)$ upper estimate for t_0 in Theorem 3.7 are tight. The fact that $2c - 1$ is tight for any fixed t is very simple, as mentioned above. Here is a sketch of the argument that for $2c - 1$ colors the $O(c^2)$ estimate for t is tight. Without making any attempt to optimize the constants, consider the family of all subsets of cardinality at least $n/2 + c^2/10000$ in an n element set $[n]$, where $n = (2c-1)^3/10000$ and c is a large integer. Consider a random coloring of $[n]$ by $2c - 1$ colors. For a fixed color i, the expected number of elements colored i is $n/(2c-1) = (2c-1)^2/10000$ and the variance is $n\frac{1}{2c-1}(1 - \frac{1}{2c-1})$ which is roughly $(2c-1)^2/10000$. Thus, the standard deviation is roughly $(2c-1)/100$. Expose the color classes in order, two at a time, $c - 1$ times, leaving the final color class to the end. It is not difficult to show that for any given history, assuming that at least some $n/2c$ elements are not yet in the color classes exposed (as is the case with high probability) when we expose the next pair of color classes the probability that the difference between their sizes is at least, say, $c/200$, exceeds $1/2$. Thus with high probability we will have at least $c/4$ pairs with difference at least $c/200$. If this is the case, then by picking the larger color class of every pair we will cover at least $c/4 \times c/200 = c^2/800$ more elements than by picking the smaller class in each pair, and as with high probability the last color class is not bigger than $2 \cdot (2c-1)^2/10000 < 8c^2/10000$ these $c - 1$ large color classes will contain, with high probability, a full edge. This shows that t_0 has to be at least $\Omega(c^2)$.
- The study of the random variant of the problem of determining $\chi(t, c)$ seems interesting. This is the problem of determining or estimating the smallest possible $f = f(t, c)$ so that a random vertex coloring of any t-intersecting hypergraph by f colors is c-strong with probability at least, say, 0.1.

  Note that the two functions f and χ differ. Indeed, the function $\chi(t, 2)$ is known for all values of t, as described in [16]. Specifically,

$\chi(0,2) = \infty$, $\chi(1,2) = 3$ and $\chi(t,2) = 2$ for all $t \geq 2$. In contrast, it is easy to see that $f(0,2) = f(1,2) = \infty$. This is because for every fixed number of colors r, a random r-coloring of the vertices of a star with $m > r$ edges will contain a monochromatic edge with probability that tends to 1 as m tends to infinity. (The same argument implies that $f(c-1,c) = \infty$ for all $c > 2$.) The arguments in [16] and here also show that $f(t,2) = 3$ for all $t \geq 2$.
The results here and the earlier ones in [16] show that the function f is somewhat better understood than χ. In particular, we have shown here that for every c and all $t > 2c^2$, $f(t,c) = 2c - 1$.

4. Applications in Theoretical Computer Science

The results and questions of Erdős have not been motivated by applications in Theoretical Computer Science (TCS), and yet the impact of his work on the development of TCS has been substantial. This short section includes some brief comments on this aspect of his work, focusing on applications of probabilistic techniques.

The Probabilistic Method plays a crucial role in the development of randomized algorithms. The quest for explicit constructions advocated time and again by Erdős is one of the early drives for derandomization – the process of converting randomized algorithms into deterministic ones. A specific problem he kept repeating over the years is that of finding explicit constructions of Ramsey graphs - graphs on n vertices in which the largest clique and largest independent set are of size $O(\log n)$, as well as explicit examples providing lower bounds for off-diagonal Ramsey number, like $r(3,n)$ – see [32].

The most successful attempts to find good explicit constructions of Ramsey graphs led to improved constructions of dispersers which are useful for derandomization, see [15]. Moreover, these constructions rely heavily on sum-product theorems initiated in the work of Erdős and Szemerédi [43] (although these are finite field analogs of the Erdős–Szemerédi results).

The method of conditional expectations, which is one of the very basic techniques in derandomization, was initiated in the paper of Erdős and Selfridge that introduced the study of combinatorial games [42].

Another useful technique which we only mention in passing is the Erdős-Rado delta-system (sunflower) method, that appears in work on circuit complexity and on matrix multiplication. A large body of work in Computational Geometry is also motivated by the results and questions of Erdős.

Finally, the area of Graph Property Testing (c.f., e.g., [13], Chapter 17), which is closely related to questions in computational learning and approximation algorithms, has its roots in old questions and results of Erdős. We do not include here a discussion of the general area, and only mention that one of the basic questions studied in it deals with the local and global nature of graph coloring. The specific question here is the ability to distinguish between graphs on n vertices that are k-colorable and graphs from which one has to delete at least εn^2 edges to get a k-colorable graph, by sampling a random induced subgraph on a small number of vertices. The first papers dealing with this question are [19] by Erdős and his collaborators and [65]. Better quantitative results appear in [48], where the systematic study of Graph Property Testing has been initiated, and in [9]. As is the case with so many other topics, the initial questions and results here can be traced back to the work of Paul Erdős.

Note added in proof: Very recently, Eberhard, Green and Manners have proved in [25] that the sum-free subset constant discussed in subsection 3.2 is in fact 1/3. The problem of deciding whether or not every set of n nonzero integers contains a sum-free subset of cardinality at least $n/3 + w(n)$, where $w(n)$ tends to infinity with n, remains open.

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