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Variational, Topological, and Partial Order Methods with Their Applications

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Zhitao Zhang
Academy of Mathematics & Systems Science
The Chinese Academy of Sciences
Beijing, P.R. China

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Preface

Nonlinear functional analysis is an important branch of contemporary mathematics; it has grown from geometry, fluid and elastic mechanics, physics, chemistry, biology, control theory and economics, etc. It is related to many areas of mathematics: topology, ordinary differential equations, partial differential equations, groups, dynamical systems, differential geometry, measure theory, etc.

We mainly present our new results on the three fundamental methods in nonlinear functional analysis: Variational, Topological and Partial Order Methods with their Applications. They have been used extensively to solve questions of the existence of solutions for elliptic equations, wave equations, Schrödinger equations, Hamiltonian systems, etc. Also they have been used to study the existence of multiple solutions and the properties of solutions.

Hilbert posed his famous 23 problems on the occasion of his speech at the centennial assembly of the International Congress 1900 in Paris. Three of these were related to the calculus of variations. Included are minimization methods, minimax methods, Morse theory, category, Ljusternik–Schnirelmann theory, etc. in the calculus of variations. We should mention that Ambrosetti and Rabinowitz's work [11] in the 1970s is the beginning of the minimax method, making it possible for people to deal with functionals that are unbounded from below, which come from the study of nonlinear elliptic equations, Hamiltonian systems, geometry, and mathematical physics. In the 1930s, Morse developed a theory which set up the relationship between critical points of a non-degenerate function and the topology of the underlying compact manifold. In the 1960s Palais [149] and Smale [164] et al. extended Morse theory to infinite-dimensional manifolds by using the Palais–Smale condition.

Topological methods and partial order methods are basic and important tools in nonlinear functional analysis too. The Brouwer degree is a powerful tool in algebraic topology; the Leray–Schauder degree is an extension of the Brouwer degree from finite-dimensional spaces to infinite-dimensional Banach spaces, which has been introduced by Leray and Schauder in the study of nonlinear partial differential equations in the 1930s. Rabinowitz's global bifurcation theorem is based on the computation of the Leray–Schauder degree. In many problems that arise in population biology, economics, and the study of infectious diseases, we need to discuss the existence of nonnegative solutions with some desired qualitative properties, so cones are used to develop partial order methods and fixed point index theory. Then one gets fixed point theorems and applications to many kinds of differential equation, etc.

In Chap. 1, we present preliminaries: some basic concepts, and useful famous theorems and results so that the reader may easily find information if need may be.

In Chap. 2, we introduce three kinds of operator: increasing operators, decreasing operators, and mixed monotone operators. Some fixed point theorems and applications to integral equations and differential equations are included. One equivalent condition of the normal cone is given.

In Chap. 3, we present the minimax methods including the Mountain Pass Theorem, linking methods, local linking methods, and critical groups; next, we treat some applications to elliptic boundary value problems.

In Chap. 4, we use bifurcation and critical point theory together to study the structure of the solutions of elliptic equations; also we have results on three sign-changing solutions.

In Chap. 5, we consider the boundary value problems for a class of Monge–Ampère equations. First we prove that any solution on the ball is radially symmetric by the moving plane argument. Then we show that there exists a critical radius such that, if the radius of a ball is smaller than this critical value, then there exists a solution, and vice versa. Using a comparison between domains we prove that this phenomenon occurs for every domain. By using the Lyapunov–Schmidt reduction method we get the local structure of the solutions near a degenerate point; by Leray–Schauder degree theory, a priori estimates, and using bifurcation theory we get the global structure.

In Chap. 6, on superlinear systems of Hammerstein integral equations and applications, we use the Leray–Schauder degree to obtain new results on the existence of solutions, and apply them to two-point boundary problems of systems of equations. We also are concerned with the existence of (component-wise) positive solutions for a semilinear elliptic system, where the nonlinear term is superlinear in one equation and sublinear in the other equation. By constructing a cone $K_1 \times K_2$, which is the Cartesian product of two cones in the space $C(\overline{\Omega})$, and computing the fixed point index in $K_1 \times K_2$, we establish the existence of positive solutions for the system.

In Chap. 7, we show some results on the Dancer–Fučik spectrum for bounded domains. We are concerned with the Fučik point spectrum for Schrödinger operators, $-\Delta + V$, in $L^2(\mathbb{R}^N)$ for certain types of potential, $V : \mathbb{R}^N \to \mathbb{R}$. We use the Dancer–Fučik spectrum to asymptotically linear elliptic problems to get one-sign solutions.

In Chap. 8, we introduce some results on sign-changing solutions of elliptic and p-Laplacian, including using Nehri manifold, invariant sets of descent flows, Morse theory, etc.

In Chap. 9, we show that if $u_0 \in W_0^{1,p}(\Omega)$ is a local minimizer of J in the C^1-topology, it is still a local minimizer of the functional J in $W_0^{1,p}(\Omega)$. This extends the famous results of Brezis–Nirenberg to $p > 2$. We thus obtain multiple solutions and structures of solutions for p-Laplacian equations. Finally, we also show uniqueness results of various kinds.

In Chap. 10, we obtain nontrivial solutions of a class of nonlocal quasilinear elliptic boundary value problems using the Yang index and critical groups, and we obtain sign-changing solutions of a class of nonlocal quasilinear elliptic boundary value problems using variational methods and invariant sets of descent flows. We also show a uniqueness result.

In Chap. 11, we study free boundary problems, Schrödinger systems from Bose–Einstein condensates, and competing systems with many species. We prove the existence and uniqueness result of the Dirichlet boundary value problem of elliptic competing systems. We show that, for the singular limit, species are spatially segregated; they satisfy a remarkable system of differential inequalities as $\kappa \to +\infty$. We also introduce optimal partition problems related to eigenvalues and nonlinear eigenvalues. Finally, some recent new results on Schrödinger systems from Bose–Einstein condensates are presented.

