John Coates
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Noncommutative Iwasawa Main Conjectures over Totally Real Fields

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Noncommutative Iwasawa Main Conjectures over Totally Real Fields

Münster, April 2011

Springer

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Preface

Hi no hikari
Yabu shi wakaneba
Isonokami-
Furinishi sato ni
Hana mo sakikeri

For the light of the sun
Shuns not the wild thickets,
Even in Isonokami-
This village grown ancient,
The flowers are in bloom.

-Furu Imamichi (Kokinshu).

The mysterious link between special values of complex zeta and L-functions and purely arithmetic problems was discovered by Dirichlet and Kummer in the nineteenth century, and spectacularly generalized in the twentieth century by Birch and Swinnerton-Dyer with the formulation of their celebrated conjecture on the arithmetic of elliptic curves. We owe to Iwasawa the great discovery that these problems can be attacked by p-adic methods, where p is any prime number, provided one is prepared to work with a class of infinite Galois extensions of the base field F (which is always supposed to be a finite extension of $\mathbb{Q}$). Iwasawa himself only considered those Galois extensions whose Galois group is isomorphic to the additive group of the ring of p-adic integers $\mathbb{Z}_p$ and the trivial Tate motive. However, it soon became apparent that his methods ought to apply to a much wider class of infinite extensions, namely those whose Galois group over F is a p-adic Lie group of dimension ≥ 1, and to a large class of motives defined over F. While it is still not known how to formulate it in complete generality, it is now widely believed that, in this general setting, the link between special values of complex L-functions and arithmetic should be expressed by what is known as a main conjecture. Very roughly speaking, such a main conjecture should assert that an appropriate p-adic L-function, interpolating special values of the relevant complex L-functions, should coincide with a certain algebraically defined invariant, usually called a characteristic element, which arises naturally from the arithmetic of the motive over the p-adic Lie extension.

This book arose from a workshop held at the University of Münster from April 25–30, 2011. The principal aim of this Workshop was to present the proof of the first key example of these general ideas, namely, the case when the motive is the trivial Tate motive and the p-adic Lie extension F_∞ of F is totally real (in addition, we always assume that F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$). The first important progress on this problem goes back to Iwasawa himself, although we owe to Mazur and Wiles the first complete proof of the most classical case of this main conjecture (when F_∞ is a the compositum of a real abelian base field F with the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$). Subsequently, Wiles discovered a deep new method, relying heavily on the theory of automorphic forms, for attacking these problems. In this way, he succeeded in proving the main conjecture when the base field F is any totally real number field, and the Galois group of F_∞ over F is abelian. This book is concerned with the problem of how one can extend Wiles' work to establish the general non-abelian totally real main conjecture for the trivial Tate motive. Two approaches for doing this were discovered independently and simultaneously, by Kakde on the one hand, generalizing ideas of Kato, and by Ritter and Weiss on the other. Both methods do in fact require one to assume a standard conjecture of Iwasawa about the vanishing of his cyclotomic μ-invariant, and so far this has only been proven when the base field F is an abelian extension of $\mathbb{Q}$ and the Galois group of F_∞ over F is assumed to be pro-p. Both approaches are discussed in this book, but, following the lectures at the Workshop, it is largely Kakde's method which is treated in detail here. One reason for doing this is that the remarkable set of congruences established by Kakde to describe the K_1- group of the Iwasawa algebra of any compact p-adic Lie group should also apply to attacking the non-commutative main conjecture for other motives. Finally, for reasons of space, the book only contains a written version of the lectures at the workshop which were closely related to the proof of the main conjecture.

The Scientific Committee for the Münster Workshop consisted of J. Coates, P. Schneider (Chairman), R. Sujatha, and O. Venjakob. It was made possible by the generous financial support of Project A2 within the DFG Collaborative Research Center 878 "Groups, Geometry, and Actions" at Münster and by some additional funding from the ERC Starting Grant IWASAWA at Heidelberg.

