Michel Marie Deza
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Encyclopedia of Distances

Second Edition

Springer

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Michel Marie Deza • Elena Deza

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In 1906, Maurice FRÉCHET submitted his outstanding thesis Sur quelques points du calcul functionnel *introducing* (*within a systematic study of functional operations) the notion of metric space* (*E*-espace*, E from écart*).
Also, in 1914, Felix HAUSDORFF published his famous Grundzüge der Mengenlehre *where the theory of topological and metric spaces* (metrische Räume) *was created.*
Let this Encyclopedia be our homage to the memory of these great mathematicians and their lives of dignity through the hard times of the first half of the XX century.

Maurice FRÉCHET (1878–1973) coined in 1906 the concept of écart (semimetric)

Felix HAUSDORFF (1868–1942) coined in 1914 the term metric space

Preface

The preparation of the second edition of Encyclopedia of Distances has presented a welcome opportunity to improve the first edition published in 2009 by updating and streamlining many sections, and by adding new items (especially in Chaps. 1, 15, 18, 23, 25, 27–29), increasing the book's size by about 70 pages. This new edition preserves, except for Chaps. 18, 23, 25 and 28, the structure of the first edition.

The first large conference with a scope matching that of this Encyclopedia is MDA 2012, the International Conference "Mathematics of Distances and Applications", held in July 2012 in Varna, Bulgaria (http://foibg.com/conf/ITA2012/2012mda.htm).

We are grateful to Jin Akiyama, Frederic Barbaresco, Pavel Chebotarev, Mathieu Dutour Sikirić, Aleksandar Jurisić, Boris Kukushkin, Victor Matrosov, Tatiana Nebesnaya, Arkadii Nedel, Michel Petitjean and Egon Schulte for their helpful advice, and to Springer-Verlag for its support in making this work a success.

Paris, France
Moscow, Russia
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Preface to the First Edition

Encyclopedia of Distances is the result of re-writing and extending of our *Dictionary of Distances* published in 2006 (and put online http://www.sciencedirect.com/science/book/9780444520876) by Elsevier. About a third of the definitions are new, and majority of the remaining ones are upgraded.

We were motivated by the growing intensity of research on metric spaces and, especially, in distance design for applications. Even if we do not address the practical questions arising during the selection of a "good" distance function, just a sheer listing of the main available distances should be useful for the distance design community.

This Encyclopedia is the first one treating fully the general notion of distance. This broad scope is useful *per se*, but it also limited our options for referencing. We give an original reference for many definitions but only when it was not too difficult to do so. On the other hand, citing somebody who well developed the notion but was not the original author may induce problems. However, with our data (usually, author name(s) and year), a reader can easily search sources using the Internet.

We found many cases when authors developed very similar distances in different contexts and, clearly, were unaware of it. Such connections are indicated by a simple "cf." in both definitions, without going into priority issues explicitly.

Concerning the style, we tried to make it a mixture of resource and coffee-table book, with maximal independence of its parts and many cross-references.

Preface to *Dictionary of Distances*, 2006

The concept of *distance* is a basic one in the whole human experience. In everyday life it usually means some degree of closeness of two physical objects or ideas, i.e., length, time interval, gap, rank difference, coolness or remoteness, while the term *metric* is often used as a standard for a measurement.

But here we consider, except for the last two chapters, the mathematical meaning of those terms which is an abstraction of measurement. The mathematical notions of *distance metric* (i.e., a function $d(x, y)$ from $X \times X$ to the set of real numbers satisfying to $d(x, y) \geq 0$ with equality only for $x = y$, $d(x, y) = d(y, x)$, and $d(x, y) \leq d(x, z) + d(z, y)$) and of *metric space* (X, d) were originated a century ago by M. Fréchet (1906) and F. Hausdorff (1914) as a special case of an infinite topological space. The *triangle inequality* above appears already in Euclid. The infinite metric spaces are usually seen as a generalization of the metric $|x - y|$ on the real numbers. Their main classes are the measurable spaces (add measure) and Banach spaces (add norm and completeness).

However, starting from K. Menger (who, in 1928, introduced metric spaces in Geometry) and L.M. Blumenthal (1953), an explosion of interest in both finite and infinite metric spaces occurred. Another trend: many mathematical theories, in the process of their generalization, settled on the level of metric space. It is an ongoing process, for example, for Riemannian geometry, Real Analysis, Approximation Theory.

Distance metrics and distances have become now an essential tool in many areas of Mathematics and its applications including Geometry, Probability, Statistics, Coding/Graph Theory, Clustering, Data Analysis, Pattern Recognition, Networks, Engineering, Computer Graphics/Vision, Astronomy, Cosmology, Molecular Biology, and many other areas of science. Devising the most suitable distance metrics and similarities, in order to quantify the proximity between objects, has become a standard task for many researchers. Especially intense ongoing search for such distances occurs, for example, in Computational Biology, Image Analysis, Speech Recognition, and Information Retrieval.

Often the same distance metric appears independently in several different areas; for example, the edit distance between words, the evolutionary distance in Biology, the Levenshtein distance in Coding Theory, and the Hamming+Gap or shuffle-Hamming distance.

This body of knowledge has become too big and disparate to operate within. The numbers of worldwide web entries offered by Google on the topics "distance", "metric space" and "distance metric" is about 216, 3 and 9 million, respectively, not to mention all the printed information outside the Web, or the vast "invisible Web" of searchable databases. About 15,000 books on Amazon.com contains "distance" in their titles. However, this huge information on distances is too scattered: the works evaluating distance from some list usually treat very specific areas and are hardly accessible for nonexperts.

Therefore many researchers, including us, keep and cherish a collection of distances for use in their areas of science. In view of the growing general need for an accessible interdisciplinary source for a vast multitude of researchers, we have expanded our private collection into this Dictionary. Some additional material was reworked from various encyclopedias, especially Encyclopedia of Mathematics [EM98], MathWorld [Weis99], PlanetMath [PM], and Wikipedia [WFE]. However, the majority of distances are extracted directly from specialist literature.

Besides distances themselves, we collected here many distance-related notions (especially in Chap. 1) and paradigms, enabling people from applications to get those (arcane for nonspecialists) research tools, in ready-to-use fashion. This and the appearance of some distances in different contexts can be a source of new research.

In the time when over-specialization and terminology fences isolate researchers, this Dictionary tries to be "centripetal" and "ecumenical", providing some access and altitude of vision but without taking the route of scientific vulgarization. This attempted balance defined the structure and style of the Dictionary.

This reference book is a specialized encyclopedic dictionary organized by subject area. It is divided into 29 chapters grouped into 7 parts of about the same length. The titles of the parts are purposely approximative: they just allow a reader to figure out her/his area of interest and competence. For example, Parts II, III and IV, V require some culture in, respectively, pure and applied Mathematics. Part VII can be read by a layman.