In preparing this manuscript I have received help and encouragement from several professors and from my students. I wish to thank Professor Shujie Li for his kind suggestions. Special thanks go to my students; to Prof. Xiyou Cheng, Dr. Kelei Wang, Dr. Yimin Sun for useful corrections, and to Dr. Yimin Sun and Liming Sun for wonderful typesetting of parts of Chaps. 1, 2, 3, and 11 of this manuscript.

I dedicate this book to my father Deren Zhang, my wife Jimin Fang and my son Fan Zhang.

Beijing, China Zhitao Zhang

Contents

Chapter 1
Preliminaries

1.1 Sobolev Spaces and Embedding Theorems

Let Ω denote a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$. For $p \geq 1$ we let $L^p(\Omega)$ denote the class Banach space consisting of measurable functions on Ω that are p-integrable. The norm of Banach space $L^p(\Omega)$ is defined by

$$\|u\|_{p;\Omega} = \|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\, dx\right)^{1/p}. \tag{1.1}$$

Hölder's inequality: For real numbers p, q satisfying $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_\Omega uv\, dx \leq \|u\|_p \|v\|_q \tag{1.2}$$

for functions $u \in L^p(\Omega)$, $v \in L^q(\Omega)$. It is a consequence of Young's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a \geq 0,\ b \geq 0.$$

As $p = q = 2$, it is the Schwarz inequality.

Generalization of Hölder's inequality:

Let $u_i \in L^{p_i}(\Omega)$, $i = 1, 2, \ldots, m$, $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1$,

$$\int_\Omega u_1 \cdots u_m\, dx \leq \|u_1\|_{p_1} \cdots \|u_m\|_{p_m}. \tag{1.3}$$

Minkowski inequality:

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}, \quad \forall f, g \in L^p(\Omega). \tag{1.4}$$

As $p = 2$, it is Cauchy inequality.

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Definition 1.1.1 (Weak derivatives) Let u be locally integrable in Ω and α any multi-index. Then a locally integrable function v is called the αth weak derivative of u if it satisfies

$$\int_\Omega \varphi v\, dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi\, dx \quad \text{for all } \varphi \in C_0^{|\alpha|}(\Omega).$$

We write $v = D^\alpha u$, and v is uniquely determined up to sets of measure zero. For a non-negative integer vector $\alpha = (\alpha_1, \dots, \alpha_n)$, we denote

$$D^\alpha = \frac{D^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

the differential operator, with $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

A function is called weakly differentiable if all its weak derivatives of first order exist and k times weakly differentiable if all its weak derivatives exist for orders up to and including k. We denote the linear space of k times weakly differentiable functions by $W^k(\Omega)$. Clearly $C^k(\Omega) \subset W^k(\Omega)$. For $p \geq 1$ and k a non-negative integer, let

$$W^{k,p}(\Omega) = \left\{u \in W^k(\Omega);\, D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\right\}, \tag{1.5}$$

with a norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha|\leq k} \left|D^\alpha u\right|^p dx\right)^{1/p}.$$

Then $W^{k,p}(\Omega)$ is a Banach space. We also have an equivalent norm

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha|\leq k} \left\|D^\alpha u\right\|_p.$$

$W_0^{k,p}(\Omega)$ is another Banach space by taking the closure of $C_0^k(\Omega)$ in $W^{k,p}(\Omega)$. $W^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$ are separable for $1 \leq p < \infty$, and reflexive for $1 < p < \infty$.

As $p = 2$, $W^{k,2}(\Omega)$, $W_0^{k,2}(\Omega)$ written as $H^k(\Omega)$, $H_0^k(\Omega)$ are Hilbert spaces under the scalar product

$$(u, v)_k = \int_\Omega \sum_{|\alpha|\leq k} D^\alpha u D^\alpha v\, dx.$$

$W_{\text{loc}}^{k,p}(\Omega)$ are local spaces to be defined to consist of functions belonging to $W^{k,p}(\Omega')$ for all $\Omega' \Subset \Omega$ (i.e., Ω' has compact closure in Ω).

Definition 1.1.2 Assume E_1, E_2 are two normed linear spaces, we call E_1 embedded in E_2, if:

(1) E_1 is a subspace of E_2,

(2) There exists an identity operator $I : E_1 \to E_2$ such that $I(u) = u$, and

$$\left\| I(u) \right\|_{E_2} \leq K \|u\|_{E_1}.$$

If $1 \leq p_1 \leq p_2 \leq \infty,\ u \in L^{p_2}(\Omega)$ then $u \in L^{p_1}(\Omega)$. Also we have

$$L^{p_2}(\Omega) \hookrightarrow L^{p_1}(\Omega).$$

Theorem 1.1.1 *The space $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Theorem 1.1.2 (Sobolev embedding theorems)

$$W_0^{1,p}(\Omega) \begin{cases} \nearrow L^{np/(n-p)}(\Omega), & p < n, \\ \longrightarrow L^\varphi(\Omega), & \varphi = \exp\left(|t|^{n/(n-1)}\right) - 1,\ p = n, \\ \searrow C^\lambda(\bar{\Omega}), & \lambda = 1 - \dfrac{n}{p},\ p > n, \end{cases}$$

where $L^\varphi(\Omega)$ denotes the Orlicz space.