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Introduction to the Work of M. Kakde on the Non-commutative Main Conjectures for Totally Real Fields

John Coates and Dohyeong Kim

Abstract These notes are aimed at providing a not too technical introduction to both the background from classical Iwasawa theory for, as well as a detailed discussion of, the principal result (see Theorem 5.1) of Mahesh Kakde's fundamental paper [K1] proving, subject to the Iwasawa conjecture, the non-commutative main conjecture for totally real p-adic Lie extensions of a number field. Kakde's work is the beautiful development of ideas initiated by Kazuya Kato in his important paper [KA]. The material covered roughly corresponds to the oral lectures given by one of us at the Workshop. We have not attempted here to discuss the detailed methods of proof used either by Kakde in his paper, or by Ritter and Weiss in their important related work [RW], leaving all of this to the written material of the subsequent lecturers at the Workshop. We would also like to particularly thank R. Greenberg and K. Ardakov for some very helpful comments which have been included in the present manuscript. In particular, we are very grateful to Greenberg for providing us with a detailed explanation of his observation (Theorem 4.5) that Wiles' work (Theorems 4.3 and 4.4) on the abelian main conjecture for totally real number fields, can be extended to include the case of abelian characters, whose order is divisible by p.

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Keywords Iwasawa algebras • main conjecture • motive

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1 Notation

Throughout, F will denote a totally real finite extension of $\mathbb{Q}$, and p an odd prime. As always μ_{p^n}, with $1 \le n \le \infty$, is the group of all p^n-th roots of unity. Write F^{cyc} for the unique $\mathbb{Z}_p$-extension of F contained in $F(\mu_{p^\infty})$, and put $\Gamma = Gal(F^{cyc}/F)$ so that $\Gamma \simeq \mathbb{Z}_p$.

Let Σ be a fixed finite set of finite primes of F which contains all the primes dividing p, and write F_Σ for the maximal extension of F, which is unramified outside the primes in Σ and the infinite primes of F. If L is any extension of F contained in F_Σ, put $G_\Sigma(L) = Gal(F_\Sigma/L)$. Also, define $M(L)$ to be the maximal abelian p-extension of L contained in F_Σ, and put

$$X(L) = Gal(M(L)/L).$$

Assume now that L is Galois over F, so that $M(L)$ is also Galois over F. There is a natural left action of $Gal(L/F)$ on $X(L)$ defined by $g \cdot x = \tilde{g}x\tilde{g}^{-1}$, where $\tilde{g}$ denotes any lifting of g in $Gal(L/F)$ to $Gal(M(L)/F)$. As usual, this left action extends to a left action of the Iwasawa algebra $\Lambda(Gal(L/F))$, which is defined by

$$\Lambda(Gal(L/F)) = \varprojlim_{U} \mathbb{Z}_p[Gal(L/F)/U],$$

where U runs over the open normal subgroups of $Gal(L/F)$. Also, if W is any abelian group, $W(p)$ will denote the p-primary subgroup of W.

A Galois extension F_∞ of F is defined to be *an admissible p-adic Lie extension of F* if (1) F_∞ is totally real, (2) the Galois group of F_∞ over F is a p-adic Lie group, (3) F_∞/F is unramified outside a finite set of primes of F, and (iv) F_∞ contains F^{cyc}. Given such an admissible p-adic Lie extension, we shall always put

$$G = Gal(F_\infty/F), \; H = Gal(F_\infty/F^{cyc}), \; \Gamma = Gal(F^{cyc}/F),$$

and take Σ to be a finite set of primes of F containing all primes which are ramified in F_∞/F. If I denotes the ring of integers of some finite extension of $\mathbb{Q}_p$, it will also be convenient to write $I[[\Gamma]]$ for the Iwasawa algebra of Γ with coefficients in I. Fixing a topological generator γ of Γ, we can, as usual, identify $I[[\Gamma]]$ with the ring $I[[T]]$ of formal power series in an indeterminate T with coefficients in I, by mapping γ to $1+T$. Finally, we shall write $\mathfrak{A}(G)$ for the set of Artin representations of G, and $L_\Sigma(\rho, s)$ for the complex Artin L-function, with the Euler factors for the primes in Σ removed, of each ρ in $\mathfrak{A}(G)$.

2 Iwasawa's Work on the Cyclotomic Theory

We use the above notation, and we stress that the base field F is always assumed to be totally real. In his fundamental paper [IW], Iwasawa proved the following basic result which is the starting point for the whole theory.