The chapters are thematic lists, by areas of Mathematics or applications, which can be read independently. When necessary, a chapter or a section starts with a short introduction: a field trip with the main concepts. Besides these introductions, the main properties and uses of distances are given, within items, only exceptionally. We also tried, when it was easy, to trace distances to their originator(s), but the proposed extensive bibliography has a less general ambition: just to provide convenient sources for a quick search.

Each chapter consists of items ordered in a way that hints of connections between them. All item titles and (with majuscules only for proper nouns) selected key terms can be traced in the large Subject Index; they are boldfaced unless the meaning is clear from the context. So, the definitions are easy to locate, by subject, in chapters and/or, by alphabetic order, in the Subject Index.

The introductions and definitions are reader-friendly and maximally independent each from another; still they are interconnected, in the 3-dimensional HTML manner, by hyperlink-like boldfaced references to similar definitions.

Many nice curiosities appear in this “Who is Who” of distances. Examples of such sundry terms are: ubiquitous Euclidean distance (“as-the-crow-flies”), flower-shop metric (shortest way between two points, visiting a “flower-shop” point first), knight-move metric on a chessboard, Gordian distance of knots, Earth Mover distance, biotope distance, Procrustes distance, lift metric, Post Office metric, Internet hop metric, WWW hyperlink quasi-metric, Moscow metric, dog-keeper distance.

Besides abstract distances, the distances having physical meaning appear also (especially in Part VI); they range from 1.6×10^{-35} m (Planck length) to 7.4×10^{26} m (the estimated size of the observable Universe, about 46×10^{60} Planck lengths).

The number of distance metrics is infinite, and therefore our Dictionary cannot enumerate all of them. But we were inspired by several successful thematic dictionaries on other infinite lists; for example, on Numbers, Integer Sequences, Inequalities, Random Processes, and by atlases of Functions, Groups, Fullerenes, etc. On the other hand, the large scope often forced us to switch to the mode of laconic tutorial.

The target audience consists of all researchers working on some measuring schemes and, to a certain degree, students and a part of the general public interested in science.

We tried to address, even if incompletely, all scientific uses of the notion of distance. But some distances did not made it to this Dictionary due to space limitations (being too specific and/or complex) or our oversight. In general, the size/interdisciplinarity cut-off, i.e., decision where to stop, was our main headache. We would be grateful to the readers who will send us their favorite distances missed here.

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Part I
Mathematics of Distances

Chapter 1
General Definitions

1.1 Basic Definitions

- **Distance**
  Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **distance** (or **dissimilarity**) on X if, for all $x, y \in X$, there holds:

  1. $d(x, y) \geq 0$ (*nonnegativity*);
  2. $d(x, y) = d(y, x)$ (*symmetry*);
  3. $d(x, x) = 0$ (*reflexivity*).

  In Topology, the distance d with $d(x, y) = 0$ implying $x = y$ is called a **symmetric**.
  For any distance d, the function D_1 defined for $x \neq y$ by $D_1(x, y) = d(x, y) + c$, where $c = \max_{x,y,z \in X}(d(x, y) - d(x, z) - d(y, z))$, and $D(x, x) = 0$, is a **metric**. Also, $D_2(x, y) = d(x, y)^c$ is a metric for sufficiently small $c \geq 0$.
  The function $D_3(x, y) = \inf \sum_i d(z_i, z_{i+1})$, where the infimum is taken over all sequences $x = z_0, \dots, z_{n+1} = y$, is the **path semimetric** of the complete weighted graph on X, where, for any $x, y \in X$, the weight of edge xy is $d(x, y)$.
- **Distance space**
  A **distance space** (X, d) is a set X equipped with a distance d.
- **Similarity**
  Let X be a set. A function $s : X \times X \to \mathbb{R}$ is called a **similarity** on X if s is nonnegative, symmetric, and if $s(x, y) \leq s(x, x)$ holds for all $x, y \in X$, with equality if and only if $x = y$.
  The main transforms used to obtain a distance (dissimilarity) d from a similarity s bounded by 1 from above are: $d = 1 - s$, $d = \frac{1-s}{s}$, $d = \sqrt{1-s}$, $d = \sqrt{2(1-s^2)}$, $d = \arccos s$, $d = -\ln s$ (cf. Chap. 4).
- **Semimetric**
  Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **semimetric** (or **écart**) on X if d is nonnegative, symmetric, if $d(x, x) = 0$ for all $x \in X$, and if

$$d(x, y) \leq d(x, z) + d(z, y)$$

for all $x, y, z \in X$ (**triangle inequality** or, sometimes, *triangular inequality*).
In Topology, it is called a **pseudo-metric** (or, rarely, **semidistance**), while the term *semimetric* is sometimes used for a **symmetric** (a distance $d(x, y)$ with $d(x, y) = 0$ only if $x = y$); cf. **symmetrizable space** in Chap. 2.
For a semimetric d, the triangle inequality is equivalent, for each fixed $n \geq 4$, to the following *n-gon inequality*

$$d(x, y) \leq d(x, z_1) + d(z_1, z_2) + \cdots + d(z_{n-2}, y),$$

for all $x, y, z_1, \ldots, z_{n-2} \in X$.
For a semimetric d on X, define an equivalence relation, called **metric identification**, by $x \sim y$ if $d(x, y) = 0$; equivalent points are equidistant from all other points. Let $[x]$ denote the equivalence class containing x; then $D([x], [y]) = d(x, y)$ is a **metric** on the set $\{[x] : x \in X\}$ of equivalence classes.

- **Metric**
  Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** on X if, for all $x, y, z \in X$, there holds:

  1. $d(x, y) \geq 0$ (*nonnegativity*);
  2. $d(x, y) = 0$ if and only if $x = y$ (*identity of indiscernibles*);
  3. $d(x, y) = d(y, x)$ (*symmetry*);
  4. $d(x, y) \leq d(x, z) + d(z, y)$ (**triangle inequality**).