The Poincaré inequality: For $u \in W_0^{1,p}(\Omega),\ 1 \leq p < \infty$

$$\|u\|_p \leq \left(\frac{1}{\omega_n} |\Omega| \right)^{1/n} \|Du\|_p \quad \left(\omega_n = \text{volume of unit ball in } \mathbb{R}^n\right).$$

After extension to the spaces $W_0^{k,p}(\Omega)$, we have

$$W_0^{k,p}(\Omega) \begin{cases} \nearrow L^{np/(n-kp)}(\Omega), & kp < n, \\ \searrow C^m(\bar{\Omega}), & 0 \leq m < k - \dfrac{n}{p}. \end{cases}$$

For $W^{k,p}(\Omega)$, if Ω satisfies a uniform interior cone condition (i.e., there exists a fixed cone K_Ω such that each $x \in \Omega$ is the vertex of a cone $K_\Omega(x) \subset \bar{\Omega}$ and congruent to K_Ω), then there is an embedding

$$W^{k,p}(\Omega) \begin{cases} \nearrow L^{np/(n-kp)}(\Omega), & kp < n, \\ \searrow C_B^m(\Omega), & 0 \leq m < k - \dfrac{n}{p}, \end{cases}$$

where $C_B^m(\Omega) = \{u \in C^m(\Omega) | D^\alpha u \in L^\infty(\Omega) \text{ for } |\alpha| \leq m\}$.

Theorem 1.1.3 (Compactly embedded theorems) *The spaces $W_0^{1,p}(\Omega)$ are compactly embedded* (i) *in the spaces $L^q(\Omega)$ for any $q < np/(n-p)$, if $p < n$, and* (ii) *in $C^0(\bar{\Omega})$, if $p > n$.*

An extension of the above theorem show that the embeddings

$$W_0^{k,p}(\Omega) \begin{array}{ll} \nearrow L^{np/(n-p)}(\Omega), & \text{for } kp < n,\ q < \dfrac{np}{n-kp}, \\ \searrow C^m(\bar{\Omega}), & \text{for } 0 \le m < k - \dfrac{n}{p} \end{array}$$

are compact.

Next define the space

$$H^1(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n)\}$$

with the inner product

$$(u, v)_1 := \int_{\mathbb{R}^n} [\nabla u \cdot \nabla v + uv]$$

and the corresponding norm

$$\|u\|_1 := \left(\int_{\mathbb{R}^n} \left[|\nabla u|^2 + u^2\right]\right)^{1/2}.$$

It is a Hilbert space.

Let $\mathcal{D}(\Omega) := \{u \in C^\infty(\Omega) : \operatorname{supp} u \text{ is a compact subset of } \Omega\}$.

Let Ω be an open subset of $\mathbb{R}^n$, the space $H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\mathbb{R}^n)$.

Let $n \ge 3$ and $2^* := 2n/(n-2)$. The space

$$\mathcal{D}^{1,2}(\mathbb{R}^n) := \{u \in L^{2^*}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n)\}$$

with the inner product $\int_{\mathbb{R}^n} \nabla u \cdot \nabla v$ and the corresponding norm $(\int_{\mathbb{R}^n} |\nabla u|^2)^{1/2}$ is a Hilbert space. The space $\mathcal{D}_0^{1,2}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. We denote $2^* = \infty$ when $n = 1, 2$.

Theorem 1.1.4 (Sobolev embedding theorem) *The following embeddings are continuous*:

$$\begin{array}{ll} H^1(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n), & 2 \le p < \infty,\ n = 1, 2, \\ H^1(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n), & 2 \le p \le 2^*,\ n \ge 3, \\ \mathcal{D}^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n), & n \ge 3. \end{array}$$

In particular, the Sobolev inequality holds:

$$S := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^n),\, |u|_{2^*} = 1} |\nabla u|_2^2 > 0. \tag{1.6}$$

Then it is clear that $H_0^1(\Omega) \subset \mathcal{D}_0^{1,2}(\Omega)$. If $|\Omega| < \infty$, Poincaré inequality implies that $H_0^1(\Omega) = \mathcal{D}_0^{1,2}(\Omega)$.

The instanton:

$$U(x) := \frac{[n(n-2)]^{(n-2)/4}}{[1+|x|^2]^{(n-2)/2}} \tag{1.7}$$

is a minimizer for $S, n \geq 3$ (Aubin and Talenti, see [193]). For every open subset Ω of $\mathbb{R}^n$,

$$S(\Omega) := \inf_{u \in \mathcal{D}^{1,2}(\Omega), |u|_{2^*}=1} |\nabla u|_2^2 = S, \tag{1.8}$$

and $S(\Omega)$ is never achieved except when $\Omega = \mathbb{R}^n$.

By Theorem 4.7.8 of [50] and [39], $U(x)$ is a minimizer for S (1.6) iff $U(x)$ has the form

$$U(x) := \frac{[n(n-2)\theta]^{(n-2)/4}}{[\theta^2 + |x-y|^2]^{(n-2)/2}}, \quad \forall \theta > 0, \ \forall y \in \mathbb{R}^n. \tag{1.9}$$

Theorem 1.1.5 (Strauss [50]) *Let $H_r^1(\mathbb{R}^n)$ be the subspace of $H^1(\mathbb{R}^n)$ consisting of radial symmetric functions. The embedding $H_r^1(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ is compact as $2 < p < 2^*, n \geq 2$.*

Remark 1.1.1 About Sobolev spaces and embedding theorems above, please see [95, 193] etc.

1.2 Critical Point

Definition 1.2.1 Let $J : U \to \mathbb{R}$ where U is an open subset of a Banach space E. The functional J has a Gateaux derivative $f \in E^*$ at $u \in U$, if for every $h \in E$,

$$\lim_{t \to 0} \frac{1}{t} \left| J(u+th) - J(u) - \langle f, th \rangle \right| = 0. \tag{1.10}$$

The functional J has a Fréchet derivative $f \in E^*$ at $u \in U$, if

$$\lim_{h \to 0} \frac{1}{\|h\|} \left| J(u+h) - J(u) - \langle f, h \rangle \right| = 0. \tag{1.11}$$

The functional J belongs to $C^1(U, \mathbb{R})$ if the Fréchet derivative of J exists and is continuous on U.