Theorem 2.1. *For all totally real number fields F, $X(F^{cyc})$ is a finitely generated and torsion $\Lambda(\Gamma)$-module, which has no non-zero finite $\Lambda(\Gamma)$-submodule. Moreover, we have*

$$H^2(G_\Sigma(F^{cyc}), \mathbb{Q}_p/\mathbb{Z}_p) = 0. \quad (1)$$

Recall that one form of Leopoldt's conjecture, which remains unproven, is the assertion that F^{cyc} is the unique $\mathbb{Z}_p$-extension of F. The above theorem is established by noting that $X(F^{cyc})$ being $\Lambda(\Gamma)$-torsion is seen, by using the full force of global class field theory, to be equivalent to the assertion that the defect in the Leopoldt conjecture (i.e. the difference between the $\mathbb{Z}$-rank of the unit group and the $\mathbb{Z}_p$-rank of its closure, in the p-adic topology, in the product of the local unit groups at the primes above p) is bounded as one mounts the finite layers of the $\mathbb{Z}_p$-extension F^{cyc}/F. This boundedness of the defect of Leopoldt is then, in turn, shown to be implied by the boundedness of capitulation of ideal classes in the extension F^{cyc}/F. Finally, Iwasawa gives an ingenious proof of the boundedness of this capitulation. The vanishing statement (1) is then a consequence of an Euler characteristic argument which shows that, in the case of a totally real base field F, the Pontrjagin duals of the two modules $H^i(G_\Sigma(F^{cyc}), \mathbb{Q}_p/\mathbb{Z}_p)(i = 1, 2)$ have the same $\Lambda(\Gamma)$-rank.

In addition, a celebrated conjecture of Iwasawa will play an important role in the non-abelian theory developed later. By the structure theory, a finitely generated $\Lambda(\Gamma)$-module W is $\Lambda(\Gamma)$-torsion if and only if $W/W(p)$ is a finitely generated $\mathbb{Z}_p$-module. Moreover, $W(p)$ is finite if and only if its Iwasawa μ-invariant is zero.

Conjecture A. For totally real F, $X(F^{cyc})$ is a finitely generated $\mathbb{Z}_p$-module.

Note that, if Conjecture A is true, Theorem 2.1 shows that $X(F^{cyc})$ is in fact a free $\mathbb{Z}_p$-module of finite rank. The classical Iwasawa $\mu = 0$ conjecture is the assertion that, for every finite extension K of $\mathbb{Q}$, the Galois group of the maximal unramified abelian p-extension of K^{cyc} is a finitely generated $\mathbb{Z}_p$-module. It is well known that, by using an argument from Kummer theory, this classical Iwasawa conjecture for the totally imaginary field $K = F(\mu_p)$ implies Conjecture A for the totally real field F.

So far, Conjecture A has only been proven when F is an abelian extension of $\mathbb{Q}$, where it is a consequence of the Ferrero-Washington for the cyclotomic $\mathbb{Z}_p$-extension of the field $F(\mu_p)$, which is again an abelian extension of $\mathbb{Q}$.

3 Admissible *p*-Adic Lie Extensions of *F*

The later material in this book will be concerned with an arbitrary admissible p-adic Lie extension F_∞/F, and the $\Lambda(G)$ module $X(F_\infty)$. We stress that this means, in particular, that F_∞ must also be totally real.

The first thing we should point out is that non-trivial examples of such admissible p-adic Lie extensions are not easy to come by. If Conjecture A is valid for F, we can always take F_∞ to be the field $M(F^{cyc})$. Moreover, assuming that (1) Conjecture A is valid, (2) that G is pro-p with no element of order p, and that (3) G has dimension at least 2 as a p-adic Lie group, it follows from Theorem 3.1 below and Theorem 5.2 of the Appendix that $X(F_\infty) \neq 0$ if and only if the $\mathbb{Z}_p$-rank of $X(F^{cyc})$ is at least 2. Perhaps the most down to earth example of such an admissible p-adic Lie extension F_∞ with $X(F_\infty) \neq 0$ is to take F to be the maximal real subfield of the field generated over $\mathbb{Q}$ by the p-th roots of unity, where p is any odd prime such that at least two of the rational numbers