  In fact, 1 follows from 3 and 4.
- **Metric space**
  A **metric space** (X, d) is a set X equipped with a metric d.
  A **metric frame** (or *metric scheme*) is a metric space with an integer-valued metric.
  A **pointed metric space** (or *rooted metric space*) (X, d, x_0) is a metric space (X, d) with a selected base point $x_0 \in X$.
  A **multimetric space** is the union of some metric spaces; cf. **bimetric theory of gravity** in Chap. 24.
- **Extended metric**
  An **extended metric** is a generalization of the notion of metric: the value ∞ is allowed for a metric d.
- **Quasi-distance**
  Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **quasi-distance** on X if d is nonnegative, and $d(x, x) = 0$ holds for all $x \in X$.
  In Topology, it is also called a **premetric** or **prametric**.
  If a quasi-distance d satisfies the **strong triangle inequality** $d(x, y) \leq d(x, z) + d(y, z)$, then (Lindenbaum, 1926) it is symmetric and so, a semimetric.
- **Quasi-semimetric**
  A function $d : X \times X \to \mathbb{R}$ is called a **quasi-semimetric** (or **hemimetric**, *ostensible metric*) on the set X if $d(x, x) = 0$, $d(x, y) \geq 0$ for all $x, y \in X$ and

$$d(x, y) \leq d(x, z) + d(z, y)$$

for all $x, y, z \in X$ (**oriented triangle inequality**).
The set X can be partially ordered by the *specialization order*: $x \preceq y$ if and only if $d(x, y) = 0$.
A **weak quasi-metric** is a quasi-semimetric d on X with *weak symmetry*, i.e., for all $x, y \in X$ the equality $d(x, y) = 0$ implies $d(y, x) = 0$.
An **Albert quasi-metric** is a quasi-semimetric d on X with *weak definiteness*, i.e., for all $x, y \in X$ the equality $d(x, y) = d(y, x) = 0$ implies $x = y$.
A **weightable quasi-semimetric** is a quasi-semimetric d on X with *relaxed symmetry*, i.e., for all $x, y, z \in X$

$$d(x, y) + d(y, z) + d(z, x) = d(x, z) + d(z, y) + d(y, x),$$

holds or, equivalently, there exists a weight function $w(x) \in \mathbb{R}$ on X with $d(x, y) - d(y, x) = w(y) - w(x)$ for all $x, y \in X$ (i.e., $d(x, y) + \frac{1}{2}(w(x) - w(y))$ is a semimetric). If d is a weightable quasi-semimetric, then $d(x, y) + w(x)$ is a **partial semimetric** (moreover, a **partial metric** if d is an Albert quasi-metric).

- **Partial metric**
  Let X be a set. A nonnegative symmetric function $p : X \times X \to \mathbb{R}$ is called a **partial metric** [Matt92] if, for all $x, y, z \in X$, it holds:

  1. $p(x, x) \le p(x, y)$ (i.e., every **self-distance** $p(x, x)$ is *small*);
  2. $x = y$ if $p(x, x) = p(x, y) = p(y, y) = 0$ (T_0 *separation axiom*);
  3. $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$ (**sharp triangle inequality**).

  If the above separation axiom is dropped, the function p is called a **partial semimetric**. The nonnegative function p is a partial semimetric if and only if $p(x, y) - p(x, x)$ is a **weightable quasi-semimetric** with $w(x) = p(x, x)$.
  If the above condition $p(x, x) \le p(x, y)$ is also dropped, the function p is called (Heckmann, 1999) a **weak partial semimetric**. The nonnegative function p is a weak partial semimetric if and only if $2p(x, y) - p(x, x) - p(y, y)$ is a semimetric.
  Sometimes, the term *partial metric* is used when a metric $d(x, y)$ is defined only on a subset of the set of all pairs x, y of points.
- **Protometric**
  A function $p : X \times X \to \mathbb{R}$ is called a **protometric** if, for all (equivalently, for all different) $x, y, z \in X$, the **sharp triangle inequality** holds:

  $$p(x, y) \le p(x, z) + p(z, y) - p(z, z).$$

  A **strong protometric** is a protometric p with $p(x, x) = 0$ for all $x \in X$. Such a protometric is exactly a quasi-semimetric, but with the condition $p(x, y) \ge 0$ (for any $x, y \in X$) being relaxed to $p(x, y) + p(y, x) \ge 0$.
  A **partial semimetric** is a **symmetric protometric** (i.e., $p(x, y) = p(y, x)$) with $p(x, y) \ge p(x, x) \ge 0$ for all $x, y \in X$. An example of a nonpositive symmetric protometric is given by $p(x, y) = -(x.y)_{x_0} = \frac{1}{2}(d(x, y) - d(x, x_0) - d(y, y_0))$, where (X, d) is a metric space with a fixed base point $x_0 \in X$; see **Gromov product similarity** $(x.y)_{x_0}$ and, in Chap. 4, **Farris transform metric** $C - (x.y)_{x_0}$.

A 0-**protometric** is a protometric p for which all sharp triangle inequalities (equivalently, all inequalities $p(x, y) + p(y, x) \geq p(x, x) + p(y, y)$ implied by them) hold as equalities. For any $u \in X$, denote by A'_u, A''_u the 0-protometrics p with $p(x, y) = 1_{x=u}, 1_{y=u}$, respectively. The protometrics on X form a flat convex cone in which the 0-protometrics form the largest linear space. For finite $|X|$, a basis of this space is given by all but one A'_u, A''_u (since $\sum_u A'_u = \sum_u A''_u$) and, for the flat subcone of all symmetric 0-protometrics on X, by all $A'_u + A''_u$.

A **weighted protometric** on X is a protometric with a point-weight function $w : X \to \mathbb{R}$. The mappings $p(x, y) = \frac{1}{2}(d(x, y) + w(x) + w(y))$ and $d(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, $w(x) = p(x, x)$ establish a bijection between the weighted strong protometrics (d, w) and the protometrics p on X, as well as between the weighted semimetrics and the symmetric protometrics. For example, a weighted semimetric (d, w) with $w(x) = -d(x, x_0)$ corresponds to a protometric $-(x.y)_{x_0}$. For finite $|X|$, the above mappings amount to the representation

$$2p = d + \sum_{u \in X} p(u, u)\big(A'_u + A''_u\big).$$

- **Quasi-metric**

  Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **quasi-metric** (or **asymmetric metric**, *directed metric*) on X if $d(x, y) \geq 0$ holds for all $x, y \in X$ with equality if and only if $x = y$, and

$$d(x, y) \leq d(x, z) + d(z, y)$$

  for all $x, y, z \in X$ (**oriented triangle inequality**). A *quasi-metric space* (X, d) is a set X equipped with a quasi-metric d.

  For any quasi-metric d, the functions $\max\{d(x, y), d(y, x)\}$, $\min\{d(x, y), d(y, x)\}$ and $\frac{1}{2}(d^p(x, y) + d^p(y, x))^{\frac{1}{p}}$ with $p \geq 1$ (usually, $p = 1$ is taken) are **equivalent metrics**.