Any Fréchet derivative is a Gateaux derivative. Using the mean value theorem, it is easy to know that if J has a continuous Gateaux derivative on U, then $J \in C^1(U, \mathbb{R})$.

Suppose J is a Fréchet differentiable functional on a Banach space E with normed dual E^* and duality pairing $\langle \cdot, \cdot \rangle : E \times E^* \to \mathbb{R}$, and let $DJ : E \to E^*$ denote the Fréchet-derivative of J. Then the directional (Gateaux-) derivative of J

at u in direction v is given by

$$\frac{d}{dt}J(u+tv)\bigg|_{t=0} = \langle v, DJ(u)\rangle = DJ(u)v.$$

For such J, we call $u \in E$ a critical point if $J'(u) := DJ(u) = 0$; otherwise u is called a regular point. A number $\alpha \in \mathbb{R}$ is a critical value of J if there exists a critical point u of J with $J(u) = \alpha$. Otherwise, α is called regular.

Let $C^1(E, \mathbb{R})$ denote the set of functionals that are Fréchet differentiable and whose Fréchet derivatives are continuous on E.

Definition 1.2.2 For $J \in C^1(E, \mathbb{R})$, we say J satisfies the *Palais–Smale condition* (henceforth denoted by (PS) condition) if any sequence $\{u_m\} \subset E$ for which $J(u_m)$ is bounded and $J'(u_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence.

Definition 1.2.3 For $J \in C^1(E, \mathbb{R})$, we say J satisfies the $(\mathrm{PS})_c$ condition if any sequence $\{u_m\} \subset E$ for which $J(u_m) \to c$ and $J'(u_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence.

It is clear that if J satisfies the $(\mathrm{PS})_c$ condition, for $\forall c$, then J satisfies the (PS) condition.

The (PS) condition is a kind of compact condition. Indeed observe that the (PS) condition implies that $K_c \equiv \{u \in E | J(u) = c, J'(u) = 0\}$, i.e., the set of critical points having critical value c, is compact for any $c \in R$.

Theorem 1.2.1 (Ekeland variational principle [50]) *Let (X, d) be a complete metric space, and let $f : X \to \mathbb{R} \cup \{+\infty\}$, but $f \not\equiv +\infty$. If f is bounded from below and lower semi-continuous* (*l.s.c.*, $\forall\lambda \in \mathbb{R}$, *the level set* $f_\lambda = \{x \in X | f(x) \leq \lambda\}$ *is closed*), *and if* $\exists\varepsilon > 0$, $\exists x_\varepsilon \in X$ *satisfying* $f(x_\varepsilon) < \inf_X f + \varepsilon$. *Then* $\exists y_\varepsilon \in X$ *such that*

1. $f(y_\varepsilon) \leq f(x_\varepsilon)$,
2. $d(x_\varepsilon, y_\varepsilon) \leq 1$,
3. $f(x) > f(y_\varepsilon) - \varepsilon d(y_\varepsilon, x)$, $\forall x \neq y_\varepsilon$.

Theorem 1.2.2 (Pohozaev identity, 1965) *For the solution of*

$$-\Delta u = f(u), \quad u \in H_0^1(\Omega), \tag{1.12}$$

where $f \in C^1(\mathbb{R}, \mathbb{R})$ *and* Ω *is a smooth bounded domain of* $\mathbb{R}^n$, $n \geq 3$. *Let* $F(u) = \int_0^u f(s)\,ds$.

Let $u \in H^2_{\mathrm{loc}}(\bar{\Omega})$ *be a solution of* (1.12) *such that* $F(u) \in L^1(\Omega)$. *Then u satisfies the following*:

$$\frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \sigma \cdot \nu \, d\sigma = n\int_\Omega F(u)\,dx - \frac{n-2}{2}\int_\Omega |\nabla u|^2\,dx,$$

where ν *denotes the unit outward normal to* $\partial\Omega$. (*For the proof see* [168, 193].)

For P.L. Lions' Concentration-Compactness Principle On the basis of this principle, for many constrained minimization problems it is possible to state necessary and sufficient conditions for the convergence of all minimizing sequences satisfying the given constraint.

Theorem 1.2.3 (Concentration-Compactness Principle, see [135, 136, 168]) *Suppose that* μ_m *is a sequence of probability measures on* $\mathbb{R}^n : \mu_m \geq 0, \int_{\mathbb{R}^n} \mu_m = 1$. *Then there is a subsequence* (μ_m) *satisfying one of the following three possibilities*:

(1) (Compactness) $\exists \{x_m\} \subset \mathbb{R}^n$ *such that for any* $\varepsilon > 0, \ \exists R > 0$ *with the property that*

$$\int_{B_R(x_m)} d\mu_m \geq 1 - \varepsilon \quad \textit{for all } m.$$

(2) (Vanishing) *For all* $R > 0$, *there holds*

$$\lim_{m \to \infty} \left(\sup_{x \in \mathbb{R}^n} \int_{B_R(x)} d\mu_m \right) = 0.$$

(3) (Dichotomy) $\exists \lambda, \ 0 < \lambda < 1$, *such that* $\forall \varepsilon > 0, \ \exists R > 0$ *and* $\exists \{x_m\}$ *with the following property*: *Given* $R' > R$ *there are non-negative measures* μ_m^1, μ_m^2 *such that*

$$0 \leq \mu_m^1 + \mu_m^2 \leq \mu_m, \qquad \operatorname{supp}(\mu_m^1) \subset B_R(x_m), \qquad \operatorname{supp}(\mu_m^2) \subset \mathbb{R}^n \setminus B_{R'}(x_m),$$

$$\limsup_{m \to \infty} \left(\left| \lambda - \int_{\mathbb{R}^n} d\mu_m^1 \right| + \left| (1 - \lambda) - \int_{\mathbb{R}^n} d\mu_m^2 \right| \right) \leq \varepsilon.$$

Let X be a topological space. A deformation of X is a continuous map $\eta : X \times [0, 1] \to X$ such that $\eta(\cdot, 0) = \mathrm{id}$.