$$\zeta(\mathbb{Q}, -1), \zeta(\mathbb{Q}, -3), \ldots, \zeta(\mathbb{Q}, 4-p)$$

have their numerators divisible by p (the smallest such prime is $p = 157$); here we take Σ to consist of the unique prime of F above p, and $\zeta(\mathbb{Q}, s)$ denotes the Riemann zeta function. It is the classical main conjecture for $X(F^{cyc})$ which guarantees that the $\mathbb{Z}_p$-rank of $X(F^{cyc})$ is at least 2 for such primes p. A much more esoteric example is given by Ramakrishna [RK], who proves the existence of infinitely many Galois extensions L_∞ of $\mathbb{Q}$, which are totally real, whose Galois group J over $\mathbb{Q}$, is either $SL_2(\mathbb{Z}_7)$ or the quotient of $SL_2(\mathbb{Z}_7)$ by the subgroup generated by $-I$ (where I is the unit matrix), and which are unramified outside a finite set T of primes of $\mathbb{Q}$. Thus we can take F_∞ to be the compositum of L_∞ and the cyclotomic $\mathbb{Z}_7$-extension of $\mathbb{Q}$. Note that if we define F to be the fixed field of the image in J of the group of matrices congruent to the identity modulo 7 in $SL_2(\mathbb{Z}_7)$, then the Galois group of F_∞/F will be pro-7, and have no element of order 7. Defining Σ to be the set of primes of F lying above either 7 or the primes in T, it follows from the above remarks that, assuming that Conjecture A is valid for F with $p = 7$, then the $\mathbb{Z}_7$-rank of $X(F^{cyc})$ is at least 2, and $X(F_\infty) \neq 0$.

The full analogue of Theorem 2.1 for any admissible p-adic Lie extension is proven in the two papers [OV, V, V1]. We say that a left $\Lambda(G)$-module W is $\Lambda(G)$-torsion if every element of W is annihilated by a non-zero divisor in $\Lambda(G)$.

Theorem 3.1. *For every admissible p-adic Lie extension F_∞/F, $X(F_\infty)$ is a finitely generated torsion $\Lambda(G)$-module. Moreover, if G has no element of order p, then $X(F_\infty)$ has no non-zero pseudo-null submodule.*

Assuming that G is both pro-p and has no element of order p, it follows from the final assertion of Theorem 3.1 and the results of [V] that there is an exact sequence of $\Lambda(G)$-modules

$$0 \to X(F_\infty)(p) \to \bigoplus_{j=1}^{j=t} \Lambda(G)/p^{n_j}\Lambda(G) \to D \to 0, \tag{2}$$

where D is a pseudo-null $\Lambda(G)$-module. One then defines $\mu_G(X(F_\infty)) = n_1 + \cdots + n_t$. In particular, we have $X(F_\infty)(p) = 0$ if and only if $\mu_G(X(F_\infty)) = 0$. We shall see below that a suitable form of Conjecture A implies a strong statement about the module $X(F_\infty)$, which shows, in particular, that $\mu_G(X(F_\infty)) = 0$.

In our present state of knowledge, we do not know how to even formulate the main conjecture using the result of this theorem alone (we cannot define a characteristic element for $X(F_\infty)$ assuming only that it is finitely generated and torsion over $\Lambda(G)$, even if we impose the additional hypothesis that $\mu_G(X(F_\infty)) = 0$). In order to overcome this difficulty, we follow [CFKSV] and introduce the category $\mathfrak{M}_H(G)$ consisting of all finitely generated $\Lambda(G)$-modules W such that $W/W(p)$ is finitely generated over $\Lambda(H)$, where we recall that $H = Gal(F_\infty/F^{cyc})$. While it seems very reasonable to conjecture that $X(F_\infty)$ always belongs to the category $\mathfrak{M}_H(G)$, we unfortunately cannot prove this unconditionally at present. Nevertheless, assuming this conjecture, the following result is proven in the Appendix.

Theorem 3.2. *Assume that the p-adic Lie extension F_∞/F is such that (i) G is pro-p and has no element of order p, (ii) G has dimension at least 2 as a p-adic Lie group, and (iii) $X(F_\infty)$ belongs to the category $\mathfrak{M}_H(G)$. Then $\mu_G(X(F_\infty)) = \mu_\Gamma(X(F^{cyc}))$, and $X(F_\infty)/X(F_\infty)(p)$ has $\Lambda(H)$-rank equal to $r - 1$, where r is the $\mathbb{Z}_p$-rank of $X(F^{cyc})/X(F^{cyc})(p)$.*

Our present inability to prove that $X(F_\infty)$ lies in the category $\mathfrak{M}_H(G)$ leads us to work with a stronger conjecture in the subsequent analytic and algebraic arguments.

Iwasawa Conjecture: The admissible p-adic Lie extension F_∞/F will be said to satisfy the *Iwasawa conjecture* if there exists a finite extension F' of F in F_∞ such that (1) the Galois group of F_∞ over F' is pro-p, and (2) $X(F'^{cyc})$ is a finitely generated $\mathbb{Z}_p$-module.
We remark that, by the theorem of Ferrero-Washington, this Iwasawa conjecture is true for all p-adic Lie extensions F_∞/F such that F is an abelian extension of $\mathbb{Q}$ and the Galois group G is pro-p. In particular, when F is an abelian extension of $\mathbb{Q}$ and $F_\infty = M(F^{cyc})$, the Iwasawa conjecture is valid.