  A **non-Archimedean quasi-metric** d is a quasi-distance on X which, for all $x, y, z \in X$, satisfies the following strengthened version of the oriented triangle inequality:

$$d(x, y) \leq \max\big\{d(x, z), d(z, y)\big\}.$$

- **Directed-metric**

  Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called (Jegede, 2005) a **directed-metric** on X if, for all $x, y, z \in X$, it holds that $d(x, y) = -d(y, x)$ and

$$\big|d(x, y)\big| \leq \big|d(x, z)\big| + \big|d(z, y)\big|.$$

  Cf. **displacement** in Chap. 24 and **rigid motion of metric space**.
- **Coarse-path metric**

  Let X be a set. A metric d on X is called a **coarse-path metric** if, for a fixed $C \geq 0$ and for every pair of points $x, y \in X$, there exists a sequence

$x = x_0, x_1, \ldots, x_t = y$ for which $d(x_{i-1}, x_i) \le C$ for $i = 1, \ldots, t$, and

$$d(x, y) \ge d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{t-1}, x_t) - C,$$

i.e., the weakened triangle inequality $d(x, y) \le \sum_{i=1}^{t} d(x_{i-1}, x_i)$ becomes an equality up to a bounded error.

- **Near-metric**
  Let X be a set. A distance d on X is called a **near-metric** (or C-*near-metric*) if $d(x, y) > 0$ for $x \ne y$ and the C-*relaxed triangle inequality*

$$d(x, y) \le C\big(d(x, z) + d(z, y)\big)$$

holds for all $x, y, z \in X$ and some constant $C \ge 1$.
A C-**inframetric** is a C-near-metric, while a C-near-metric is a $2C$-inframetric.
Some recent papers use the term *quasi-triangle inequality* for the above inequality and so, *quasi-metric* for the notion of near-metric.
The **power transform** (cf. Chap. 4) $(d(x, y))^\alpha$ of any near-metric is a near-metric for any $\alpha > 0$. Also, any near-metric d admits a **bi-Lipschitz mapping** on $(D(x, y))^\alpha$ for some semimetric D on the same set and a positive number α.
A near-metric d on X is called a **Hölder near-metric** if the inequality

$$\big|d(x, y) - d(x, z)\big| \le \beta d(y, z)^\alpha \big(d(x, y) + d(x, z)\big)^{1-\alpha}$$

holds for some $\beta > 0$, $0 < \alpha \le 1$ and all points $x, y, z \in X$. Cf. **Hölder mapping**.

- **Weak ultrametric**
  A **weak ultrametric** (or C-**inframetric**, C-*pseudo-distance*) d is a distance on X such that $d(x, y) > 0$ for $x \ne y$ and the C-*inframetric inequality*

$$d(x, y) \le C \max\big\{d(x, z), d(z, y)\big\}$$

holds for all $x, y, z \in X$ and some constant $C \ge 1$.
The term **pseudo-distance** is also used, in some applications, for any of a **pseudo-metric**, a **quasi-distance**, a **near-metric**, a distance which can be infinite, a distance with an error, etc. Another unsettled term is **weak metric**: it is used for both a **near-metric** and a **quasi-semimetric**.

- **Ultrametric**
  An **ultrametric** (or *non-Archimedean metric*) is (Krasner, 1944) a metric d on X which satisfies, for all $x, y, z \in X$, the following strengthened version of the triangle inequality (Hausdorff, 1934), called the **ultrametric inequality**:

$$d(x, y) \le \max\big\{d(x, z), d(z, y)\big\}.$$

So, at least two of $d(x, y), d(z, y), d(x, z)$ are equal, and an ultrametric space is also called an *isosceles space*. An ultrametric on set V has at most $|V|$ different values.
A metric d is an ultrametric if and only if its **power transform** (see Chap. 4) d^α is a metric for any real positive number α. Any ultrametric satisfies the **four-point**

inequality. A metric d is an ultrametric if and only if it is a **Farris transform metric** (cf. Chap. 4) of a **four-point inequality metric**.

- **Robinsonian distance**
A distance d on X is called a **Robinsonian distance** (or *monotone distance*) if there exists a total order $\preceq$ on X *compatible* with it, i.e., for $x, y, w, z \in X$,

$$x \preceq y \preceq w \preceq z \quad \text{implies} \quad d(y, w) \leq d(x, z),$$

or, equivalently, for $x, y, z \in X$,

$$x \preceq y \preceq z \quad \text{implies} \quad d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

Any **ultrametric** is a Robinsonian distance.

- **Four-point inequality metric**
A metric d on X is a **four-point inequality metric** (or **additive metric**) if it satisfies the following strengthened version of the triangle inequality called the **four-point inequality** (Buneman, 1974): for all $x, y, z, u \in X$

$$d(x, y) + d(z, u) \leq \max\{d(x, z) + d(y, u), d(x, u) + d(y, z)\}$$

holds. Equivalently, among the three sums $d(x, y) + d(z, u)$, $d(x, z) + d(y, u)$, $d(x, u) + d(y, z)$ the two largest sums are equal.
A metric satisfies the four-point inequality if and only if it is a **tree-like metric**.
Any metric, satisfying the four-point inequality, is a **Ptolemaic metric** and an L_1-*metric*. Cf. L_p-**metric** in Chap. 5.
A **bush metric** is a metric for which all four-point inequalities are equalities, i.e., $d(x, y) + d(u, z) = d(x, u) + d(y, z)$ holds for any $u, x, y, z \in X$.

- **Relaxed four-point inequality metric**
A metric d on X satisfies the **relaxed four-point inequality** if, for all $x, y, z, u \in X$, among the three sums

$$d(x, y) + d(z, u), d(x, z) + d(y, u), d(x, u) + d(y, z)$$

at least two (not necessarily the two largest) are equal.
A metric satisfies the relaxed four-point inequality if and only if it is a **relaxed tree-like metric**.

- **Ptolemaic metric**
A **Ptolemaic metric** d is a metric on X which satisfies the **Ptolemaic inequality**

$$d(x, y)d(u, z) \leq d(x, u)d(y, z) + d(x, z)d(y, u)$$

(shown by Ptolemy to hold in Euclidean space) for all $x, y, u, z \in X$.
A *Ptolemaic space* is a *normed vector space* $(V, \|.\|)$ such that its norm metric $\|x - y\|$ is a Ptolemaic metric. A normed vector space is a Ptolemaic space if and only if it is an **inner product space** (cf. Chap. 5); so, a **Minkowskian metric** (cf. Chap. 6) is Euclidean if and only if it is Ptolemaic.

The *involution space* $(X\backslash z, d_z)$, where $d_z(x,y) = \frac{d(x,y)}{d(x,z)d(y,z)}$, is a metric space, for any $z \in X$, if and only if d is Ptolemaic [FoSc06].
For any metric d, the metric $\sqrt{d}$ is Ptolemaic [FoSc06].