Definition 1.2.4 For a topological pair $Y \subset X$. A continuous map $r : X \to Y$ is called a deformation retract, if $r \circ i = \mathrm{id}_Y$ and $i \circ r \sim \mathrm{id}_X$, where $i : Y \to X$ is the injection. In this case Y is called a deformation retraction of X.

Definition 1.2.5 A deformation retract r is called a strong deformation retract, if there exists a deformation $\eta : X \times [0, 1] \to X$, such that $\eta(\cdot, t)|_Y = \mathrm{id}_Y, \forall t \in [0, 1]$ and $\eta(\cdot, 1) = i \circ r$. Then Y is called a strong deformation retraction of X.

Definition 1.2.6 Let E be a real Banach space, $U \subset E$, and $I \in C^1(U, \mathbb{R})$. Then $v \in E$ is called a pseudo-gradient vector for I at $u \in U$ if

$$\text{(i)} \quad \|v\| \leq 2 \|I'(u)\|,$$

$$\text{(ii)} \quad I'(u)v \geq \|I'(u)\|^2.$$

Note that a pseudo-gradient vector is not unique in general and any convex combination of pseudo-gradient vectors for I at u is also a pseudo-gradient vector for I at u.

Let $I \in C^1(E, \mathbb{R})$ and $K \equiv \{u \in E | I'(u) = 0\}$, $\tilde{E} \equiv E \setminus K \equiv \{u \in E | I'(u) \neq 0\}$. Then $V : \tilde{E} \to E$ is called a pseudo-gradient vector field on $\tilde{E}$ if V is locally Lipschitz continuous and $V(u)$ is a pseudo-gradient vector for I for all $u \in \tilde{E}$.

Theorem 1.2.4 (See [159]) *If $I \in C^1(E, \mathbb{R})$, there exists a pseudo-gradient vector field for I on $\tilde{E}$. If $I(u)$ is even in u, I has a pseudo-gradient vector field on $\tilde{E}$ given by an odd function W.*

Using the pseudo-gradient vector field, one can construct a deformation by modified negative gradient flow for I.

Recall that $I_s \equiv \{u \in E | I(u) \leq s\}$ for $s \in \mathbb{R}$, sometimes also write $I^s := \{u \in E | I(u) \leq s\}$ and we recall the following version of Deformation Theorems.

Theorem 1.2.5 (Noncritical interval theorem, see [50]) *If $I \in C^1(E, \mathbb{R})$ satisfies* $(\mathrm{PS})_c, \forall c \in [a, b]$ *and if $K \cap I^{-1}[a, b] = \emptyset$, then I_a is a strong deformation retraction of I_b.*

Theorem 1.2.6 (Second deformation theorem, see [50]) *If $I \in C^1(E, \mathbb{R})$ satisfies* $(\mathrm{PS})_c$, $\forall c \in [a, b]$, *if $K \cap I^{-1}(a, b] = \emptyset$ and the connected components of $K \cap I^{-1}(a)$ are only isolated points, then I_a is a strong deformation retraction of I_b.*

Theorem 1.2.7 (See [159]) *Let E be a real Banach space and let $I \in C^1(E, \mathbb{R})$ and satisfy* (PS) *condition. If $c \in R$, $\bar{\varepsilon} > 0$, and Θ is any neighborhood of K_c, then there exist an $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0, 1] \times E, E)$ such that*

(1) *$\eta(0, u) = u$ for all $u \in E$.*
(2) *$\eta(t, u) = u$ for all $t \in [0, 1]$ if $I(u) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}]$.*
(3) *$\eta(t, u)$ is a homeomorphism of E onto E for each $t \in [0, 1]$.*
(4) *$\|\eta(t, u) - u\| \leq 1$ for all $t \in [0, 1]$ and $u \in E$.*
(5) *$I(\eta(t, u)) \leq I(u)$ for all $t \in [0, 1]$ and $u \in E$.*
(6) *$\eta(1, I_{c+\varepsilon} \setminus \Theta) \subset I_{c-\varepsilon}$.*
(7) *If $K_c = \emptyset$, $\eta(1, I_{c+\varepsilon}) \subset I_{c-\varepsilon}$.*
(8) *If $I(u)$ is even in u, $\eta(t, u)$ is odd in u.*

Definition 1.2.7 The action of a topological group G on a normed space X is a continuous map

$$G \times X \to X : [g, u] \to gu$$

such that $1 \cdot u = u$, $(gh)u = g(hu)$, $u \to gu$ is linear.

The action is isometric if

$$\|gu\| = \|u\|.$$

The space of invariant points is defined by

$$\mathrm{Fix}(G) := \{u \in X : gu = u, \forall g \in G\}.$$

A set $A \subset X$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : X \to \mathbb{R}$ is invariant if $\varphi \circ g = \varphi$ for every $g \in G$. A map $f : X \to X$ is equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Theorem 1.2.8 (Principle of symmetric criticality, Palais [150]) *Assume that the action of the topological group G on the Hilbert space X is isometric. If $\varphi \in C^1(X, \mathbb{R})$ is invariant and if u is a critical point of φ restricted to* Fix(G), *then u is a critical point of φ.*

The following part of this section can be seen in [49, 50].

Definition 1.2.8 Let I be a C^1 function defined on a Banach space E, let p be an isolated critical point of I, and let $c = I(p)$.

$$C_q(I, p) = H_q\big(I_c \cap U, \big(I_c \setminus \{p\}\big) \cap U; G\big) \tag{1.13}$$

is called the qth critical group of I at $p, q = 0, 1, 2, \ldots,$ where G is the coefficient group, U is a neighborhood of p such that $K \cap (I_c \cap U) = \{p\}$, and $H_*(X, Y; G)$ stands for the singular relative homology groups with the Abelian coefficient group G.