Theorem 3.3. *Assume that the p-adic Lie extension F_∞/F satisfies the Iwasawa Conjecture. Then $X(F_\infty)$ is finitely generated over $\Lambda(H)$, and $X(F_\infty)(p) = 0$.*

Proof. Put $H' = Gal(F_\infty/F'^{cyc})$. Then we have the exact sequence of inflation restriction

$$\begin{aligned} 0 &\to H^1(H', \mathbb{Q}_p/\mathbb{Z}_p) \to Hom(X(F'^{cyc}), \mathbb{Q}_p/\mathbb{Z}_p) \\ &\to Hom(X(F_\infty)_{H'}, \mathbb{Q}_p/\mathbb{Z}_p) \to H^2(H', \mathbb{Q}_p/\mathbb{Z}_p). \end{aligned}$$

Now $H^i(H', \mathbb{Q}_p/\mathbb{Z}_p)$ is a cofinitely generated $\mathbb{Z}_p$-module for all $i \geq 0$. Hence, assuming that $X(F'^{cyc})$ is a finitely generated $\mathbb{Z}_p$-module, it follows that $X(F_\infty)_{H'}$ is also a finitely generated $\mathbb{Z}_p$-module. But, as H' is pro-p, $\Lambda(H')$ is a local ring, and so it follows from Nakayama's lemma that $X(F_\infty)$ is finitely generated over $\Lambda(H')$, and so all the more so over $\Lambda(H)$. To prove the final assertion of the theorem, we note that we can find an open subgroup H'' of H' such that H'' is pro-p and has no element of order p (possibly $H'' = 0$). Since $X(F_\infty)$ is also finitely generated over $\Lambda(H'')$, a theorem of Venjakob asserts that every $\Lambda(G)$ submodule of $X(F_\infty)$, which is $\Lambda(H'')$-torsion, is pseudo-null as a $\Lambda(G)$-module, and so must be zero by Theorem 3.1. In particular, this shows that $X(F_\infty)(p) = 0$.

4 The Classical Abelian Main Conjecture

In this section, we discuss the classical abelian main conjecture for an arbitrary admissible p-adic Lie extension F_∞/F which will be assumed throughout this section to satisfy the:

Abelian Hypothesis. $G = Gal(F_\infty/F)$ is an abelian p-adic Lie group of dimension 1.

As before, Σ will denote the set of primes of F which ramify in F_∞. We fix a lifting of $\Gamma = Gal(F^{cyc}/F)$ to G, which we denote by the same symbol Γ. Thus, since G is abelian, this means that we have $G = H \times \Gamma$. We define K to be the fixed field of the subgroup Γ of G, so that $K \cap F^{cyc} = F$, and F_∞ is the compositum of K and F^{cyc}. Let $\hat{H}$ be the group of 1-dimensional characters of H. Write

$$\kappa_F : Gal(F(\mu_{p^\infty})/F) \to \mathbb{Z}_p^\times$$

for the cyclotomic character. As we can view Γ as a subgroup of $Gal(F(\mu_{p^\infty})/F)$, it makes sense to consider the restriction of κ_F to Γ. In what follows, we then consider κ_F as a character of G by defining it always to be trivial on H.

While complex L-functions can be defined in great generality via Euler products, nothing like this seems to be true in the p-adic world, and, at present, our only way to define p-adic L-function is via p-adic interpolation of essentially algebraic special values of complex L-functions. Viewing an element χ in $\hat{H}$ as being complex valued, let $L_\Sigma(\chi, s)$ be the imprimitive complex L-function attached to χ, with the Euler factors corresponding to the primes in Σ omitted from its Euler product. The following basic result is due to Siegel.

Theorem 4.1. *For each χ in $\hat{H}$, and each even integer $n > 0$, $L_\Sigma(\chi, 1-n)$ belongs to the field $\mathbb{Q}(\chi)$, which is generated over $\mathbb{Q}$ by the values of χ.*

In fact, Siegel's proof shows that

$$L_\Sigma(\chi^\sigma, 1-n) = L_\Sigma(\chi, 1-n)^\sigma \tag{3}$$

for all σ in the absolute Galois group of $\mathbb{Q}$, and all even integers $n > 0$, and even allows us to define this value intrinsically when the character χ is no longer assumed to have complex values. In fact, we shall assume from now on that the χ in $\hat{H}$ all have values in the algebraic closure of $\mathbb{Q}_p$.