- **δ-hyperbolic metric**
Given a number $\delta \geq 0$, a metric d on a set X is called **δ-hyperbolic** if it satisfies the **Gromov δ-hyperbolic inequality** (another weakening of the **four-point inequality**): for all $x, y, z, u \in X$, it holds that

$$d(x,y) + d(z,u) \leq 2\delta + \max\{d(x,z) + d(y,u), d(x,u) + d(y,z)\}.$$

A metric space (X, d) is δ-hyperbolic if and only if for all $x_0, x, y, z \in X$ it holds that

$$(x.y)_{x_0} \geq \min\{(x.z)_{x_0}, (y.z)_{x_0}\} - \delta,$$

where $(x.y)_{x_0} = \frac{1}{2}(d(x_0,x) + d(x_0,y) - d(x,y))$ is the **Gromov product** of the points x and y of X with respect to the base point $x_0 \in X$.
A metric space (X, d) is 0-hyperbolic exactly when d satisfies the **four-point inequality**. Every bounded metric space of diameter D is D-hyperbolic. The n-dimensional *hyperbolic space* is $\ln 3$-hyperbolic.
Every δ-hyperbolic metric space is isometrically embeddable into a **geodesic metric space** (Bonk and Schramm, 2000).

- **Gromov product similarity**
Given a metric space (X, d) with a fixed point $x_0 \in X$, the **Gromov product similarity** (or *Gromov product*, *covariance*, *overlap function*) $(.)_{x_0}$ is a similarity on X defined by

$$(x.y)_{x_0} = \frac{1}{2}\big(d(x,x_0) + d(y,x_0) - d(x,y)\big).$$

The triangle inequality for d implies $(x.y)_{x_0} \geq (x.z)_{x_0} + (y.z)_{x_0} - (z.z)_{x_0}$ (**covariance triangle inequality**), i.e., the **sharp triangle inequality** for a **protometric** $-(x.y)_{x_0}$.
If (X, d) is a tree, then $(x.y)_{x_0} = d(x_0, [x,y])$. If (X, d) is a **measure semimetric space**, i.e., $d(x,y) = \mu(x \triangle y)$ for a Borel measure μ on X, then $(x.y)_{\emptyset} = \mu(x \cap y)$. If d is a **distance of negative type**, i.e., $d(x,y) = d_E^2(x,y)$ for a subset X of a Euclidean space $\mathbb{E}^n$, then $(x.y)_0$ is the usual *inner product* on $\mathbb{E}^n$.
Cf. **Farris transform metric** $d_{x_0}(x,y) = C - (x.y)_{x_0}$ in Chap. 4.

- **Cross-difference**
Given a metric space (X, d) and quadruple (x, y, z, w) of its points, the **cross-difference** is the real number cd defined by

$$cd(x,y,z,w) = d(x,y) + d(z,w) - d(x,z) - d(y,w).$$

In terms of the **Gromov product similarity**, for all $x, y, z, w, p \in X$, it holds

$$\frac{1}{2}cd(x,y,z,w) = -(x.y)_p - (z.w)_p + (x.z)_p + (y.w)_p;$$

in particular, it becomes $(x.y)_p$ if $y = w = p$.

Given a metric space (X, d) and quadruple (x, y, z, w) of its points with $x \neq z$ and $y \neq w$, the **cross-ratio** is the real number cr defined by

$$cr(x, y, z, w) = \frac{d(x, y)d(z, w)}{d(x, z)d(y, w)} \geq 0.$$

- **$2k$-gonal distance**
  A $2k$-**gonal distance** d is a distance on X which satisfies, for all distinct elements $x_1, \dots, x_n \in X$, the $2k$-**gonal inequality**

$$\sum_{1 \leq i < j \leq n} b_i b_j d(x_i, x_j) \leq 0$$

  for all $b \in \mathbb{Z}^n$ with $\sum_{i=1}^{n} b_i = 0$ and $\sum_{i=1}^{n} |b_i| = 2k$.
- **Distance of negative type**
  A **distance of negative type** d is a distance on X which is $2k$-**gonal** for any $k \geq 1$, i.e., satisfies the **negative type inequality**

$$\sum_{1 \leq i < j \leq n} b_i b_j d(x_i, x_j) \leq 0$$

  for all $b \in \mathbb{Z}^n$ with $\sum_{i=1}^{n} b_i = 0$, and for all distinct elements $x_1, \dots, x_n \in X$.
  A distance can be of negative type without being a semimetric. Cayley proved that a metric d is an L_2-**metric** if and only if d^2 is a distance of negative type.
- **$(2k+1)$-gonal distance**
  A $(2k+1)$-**gonal distance** d is a distance on X which satisfies, for all distinct elements $x_1, \dots, x_n \in X$, the $(2k+1)$-**gonal inequality**

$$\sum_{1 \leq i < j \leq n} b_i b_j d(x_i, x_j) \leq 0$$

  for all $b \in \mathbb{Z}^n$ with $\sum_{i=1}^{n} b_i = 1$ and $\sum_{i=1}^{n} |b_i| = 2k+1$.
  The $(2k+1)$-gonal inequality with $k = 1$ is the usual triangle inequality. The $(2k+1)$-gonal inequality implies the $2k$-**gonal inequality**.
- **Hypermetric**
  A **hypermetric** d is a distance on X which is $(2k+1)$-**gonal** for any $k \geq 1$, i.e., satisfies the **hypermetric inequality** (Deza, 1960)

$$\sum_{1 \leq i < j \leq n} b_i b_j d(x_i, x_j) \leq 0$$

  for all $b \in \mathbb{Z}^n$ with $\sum_{i=1}^{n} b_i = 1$, and for all distinct elements $x_1, \dots, x_n \in X$.
  Any hypermetric is a semimetric, a **distance of negative type** and, moreover, it can be isometrically embedded into some n-sphere $\mathbb{S}^n$ with squared Euclidean distance. Any L_1-*metric* (cf. L_p-**metric** in Chap. 5) is a hypermetric.

- P**-metric**
  A P**-metric** d is a metric on X with values in $[0, 1]$ which satisfies the **correlation triangle inequality**

$$d(x, y) \leq d(x, z) + d(z, y) - d(x, z)d(z, y).$$

  The equivalent inequality $1 - d(x, y) \geq (1 - d(x, z))(1 - d(z, y))$ expresses that the probability, say, to reach x from y via z is either equal to $(1 - d(x, z))(1 - d(z, y))$ (independence of reaching z from x and y from z), or greater than it (positive correlation). A metric is a P-metric if and only if it is a **Schoenberg transform metric** (cf. Chap. 4).