Definition 1.2.9 Let p be a non-degenerate critical point of I, we call the dimension of the negative space corresponding to the spectral decomposition of $I''(p)$, the Morse index of p, and denote it by ind(I, p).

Example 1.2.1 If p is an isolated minimum point of I, then

$$C_q(I, p) = \delta_{q0} \cdot G.$$

Example 1.2.2 If E is n-dimensional, and p is an isolated local maximum point of I, then

$$C_q(I, p) = \delta_{qn} \cdot G.$$

Example 1.2.3 If $I \in C^2(E, \mathbb{R})$ and p is a non-degenerate critical point of I with Morse index j; then

$$C_q(I, p) = \delta_{qj} \cdot G.$$

Suppose that $f \in C^1(E, \mathbb{R})$ has only isolated critical values, and that each of them corresponds to a finite number of critical points; say

$$\cdots < c_{-2} < c_{-1} < c_0 < c_1 < c_2 < \cdots$$

are critical values with

$$K \cap f^{-1}(c_i) = \{z_j^i\}_{j=1}^{m_i}, \quad i = 0, \pm 1, \pm 2, \ldots.$$

One chooses

$$0 < \varepsilon_i < \max\{c_{i+1} - c_i, c_i - c_{i-1}\}, \quad i = 0, \pm 1, \pm 2, \ldots.$$

Definition 1.2.10 For a pair of regular values $a < b$, we call

$$M_q(a,b) = \sum_{a<c_i<b} \operatorname{rank} H_q(f_{c_i+\varepsilon_i}, f_{c_i-\varepsilon_i}; G)$$

the qth Morse type number of the function f on (a, b), $q = 0, 1, 2, \ldots$.

Theorem 1.2.9 (See [50]) *Assume that $f \in C^1(E, \mathbb{R})$ satisfies the* (PS) *condition, and has an isolated critical value c, with $K \cap f^{-1}(c) = \{z_j\}_{j=1}^m$. Then for sufficiently small $\varepsilon > 0$ we have*

$$H_*(f_{c+\varepsilon}, f_{c-\varepsilon}; G) = \bigoplus_{j=1}^{m} C_*(f, z_j) \quad \textit{and} \quad M_*(a,b) = \sum_{a<c_i<b} \sum_{j=1}^{m_i} \operatorname{rank} C_*\big(f, z_j^i\big).$$

Define the qth Betti number

$$\beta_q = \beta_q(a,b) = \operatorname{rank} H_q(f_b, f_a; G), \quad q = 0, 1, \ldots.$$

Theorem 1.2.10 (Morse relation [50]) *Suppose that $f \in C^1(E, \mathbb{R})$ satisfies* $(\mathrm{PS})_c$, $\forall c \in [a, b]$, *where a and b are regular values. Assume $(K \cap f^{-1}[a, b])$ is finite. Moreover, if all $M_q(a, b)$ and $\beta_q(a, b)$ are finite, and only finitely many of them are non-zeroes, then*

$$\sum_{q=0}^{\infty} \big(M_q(a,b) - \beta_q(a,b)\big) t^q = (1+t) Q(t), \tag{1.14}$$

where $Q(t)$ is a formal series with non-negative coefficients. In particular, $\forall p = 0, 1, 2, \ldots$,

$$\sum_{q=0}^{p} (-1)^{p-q} M_q(a,b) \geq \sum_{q=0}^{p} (-1)^{p-q} \beta_q(a,b). \tag{1.15}$$

More specifically,

$$\sum_{q=0}^{\infty} (-1)^q M_q(a,b) = \sum_{q=0}^{\infty} (-1)^q \beta_q(a,b). \tag{1.16}$$

1.3 Cone and Partial Order

Definition 1.3.1 Let E be a real Banach space. A nonempty convex closed set $P \subset E$ is called a cone if it satisfies the following two conditions:

(i) $x \in P,\ \lambda \geq 0$ implies $\lambda x \in P$,
(ii) $x \in P,\ -x \in P$ implies $x = \theta$, where θ denotes the zero element of E.

Every cone P in E defines a partial ordering in E given by $x \leq y$ iff $y - x \in P$. If $x \leq y$ and $x \neq y$, we write $x < y$.

Definition 1.3.2 A cone P is called solid if it contains interior points, i.e., $\text{int}(P) \neq \emptyset$, or denote $\mathring{P} \neq \emptyset$.

Definition 1.3.3 A cone P is called generating if $E = P - P$, i.e., every element $x \in E$ can be represented in the form $x = u - v$, where $u, v \in P$.

Definition 1.3.4 A cone $P \subset E$ is said to be normal if there exists a positive constant δ such that $\|x + y\| \geq \delta,\ \forall x, y \in P$ and $\|x\| = \|y\| = 1$.

If cone P is solid and $y - x \in \mathring{P} \neq \emptyset$, we write $x \ll y$.
Here we list the definitions of different cones:

(a) A cone $P \subset E$ is called regular if every increasing and bounded in order sequence in E has a limit, i.e., if $\{x_n\} \subset E$ and $y \in E$ satisfy

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y,$$

then there exists $x^* \in E$ such that $\|x_n - x^*\| \to 0$.
(b) A cone $P \subset E$ is called fully regular if every increasing and bounded in norm sequence in E has a limit, i.e., if $\{x_n\} \subset E$ satisfies

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots, \qquad M = \sup_n \|x_n\| < \infty,$$

then there exists $x^* \in E$ such that $\|x_n - x^*\| \to 0$.
(c) A cone $P \subset E$ is called minihedral if $\sup\{x, y\}$ exists for any pair x, y, where $\sup D$ is the least upper bound of a set D.
(d) A cone $P \subset E$ is called strongly minihedral if $\sup D$ exists for any bounded above in order set $D \subset E$.