Let $\mathcal{O}$ be the ring of integers of the field obtained by adjoining the values of all χ in $\hat{H}$ to $\mathbb{Q}_p$, and let $\Lambda_{\mathcal{O}}(G)$ be the Iwasawa algebra of G with coefficients in $\mathcal{O}$. Write $Q_{\mathcal{O}}(G)$ for the ring of fractions of $\Lambda_{\mathcal{O}}(G)$ (i.e. the localization of this ring with respect to its set of non-zero divisors). An element μ of $Q_{\mathcal{O}}(G)$ is defined to be a *pseudo-measure* on G if $(\sigma - 1)\mu$ is in $\Lambda_{\mathcal{O}}(G)$ for all σ in G. If $\psi : G \to \mathcal{O}^{\times}$ is any continuous homomorphism, which is distinct from the trivial homomorphism of G, which we denote by $\mathbf{1}$, it is easily seen that one can define the integral of ψ against μ, which we denote by

$$\int_G \psi d\mu,$$

and which is a well defined element of the fraction field of $\mathcal{O}$. The following theorem, which generalizes many earlier results starting with Kummer, is due to Cassou-Nogues and Deligne-Ribet.

Theorem 4.2. *Assume that F_∞/F satisfies the abelian hypothesis. Then there exists a unique pseudo-measure $\zeta_{F_\infty/F}$ on $G = H \times \Gamma$ such that, for all χ in $\hat{H}$, we have*

$$\int_G \chi {\kappa_F}^n d\zeta_{F_\infty/F} \;=\; L_\Sigma(\chi, 1-n), \tag{4}$$

for all integers $n > 0$ with $n \equiv 0$ mod δ, where $\delta = [F(\mu_p) : F]$.

This theorem is easily seen to imply the following assertion. For each character χ in $\hat{H}$, let $\mathcal{O}_\chi$ be the ring of integers of the field obtained by adjoining the values of χ to $\mathbb{Q}_p$, and let $\mathcal{O}_\chi$[[T]] be the ring of formal power series in an indeterminate T with coefficients in $\mathcal{O}_\chi$. Fix, for the remainder of this section, a topological generator γ of Γ. Then, if $\chi \neq \mathbf{1}$, there exists a unique formal power series $W_\chi(T)$ in $\mathcal{O}_\chi$[[T]] such that

$$W_\chi(\kappa_F(\gamma)^n - 1) = L_\Sigma(\chi, 1-n),$$

for all integers $n > 0$ with $n \equiv 0 \bmod \delta$. In addition, if $\chi = \mathbf{1}$, there exists a unique power series $W_{\mathbf{1}}(T)$ in $\mathbb{Z}_p$[[T]] such that

$$W_{\mathbf{1}}(\kappa_F(\gamma)^n - 1)/(\kappa_F(\gamma)^n - 1) = \zeta_\Sigma(F, 1-n),$$

where $\zeta_\Sigma(F, s)$ denotes the complex zeta function of F, with the Euler factors removed at the primes in Σ. Let π_χ be any fixed local parameter for the ring $\mathcal{O}_\chi$. We plainly can write

$$W_\chi(T) = \pi_\chi^{\mu_\chi} V_\chi(T), \tag{5}$$

where μ_χ is a non-negative integer, and $V_\chi(T)$ is a power series in $\mathcal{O}_\chi$[[T]], with at least one of its coefficients a unit in $\mathcal{O}_\chi$. It is conjectured that we always have $\mu_\chi = 0$ for every F_∞/F and every χ in $\hat{H}$, but this has only been proven in the case $F = \mathbb{Q}$ by Ferrero-Washington, and it is unknown for every other totally real base field other than $\mathbb{Q}$.