1.2 Main Distance-Related Notions

- **Metric ball**
  Given a metric space (X, d), the **metric ball** (or *closed metric ball*) with center $x_0 \in X$ and radius $r > 0$ is defined by $\overline{B}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}$, and the **open metric ball** with center $x_0 \in X$ and radius $r > 0$ is defined by $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$.
  The **metric sphere** with center $x_0 \in X$ and radius $r > 0$ is defined by $S(x_0, r) = \{x \in X : d(x_0, x) = r\}$.
  For the **norm metric** on an n-dimensional *normed vector space* $(V, \|.\|)$, the metric ball $\overline{B}^n = \{x \in V : \|x\| \leq 1\}$ is called the *unit ball*, and the set $S^{n-1} = \{x \in V : \|x\| = 1\}$ is called the *unit sphere*. In a two-dimensional vector space, a metric ball (closed or open) is called a **metric disk** (closed or open, respectively).
- **Metric hull**
  Given a metric space (X, d), let M be a **bounded** subset of X.
  The **metric hull** $H(M)$ of M is the intersection of all metric balls containing M.
  The set of *surface points* $S(M)$ of M is the set of all $x \in H(M)$ such that x lies on the sphere of one of the metric balls containing M.
- **Distance-invariant metric space**
  A metric space (X, d) is **distance-invariant** if all **metric balls** $\overline{B}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}$ of the same radius have the same number of elements.
  Then the **growth rate of a metric space** (X, d) is the function $f(n) = |\overline{B}(x, n)|$.
  (X, d) is a *metric space of polynomial growth* if there are some positive constants k, C such that $f(n) \leq Cn^k$ for all $n \geq 0$. Cf. **graph of polynomial growth**, including the group case, in Chap. 15.
  For a **metrically discrete metric space** (X, d) (i.e., with $a = \inf_{x,y \in X, x \neq y} d(x, y) > 0$), its *growth rate* was defined also (Gordon, Linial and Rabinovich, 1998) by

$$\max_{x \in X, r \geq 2} \frac{\log |\overline{B}(x, ar)|}{\log r}.$$

- **Ahlfors q-regular metric space**
  A metric space (X, d) endowed with a Borel measure μ is called **Ahlfors q-regular** if there exists a constant $C \geq 1$ such that for every ball in (X, d) with radius $r < \text{diam}(X, d)$ it holds

$$C^{-1}r^q \leq \mu\big(\overline{B}(x_0, r)\big) \leq Cr^Q.$$

  If such an (X, d) is locally compact, then the **Hausdorff q-measure** can be taken as μ.
- **Closed subset of metric space**
  Given a subset M of a metric space (X, d), a point $x \in X$ is called a *limit point of M* (or *accumulation point*) if every **open metric ball** $B(x, r) = \{y \in X : d(x, y) < r\}$ contains a point $x' \in M$ with $x' \neq x$. The *closure of M*, denoted by $\overline{M}$, is the set M together with all its limit points. The subset M is called **closed** if $M = \overline{M}$.
  A closed subset M is **perfect** if every point of M is a limit point of M.
  Every point of M which is not a limit point of M, is called an *isolated point*. The *interior* $\text{int}(M)$ of M is the set of all its isolated points; the *exterior* $\text{ext}(M)$ of M is $\text{int}(X \backslash M)$ and the *boundary* $\vartheta(M)$ of M is $X \backslash (\text{int}(M) \cup \text{ext}(M))$.
  A subset M is called **topologically discrete** if $M = \text{int}(M)$.
- **Open subset of metric space**
  A subset M of a metric space (X, d) is called *open* if, given any point $x \in M$, the **open metric ball** $B(x, r) = \{y \in X : d(x, y) < r\}$ is contained in M for some positive number r. The family of open subsets of a metric space forms a natural topology on it.
  An open subset of a metric space is called *clopen* if it is **closed**. An open subset of a metric space is called a *domain* if it is **connected**.
  A *door space* is a metric (in general, topological) space in which every subset is either open or closed.
- **Connected metric space**
  A metric space (X, d) is called **connected** if it cannot be partitioned into two nonempty **open** sets. Cf. **connected space** in Chap. 2.
  The maximal connected subspaces of a metric space are called its *connected components*. A **totally disconnected metric space** is a space in which all connected subsets are $\emptyset$ and one-point sets.
  A **path-connected metric space** is a connected metric space such that any two its points can be joined by an **arc** (cf. **metric curve**).
- **Cantor connected metric space**
  A metric space (X, d) is called **Cantor connected** (or pre-connected) if, for any two its points x, y and any $\epsilon > 0$, there exists an *ϵ-chain* joining them, i.e., a sequence of points $x = z_0, z_1, \dots, z_{n-1}, z_n = y$ such that $d(z_k, z_{k+1}) \leq \epsilon$ for every $0 \leq k \leq n$. A metric space (X, d) is Cantor connected if and only if it cannot be partitioned into two *remote parts* A and B, i.e., such that $\inf\{d(x, y) : x \in A, y \in B\} > 0$.

The maximal Cantor connected subspaces of a metric space are called its *Cantor connected components*. A **totally Cantor disconnected metric** is the metric of a metric space in which all Cantor connected components are one-point sets.

- **Indivisible metric space**
  A metric space (X, d) is called **indivisible** if it cannot be partitioned into two parts, neither of which contains an isometric copy of (X, d). Any indivisible metric space with $|X| \geq 2$ is infinite, bounded and **totally Cantor disconnected** (Delhomme, Laflamme, Pouzet and Sauer, 2007).
  A metric space (X, d) is called an **oscillation stable metric space** (Nguyen Van Thé, 2006) if, given any $\epsilon > 0$ and any partition of X into finitely many pieces, the ϵ-**neighborhood** of one of the pieces includes an isometric copy of (X, d).
- **Metric topology**
  A **metric topology** is a *topology* on X induced by a metric d on X; cf. **equivalent metrics**.
  More exactly, given a metric space (X, d), define the *open set* in X as an arbitrary union of (finitely or infinitely many) open metric balls $B(x, r) = \{y \in X : d(x, y) < r\}$, $x \in X$, $r \in \mathbb{R}$, $r > 0$. A *closed set* is defined now as the complement of an open set. The metric topology on (X, d) is defined as the set of all open sets of X. A topological space which can arise in this way from a metric space is called a **metrizable space** (cf. Chap. 2).
  Metrization theorems are theorems which give sufficient conditions for a topological space to be metrizable.
  On the other hand, the adjective *metric* in several important mathematical terms indicates connection to a measure, rather than distance, for example, *metric Number Theory*, *metric Theory of Functions*, *metric transitivity*.
- **Equivalent metrics**
  Two metrics d_1 and d_2 on a set X are called **equivalent** if they define the same *topology* on X, i.e., if, for every point $x_0 \in X$, every open metric ball with center at x_0 defined with respect to d_1, contains an open metric ball with the same center but defined with respect to d_2, and conversely.
  Two metrics d_1 and d_2 are equivalent if and only if, for every $\epsilon > 0$ and every $x \in X$, there exists $\delta > 0$ such that $d_1(x, y) \leq \delta$ implies $d_2(x, y) \leq \epsilon$ and, conversely, $d_2(x, y) \leq \delta$ implies $d_1(x, y) \leq \epsilon$.
  All metrics on a finite set are equivalent; they generate the *discrete topology*.
- **Metric betweenness**
  The **metric betweenness** of a metric space (X, d) is (Menger, 1928) the set of all ordered triples (x, y, z) such that x, y, z are (not necessarily distinct) points of X for which the **triangle equality** $d(x, y) + d(y, z) = d(x, z)$ holds.
- **Closed metric interval**
  Given two different points $x, y \in X$ of a metric space (X, d), the **closed metric interval** between them is the set

$$I(x, y) = \{z \in X : d(x, y) = d(x, z) + d(z, y)\}$$

of the points z, for which the **triangle equality** (or **metric betweenness** (x, z, y)) holds. Cf. examples in Chap. 5 (**inner product space**) and Chap. 15 (**graph-geodetic metric**).