For normal cones, we have

Theorem 1.3.1 *Assume P is a cone of E, the following conclusions are equivalent*:

(a) *P is normal*;
(b) *there exists a constant $\delta > 0$ such that $\|x + y\| \geq \delta \max\{\|x\|, \|y\|\}$ for all x, $y \in P$*;

(c) *there exists a constant* $N > 0$ *such that* $\theta \le x \le y$ *implies* $\|x\| \le N\|y\|$;
(d) $x_n \le z_n \le y_n$ $(n = 1, 2, 3, \ldots)$ *and* $\|x_n - x\| \to 0$, $\|y_n - x\| \to 0$ *imply* $\|z_x - x\| \to 0$;
(e) *set* $(B + P) \cap (B - P)$ *is bounded, where* $B = \{x \in E \,|\, \|x\| \le 1\}$;
(f) *every order interval* $[x, y] = \{z \in E | x \le z \le y\}$ *is bounded.*

Theorem 1.3.2 *The following assertions hold*:

(i) *If* E *is reflexive, then* P *is normal* $\Leftrightarrow$ P *is regular* $\Leftrightarrow$ P *is fully regular.*
(ii) *If* E *is separable and reflexive and the cone* $P \subset E$ *is normal and minihedral,* P *is strongly minihedral.*

Zorn's lemma *Suppose a partially ordered set* S *has the property that every chain (i.e., totally ordered subset) has an upper bound in* S*. Then the set* S *contains at least one maximal element.*

Note that the content of this section can be seen in [100, 110].

1.4 Brouwer Degree

Theorem 1.4.1 (Sard, see [165]) *Let* U *be an open set of* $\mathbb{R}^p$ *and* $f : U \to \mathbb{R}^q$ *be a* C^s *map where* $s > \max\{p - q, 0\}$*. Then the set of critical values in* $\mathbb{R}^q$ *has measure zero.*

(This section is included in [81].)

Definition 1.4.1 Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in C^1(\bar{\Omega})$ and $y \in \mathbb{R}^n \backslash f(\partial\Omega \cup S_f)$, where $S_f(\Omega) = \{x \in \Omega : J_f(x) = 0\}$. Then we define

$$d(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn} J_f(x) \quad \left(\text{agreement: } \sum_{\emptyset} = 0\right).$$

If y is a regular value of f then $f(x) = y$ has at most finitely many solutions. So Definition 1.4.1 is reasonable. When y is a singular value of f, we have

Definition 1.4.2 Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in C^2(\bar{\Omega})$ and $y \notin f(\partial\Omega)$. Then we define $d(f, \Omega, y) = d(f, \Omega, y^1)$, where y^1 is any regular value of f such that $|y^1 - y| < \varrho(y, f(\partial\Omega))$ and $d(f, \Omega, y^1)$ is given by Definition 1.4.1.

In fact, the smooth assumption of f in Definitions 1.4.1 and 1.4.2 can be relaxed to $C(\bar{\Omega})$.

Definition 1.4.3 Let $f \in C(\bar{\Omega})$ and $y \in \mathbb{R}^n \backslash f(\partial\Omega)$. Then we define $d(f, \Omega, y) := d(g, \Omega, y)$, where $g \in C^2(\bar{\Omega})$ is any map such that $|g - f|_0 < \varrho(y, f(\partial\Omega))$ and $d(g, \Omega, y)$ is given by Definition 1.4.2.

Theorem 1.4.2 *Let $M = \{(f, \Omega, y) : \Omega \subset \mathbb{R}^n$ open bounded, $f \in C(\bar{\Omega})$ and $y \notin f(\partial\Omega)\}$ and $d : M \to \mathbb{Z}$ the topological degree defined by Definition 1.4.3. Then d has the following properties*:

(d1) $d(\mathrm{id}, \Omega, y) = 1$ *for* $y \in \Omega$.
(d2) $d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$ *whenever* Ω_1 *and* Ω_2 *are disjoint open subsets of* Ω *such that* $y \notin \bar{\Omega}\backslash(\Omega_1 \cup \Omega_2)$.
(d3) $d(h(t, \cdot), \Omega, y(t))$ *is independent of* $t \in [0, 1]$ *whenever* $h : [0, 1] \times \bar{\Omega} \to \mathbb{R}^n$ *is continuous,* $y : [0, 1] \to \mathbb{R}^n$ *is continuous and* $y(t) \notin h(t, \cdot)(\partial\Omega)$ *on* $[0, 1]$.
(d4) $d(f, \Omega, y) \neq 0$ *implies* $f^{-1}(y) \neq \emptyset$.
(d5) $d(\cdot, \Omega, y)$ *and* $d(f, \Omega, \cdot)$ *are constant on* $\{g \in C(\bar{\Omega}) : |g - f|_0 < r\}$ *and* $B_r(y) \subset \mathbb{R}^n$, *respectively, where* $r = \varrho(y, f(\partial\Omega))$. *Moreover,* $d(f, \Omega, \cdot)$ *is constant on every connected component of* $\mathbb{R}^n \backslash f(\partial\Omega)$.
(d6) $d(g, \Omega, y) = d(f, \Omega, y)$ *whenever* $g|_{\partial\Omega} = f|_{\partial\Omega}$.
(d7) $d(f, \Omega, y) = d(f, \Omega_1, y)$ *for every open subset* Ω_1 *of* Ω *such that* $y \notin f(\bar{\Omega}\backslash\Omega_1)$.

Theorem 1.4.3 *Let X_n be a real topological vector space of $\dim X_n = n$, X_m a subspace with $\dim X_m = m < n$, $\Omega \subset X_n$ open bounded, $f : \bar{\Omega} \to X_m$ continuous and $y \in X_m\backslash g(\partial\Omega)$, where $g = \mathrm{id} - f$. Then $d(g, \Omega, y) = d(g|_{\overline{\Omega \cap X_m}}, \Omega \cap X_m, y)$.*

1.5 Compact Map and Leray–Schauder Degree

This section is included in Deimling [81].