The aim of the abelian main conjecture is to give a precise relation between the analytic pseudo-measure $\zeta_{F_\infty/F}$ on the one hand, and the algebraic structure of the arithmetic $\Lambda_{\mathcal{O}}(G)$-module $X(F_\infty)$ on the other hand. However, the exact formulation of this relationship is not straightforward from a classical point of view, because there is no known structure theory for finitely generated torsion $\Lambda_{\mathcal{O}}(G)$-modules when p divides the order of H. For each χ in $\hat{H}$, let

$$e_\chi = \#(H)^{-1} \sum_{h \in H} \chi(h) h^{-1}$$

be the orthogonal idempotent of χ in the group ring of H with coefficients in the field of fractions $\mathcal{L}$ of $\mathcal{O}_\chi$. The simplest thing to do is to simply consider

$$Z(F_\infty) = X(F_\infty) \otimes_{\mathbb{Z}_p} \mathcal{L}, \; Z(F_\infty)_\chi = e_\chi Z(F_\infty), \tag{6}$$

which are both finite dimensional vector spaces over $\mathcal{L}$ by Theorem 2.1. We then define $R_\chi(T)$ to be the characteristic polynomial of $\gamma - 1$ acting on $Z(F_\infty)_\chi$. We omit the proof of the following technical lemma, which is due to Greenberg (see [G1], Proposition 1).

Lemma 4.1. *Let χ be any element of $\hat{H}$, and let K' be any intermediate field between F and K such that χ is trivial on $Gal(K/K')$. Write χ' for χ, when viewed as a character of $Gal(K'/F)$, and let F'_∞ be the compositum of K' and F^{cyc}. Then $Z(F_\infty)_\chi$ is isomorphic to $Z(F'_\infty)_{\chi'}$ as representations of Γ.*

In particular, this lemma shows that the polynomial $R_\chi(T)$ depends only the character χ of H, and not on the particular finite extension of F such that χ factors through the Galois group over F of this extension.

The first fundamental result of Wiles (see Theorem 1.3 of [W1]) in the direction of the main conjecture for all totally real number fields F is the following.

Theorem 4.3. *Assume that F_∞/F satisfies the abelian hypothesis. Then, for all characters χ of H, we have*

$$V_\chi(T)\mathcal{O}_\chi[[\mathrm{T}]] = R_\chi(T)\mathcal{O}_\chi[[\mathrm{T}]]. \tag{7}$$

The problem with this result is that it does not tell us anything about the μ-invariants on either the analytic or the algebraic sides. Of course, the analytic μ-invariant is the integer μ_χ appearing in (5), and is valid for all characters χ of H, irrespective of whether the order of χ is divisible by p or not. The definition of the algebraic μ-invariant is much more delicate. We first explain what to do in the easy case, when the order of χ is prime to p. Assuming this to be the case, we may also suppose that

K is exactly the fixed field of the kernel of χ, as this does not change the polynomial $R_\chi(T)$ by Lemma 4.1. Define

$$X(F_\infty)_\chi = e_\chi(X(F_\infty) \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi).$$

Now, by Theorem 2.1, $X(F_\infty)_\chi$ is a finitely generated torsion $\mathcal{O}_\chi$[[T]]-module, and thus, by the well known structure theory for such modules and the Weierstrass preparation theorem, it has a characteristic ideal of the form $C_\chi(T)\mathcal{O}_\chi$[[T]], where C_χ(T) is a polynomial in $\mathcal{O}_\chi[T]$ of such that

$$C_\chi(T) = \pi_\chi^{\nu_\chi} R_\chi(T), \tag{8}$$

for some integer $\nu_\chi \geq 0$; here $R_\chi(T)$ is, as above, the characteristic polynomial of $\gamma - 1$ acting on $Z(F_\infty)_\chi$. The second fundamental result of Wiles (see Theorem 1.4 of [W1]) is the following.

Theorem 4.4. *Assume that F_∞/F satisfies the abelian hypothesis. If $\chi \in \hat{H}$ has order prime to p, then we have*

$$\mu_\chi = \nu_\chi. \tag{9}$$

In particular, when combined with Theorem 4.3, this result proves the main conjecture asserting that

$$W_\chi(T)\mathcal{O}_\chi[[\mathrm{T}]] = C_\chi(T)\mathcal{O}_\chi[[\mathrm{T}]], \tag{10}$$

for all characters χ of H of order prime to p.