- **Underlying graph of a metric space**
  The **underlying graph** (or *neighborhood graph*) of a metric space (X, d) is a graph with the vertex-set X and xy being an edge if $I(x, y) = \{x, y\}$, i.e., there is no third point $z \in X$, for which $d(x, y) = d(x, z) + d(z, y)$.
- **Distance monotone metric space**
  A metric space (X, d) is called **distance monotone** if any interval $I(x, x')$ is *closed*, i.e., for any $y \in X \backslash I(x, x')$, there exists $x'' \in I(x, x')$ with $d(y, x'') > d(x, x')$.
- **Metric triangle**
  Three distinct points $x, y, z \in X$ of a metric space (X, d) form a **metric triangle** if the **closed metric intervals** $I(x, y)$, $I(y, z)$ and $I(z, x)$ intersect only in the common endpoints.
- **Metric space having collinearity**
  A metric space (X, d) has **collinearity** if for any $\epsilon > 0$ each of its infinite subsets contains distinct ϵ-*collinear* (i.e., with $d(x, y) + d(y, z) - d(x, z) \leq \epsilon$) points x, y, z.
- **Modular metric space**
  A metric space (X, d) is called **modular** if, for any three different points $x, y, z \in X$, there exists a point $u \in I(x, y) \cap I(y, z) \cap I(z, x)$. This should not be confused with **modular distance** in Chap. 10 and **modulus metric** in Chap. 6.
- **Median metric space**
  A metric space (X, d) is called a **median metric space** if, for any three points $x, y, z \in X$, there exists a unique point $u \in I(x, y) \cap I(y, z) \cap I(z, x)$.
  Any median metric space is an L_1-*metric*; cf. L_p-**metric** in Chap. 5 and **median graph** in Chap. 15.
  A metric space (X, d) is called an **antimedian metric space** if, for any three points $x, y, z \in X$, there exists a unique point $u \in X$ maximizing $d(x, u) + d(y, u) + d(z, u)$.
- **Metric quadrangle**
  Four different points $x, y, z, u \in X$ of a metric space (X, d) form a **metric quadrangle** if $x, z \in I(y, u)$ and $y, u \in I(x, z)$; then $d(x, y) = d(z, u)$ and $d(x, u) = d(y, z)$.
  A metric space (X, d) is called *weakly spherical* if, for any three different points $x, y, z \in X$ with $y \in I(x, z)$, there exists $u \in X$ such that x, y, z, u form a metric quadrangle.
- **Metric curve**
  A **metric curve** (or, simply, *curve*) γ in a metric space (X, d) is a continuous mapping $\gamma : I \rightarrow X$ from an interval I of $\mathbb{R}$ into X. A curve is called an **arc** (or **path**, *simple curve*) if it is injective. A curve $\gamma : [a, b] \rightarrow X$ is called a *Jordan curve* (or *simple closed curve*) if it does not cross itself, and $\gamma(a) = \gamma(b)$.

The **length of a curve** $\gamma : [a, b] \to X$ is the number $l(\gamma)$ defined by

$$l(\gamma) = \sup\left\{ \sum_{1 \le i \le n} d\big(\gamma(t_i), \gamma(t_{i-1})\big) : n \in \mathbb{N}, a = t_0 < t_1 < \cdots < t_n = b \right\}.$$

A *rectifiable curve* is a curve with a finite length. A metric space (X, d), where every two points can be joined by a rectifiable curve, is called a **quasi-convex metric space** (or, specifically, C**-quasi-convex metric space**) if there exists a constant $C \ge 1$ such that every pair $x, y \in X$ can be joined by a rectifiable curve of length at most $Cd(x, y)$. If $C = 1$, then this length is equal to $d(x, y)$, i.e., (X, d) is a **geodesic metric space** (cf. Chap. 6).

In a quasi-convex metric space (X, d), the infimum of the lengths of all rectifiable curves, connecting $x, y \in X$ is called the **internal metric**.

The metric d on X is called the **intrinsic metric** (and then (X, d) is called a **length space**) if it coincides with the internal metric of (X, d).

If, moreover, any pair x, y of points can be joined by a curve of length $d(x, y)$, the metric d is called **strictly intrinsic**, and the length space (X, d) is a geodesic metric space. Hopf and Rinow, 1931, showed that any complete locally compact length space is geodesic and **proper**. The **punctured plane** $(\mathbb{R}^2 \setminus \{0\}, \|x - y\|_2)$ is **locally compact** and **path-connected** but not geodesic: the distance between $(-1, 0)$ and $(1, 0)$ is 2 but there is no geodesic realizing this distance.

The **metric derivative** of a metric curve $\gamma : [a, b] \to X$ at a limit point t of $[a, b]$ is, if it exists,

$$\lim_{s \to 0} \frac{d(\gamma(t + s), \gamma(t))}{|s|}.$$

It is the rate of change, with respect to t, of the length of the curve at almost every point, i.e., a generalization of the notion of *speed* to metric spaces.

- **Geodesic**

Given a metric space (X, d), a **geodesic** is a locally shortest **metric curve**, i.e., it is a locally isometric embedding of $\mathbb{R}$ into X; cf. Chap. 6.

A subset S of X is called a **geodesic segment** (or **metric segment**, *shortest path*, *minimizing geodesic*) between two distinct points x and y in X, if there exists a *segment* (closed interval) $[a, b]$ on the real line $\mathbb{R}$ and an isometric embedding $\gamma : [a, b] \to X$, such that $\gamma[a, b] = S$, $\gamma(a) = x$ and $\gamma(b) = y$.

A **metric straight line** is a geodesic which is minimal between any two of its points; it is an isometric embedding of the whole of $\mathbb{R}$ into X. A **metric ray** and **metric great circle** are isometric embeddings of, respectively, the half-line $\mathbb{R}_{\ge 0}$ and a circle $S^1(0, r)$ into X.

A **geodesic metric space** (cf. Chap. 6) is a metric space in which any two points are joined by a geodesic segment. If, moreover, the geodesic is unique, the space is called *totally geodesic* (or *uniquely geodesic*).