We are very grateful to R. Greenberg (private communication) for the following explanation of how one can define the analogue of the algebraic μ-invariant ν_χ appearing in (8) even for characters χ of H whose order is divisible by p, and then show that the main conjecture (10) still remains valid for such characters. As we shall need to vary the base field F in this argument, for the remainder of this section we shall write $W_{F,\chi}(T)$, $\mu_{F,\chi}, \ldots$ to indicate the dependence of the above quantities on the base field F. Fix a character χ of H, whose order is divisible by p. We shall assume that K is the fixed field of the kernel of χ. Now we can write χ in the form $\chi = \psi\rho$, where ψ is a character of H of order prime to p, and ρ has p-power order. Define $\rho' = \rho^p$, and write L', L for the fixed fields of Ker(ρ'), Ker(ρ), respectively. We can now take the restriction ψ_L (resp. $\psi_{L'}$) of ψ to the absolute Galois group of L (resp. the absolute Galois group of L'). Then K is the fixed field of Ker(ψ_L), and we define K' to be the fixed field of Ker($\psi_{L'}$). Thus we have the tower of fields

$$F \subset L' \subset L \subset K' \subset K. \tag{11}$$

Write F'_∞ for the compositum of K' with F^{cyc}, and, as before, let F_∞ be the compositum of K with F^{cyc}. To lighten our notation, put

$$\mathrm{J} = \mathcal{O}_\psi,\ \mathrm{I} = O_\rho,\ \mathrm{E} = \mathcal{O}_\chi, \tag{12}$$

so that E is the ring generated over J by the values of ρ. We first observe that, up to a pseudo-isomorphism of Γ-modules, we can identify $X(F'_\infty)$ with a quotient of $X(F_\infty)$. Indeed, let P (resp. P') be the Sylow p-subgroup of $Gal(K/F)$ (resp. $Gal(K'/F)$), and put

$$\Omega = Ker(P \to P'), \tag{13}$$

so that Ω has order p. Then the natural map from $X(F_\infty)_\Omega$ to $X(F'_\infty)$, which is the dual of the restriction map on Galois cohomology, has finite kernel and cokernel. Indeed, by the usual inflation restriction sequence, the cokernel is finite because it is dual to $H^1(Gal(F_\infty/F'_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$, and the kernel is finite because it is dual to a submodule of $H^2(Gal(F_\infty/F'_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$; both these cohomology groups are obviously finite because $Gal(F_\infty/F'_\infty)$ is a finite cyclic group. In particular, it follows that $X(F_\infty)_\Omega$ and $X(F'_\infty)$ have the same characteristic power series as Γ-modules. We then define

$$\Pi(F_\infty) = Ker(X(F_\infty) \to X(F_\infty)_\Omega). \tag{14}$$

Explicitly, we have $\Pi(F_\infty) = (\tau - 1)X(F_\infty)$, where τ is any generator of Ω. Now the group ring $\mathbb{Z}_p[P]$ acts on $\Pi(F_\infty)$, and this action factors through an action of the ring

$$B = \mathbb{Z}_p[P]/(1 + \tau + \ldots + \tau^{p-1})\mathbb{Z}_p[P].$$

But evaluation at the character ρ defines an isomorphism from B onto the ring I. Thus we see that $\Pi(F_\infty)$ has a natural structure as an $I[[\Gamma]]$-module. Now ψ is a faithful character of $Gal(K/L)$ of order prime to p, and thus, for any $\mathbb{Z}_p[Gal(K/L)]$-module A, we may define

$$A_\psi = e_\psi(A \otimes_{\mathbb{Z}_p} J).$$

In particular, we have

$$\Pi(F_\infty)_\psi = Ker(X(F_\infty)_\psi \to (X(F_\infty)_\psi)_\Omega). \tag{15}$$

It is clear that $\Pi(F_\infty)_\psi$ has a structure as an $E[[\Gamma]]$-module, because I acts on $\Pi(F_\infty)$. Moreover, since $X(F_\infty)$ is a finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$-module, it follows that $\Pi(F_\infty)_\psi$ is a finitely generated torsion $E[[\Gamma]]$-module. As before, let π_χ be any local parameter of the ring $E = \mathcal{O}_\chi$. Then, by the structure theory for finitely generated torsion $E[[\Gamma]]$-modules, $\Pi(F_\infty)_\psi$ will have a characteristic ideal of the form $\mathfrak{C}_\chi(T)E[[\mathrm{T}]]$, where $\mathfrak{C}_\chi(T)$ is a polynomial such that

$$\mathfrak{C}_\chi(T) = \pi_\chi^{\nu_\chi}\mathfrak{R}_\chi(\mathrm{T}), \tag{16}$$

where ν_χ is some integer ≥ 0, and $\mathfrak{R}_\chi(\mathrm{T})$ is a monic polynomial in $E[T]$. It is this integer ν_χ which we define to be the algebraic μ-invariant of χ when p divides the order of χ. On the other hand, since $(X(F_\infty)_\psi)_\Omega$ is pseudo-isomorphic as a