A geodesic metric space (X, d) is called *geodesically complete* if every geodesic is a subarc of a metric straight line. If (X, d) is a **complete metric space**, then it is geodesically complete. The **punctured plane** $(\mathbb{R}^2 \setminus \{0\}, \|x - y\|_2)$ is not geodesically complete: any geodesic going to 0 is not a subarc of a metric straight line.

- **Length spectrum**
Given a metric space (X, d), a *closed geodesic* is a map $\gamma : \mathbb{S}^1 \to X$ which is locally minimizing around every point of $\mathbb{S}^1$.
If (X, d) is a compact **length space**, its **length spectrum** is the collection of lengths of closed geodesics. Each length is counted with *multiplicity* equal to the number of distinct *free homotopy* classes that contain a closed geodesic of such length. The **minimal length spectrum** is the set of lengths of closed geodesics which are the shortest in their free homotopy class. Cf. the **distance list**.
- **Systole of metric space**
For any compact metric space (X, d) its **systole** $sys(X, d)$ is the length of the shortest noncontractible loop in X; such a loop is necessarily a closed geodesic. So, $sys(X, d) = 0$ exactly if (X, d) is **simply connected**. Cf. **connected space** in Chap. 2.
If (X, d) is a graph with path metric, then its systole is referred to as the *girth*.
If (X, d) is a closed surface, then its *systolic ratio* is defined to be the ratio $\frac{sys^2(X,d)}{area(X,d)}$.
- **Shankar–Sormani radii**
Given a **geodesic metric space** (X, d), Shankar and Sormani, 2009, defined its **unique injectivity radius** $Uirad(X)$ as the supremum over all $r \geq 0$ such that any two points at distance at most r are joined by a unique geodesic, and its **minimal radius** $Mrad(X)$ as $\inf_{p\in X} d(p, MinCut(p))$.
Here the *minimal cut locus of p* $MinCut(p)$ is the set of points $q \in X$ for which there is a geodesic γ running from p to q such that γ extends past q but is not minimizing from p to any point past q. If (X, d) is a Riemannian space, then the distance function from p is a smooth function except at p itself and the cut locus. Cf. **medial axis and skeleton** in Chap. 21.
It holds $Uirad(X) \leq Mrad(X)$ with equality if (X, d) is a Riemannian space in which case it is the **injectivity radius**. It holds $Uirad(X) = \infty$ for a flat disk but $Mrad(X) < \infty$ if (X, d) is compact and at least one geodesic is extendible.
- **Geodesic convexity**
Given a **geodesic metric space** (X, d) and a subset $M \subset X$, the set M is called **geodesically convex** (or *convex*) if, for any two points of M, there exists a geodesic segment connecting them which lies entirely in M; the space is **strongly convex** if such a segment is unique and no other geodesic connecting those points lies entirely in M. The space is called **locally convex** if such a segment exists for any two sufficiently close points in M.
For a given point $x \in M$, the **radius of convexity** is $r_x = \sup\{r \geq 0 : B(x, r) \subset M\}$, where the **metric ball** $B(x, r)$ is convex. The point x is called the *center of mass* of points $y_1, \dots, y_k \in M$ if it minimizes the function $\sum_i d(x, y_i)^2$ (cf. **Frechét mean**); such point is unique if $d(y_i, y_j) < r_x$ for all $1 \leq i < j \leq k$.
The **injectivity radius** of the set M is the supremum over all $r \geq 0$ such that any two points in M at distance $\leq r$ are joined by unique geodesic segment which lies entirely in M. The **Hawaiian Earring** is a compact complete metric space

consisting of a collection of circles of radius $\frac{1}{i}$ for each $i \in \mathbb{N}$ all joined at a common point; its injectivity radius is 0. It is **path-connected** but not **simply connected**.

The set $M \subset X$ is called a **totally convex metric subspace** of (X, d) if, for any two points of M, any geodesic segment connecting them lies entirely in M.

- **Busemann convexity**

A **geodesic metric space** (X, d) is called **Busemann convex** (or **globally nonpositively Busemann curved**) if, for any three points $x, y, z \in X$ and *midpoints* $m(x, z)$ and $m(y, z)$ (i.e., $d(x, m(x, z)) = d(m(x, z), z) = \frac{1}{2}d(x, z)$ and $d(y, m(y, z)) = d(m(y, z), z) = \frac{1}{2}d(y, z)$), there holds

$$d\big(m(x, z), m(y, z)\big) \le \frac{1}{2}d(x, y).$$

Equivalently, the distance $D(c_1, c_2)$ between any geodesic segments $c_1 = [a_1, b_1]$ and $c_2 = [a_2, b_2]$ is a *convex function*; cf. **metric between intervals** in Chap. 10. (A real-valued function f defined on an interval is called *convex* if $f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda) f(y)$ for any x, y and $\lambda \in (0, 1)$.)

The *flat Euclidean strip* $\{(x, y) \in \mathbb{R}^2 : 0 < x < 1\}$ is **Gromov hyperbolic** but not Busemann convex. In a complete Busemann convex metric space any two points are joined by a unique geodesic segment.

A metric space is **CAT**(0) (cf. Chap. 6) if and only if it is Busemann convex and Ptolemaic (Foertsch, Lytchak and Schroeder, 2007).

A geodesic metric space (X, d) is **Busemann locally convex** (Busemann, 1948) if the above inequality holds locally. Any geodesic **locally CAT**(0) metric space (cf. Chap. 6) is Busemann locally convex, and any geodesic **CAT**(0) metric space is Busemann convex but not vice versa.

- **Menger convexity**

A metric space (X, d) is called **Menger convex** if, for any different points $x, y \in X$, there exists a third point $z \in X$ for which $d(x, y) = d(x, z) + d(z, y)$, i.e., $|I(x, y)| > 2$ holds for the **closed metric interval** $I(x, y) = \{z \in X : (x, y) = d(x, z) + d(z, y)\}$. It is called **strictly Menger convex** if such a z is unique for all $x, y \in X$.

Geodesic convexity implies Menger convexity. The converse holds for **complete** metric spaces.

A subset $M \subset X$ is called (Menger, 1928) a *d-convex set* (or *interval-convex set*) if $I(x, y) \subset M$ for any different points $x, y \in M$. A function $f : M \to \mathbb{R}$ defined on a d-convex set $M \subset X$ is a d-**convex function** if for any $z \in I(x, y) \subset M$

$$f(z) \le \frac{d(y, z)}{d(x, y)} f(x) + \frac{d(x, z)}{d(x, y)} f(y).$$

A subset $M \subset X$ is a *gated set* if for every $x \in X$ there exists a unique $x' \in M$, the *gate*, such that $d(x, y) = d(x, x') + d(x', y)$ for $y \in M$. Any such set is d-convex.