1.5.1 Definitions

Consider two Banach spaces X and Y, a subset Ω of X and a map $F : \Omega \to Y$. Then F is said to be compact if it is continuous and such that $F(\Omega)$ is relatively compact. $\mathcal{K}(\Omega, Y)$ will denote the class of compact maps and we shall write $\mathcal{K}(\Omega)$ instead of $\mathcal{K}(\Omega, X)$.

F is said to be completely continuous if it is continuous and maps bounded subsets of Ω into relatively compact sets. F is said to be finite-dimensional if $F(\Omega)$ is contained in a finite-dimensional subspace of Y. The class of all finite-dimensional compact maps will be denoted by $\mathcal{F}(\Omega, Y)$ and we shall write $\mathcal{F}(\Omega)$ instead of $\mathcal{F}(\Omega, X)$. Instead of "maps" we shall also speak of "operators".

If $F : X \to Y$ is linear and maps bounded sets into relatively compact sets then it is automatically continuous, and if it is linear and finite-dimensional then it is automatically compact.

Finally, let $\Omega \subset X$ be closed and bounded. Then $F : \Omega \to Y$ is said to be proper if $F^{-1}(K)$ is compact whenever K is compact. Let us note that a continuous proper map is closed, that is, $F(A)$ is closed whenever $A \subset \Omega$ is closed. In fact, if $(x_n) \subset A$ and $Fx_n \to y$ then $(x_n) \subset F^{-1}(\{Fx_n : n \geq 1\} \cup \{y\})$ and therefore (x_n) has a cluster point $x_0 \in A$, and $y = Fx_0 \in F(A)$. Next, we introduce some useful properties.

1.5.2 Properties of Compact Maps

Definition 1.5.1 Let X be a Banach space and $\mathbb{B}$ its bounded sets. Then $\alpha : \mathbb{B} \to \mathbb{R}^+$, defined by

$$\alpha(B) = \inf\{d > 0 : B \text{ admits a finite cover by sets of diameter } \le d\},$$

is called the (Kuratowski-) measure of noncompactness, the α-MNC for short, and $\beta : \mathbb{B} \to \mathbb{R}^+$ defined by

$$\beta(B) = \inf\{r > 0 : B \text{ can be covered by finitely many balls of radius } r\},$$

is called the ball measure of noncompactness. (Here $\operatorname{diam} B = \sup\{|x - y| : x \in B, y \in B\}$.)

Proposition 1.5.1 *Let X be a Banach space with $\dim X = \infty$, $\mathbb{B}$ the family of all bounded sets of X, and $\gamma : \mathbb{B} \to \mathbb{R}^+$ be either α or β. Then*

(a) $\gamma(B) = 0$ *iff $\bar{B}$ is compact.*
(b) *γ is a seminorm, i.e., $\gamma(\lambda B) = |\lambda|\gamma(B)$ and $\gamma(B_1 + B_2) \le \gamma(B_1) + \gamma(B_2)$.*
(c) *$B_1 \subset B_2$ implies $\gamma(B_1) \le \gamma(B_2)$; $\gamma(B_1 \cup B_2) = \max\{\gamma(B_1), \gamma(B_2)\}$.*
(d) $\gamma(\operatorname{conv} B) = \gamma(B)$.
(e) *γ is continuous with respect to the Hausdorff distance ϱ_H, defined by*

$$\varrho_H(B_1, B_2) = \max\left\{\sup_{B_1} \varrho(x, B_2),\ \sup_{B_2} \varrho(x, B_1)\right\};$$

in particular $\gamma(\bar{B}) = \gamma(B)$.

Together with the degree for finite-dimensional spaces the following proposition will be essential to obtain a degree for compact perturbations of the identity.

Proposition 1.5.2 *Let X and Y be Banach spaces, and $B \subset X$ closed bounded. Then*

(a) *$\mathcal{F}(B, Y)$ is dense in $\mathcal{K}(B, Y)$ with respect to the sup norm, i.e. for $F \in \mathcal{K}(B, Y)$ and $\varepsilon > 0$ there exists $F_\varepsilon \in \mathcal{F}(B, Y)$ such that $\sup_B |Fx - F_\varepsilon x| \le \varepsilon$.*
(b) *If $F \in \mathcal{K}(B)$ then $I - F$ is proper.*

Proof To prove (a), let $F \in \mathcal{K}(B, Y)$, $\varepsilon > 0$ and $y_1, y_2, \ldots, y_p$ such that $\overline{F(B)} \subset \bigcup_{i=1}^p B_i(y_i)$. Let $\varphi_i(y) = \max\{0, \varepsilon - |y - y_i|\}$ and $\psi_i(y) = \varphi_i(y) / \sum_{j=1}^p \varphi_j(y)$ for $y \in \overline{F(B)}$, and define $F_\varepsilon(x) = \sum_{i=1}^p \psi_i(Fx) y_i$ for $x \in B$. Then F_ε is continuous, $F_\varepsilon(B) \subset \{y_1, \ldots, y_p\}$, $F_\varepsilon(B)$ is relatively compact and $\sup_B |F_\varepsilon x - Fx| \le \varepsilon$.

To prove (b), let $A = (I - F)^{-1}(K)$ and K compact. Then $\alpha(A) \le \alpha(F(A)) + \alpha(K) = 0$ and A is closed, and therefore compact. □

For differentiable compact maps we have

Proposition 1.5.3 *Let X, Y be Banach space, $\Omega \subset X$ be open, $F \in \mathcal{K}(\Omega, Y)$ and F is differentiable at $x_0 \in \Omega$. Then $F'(x_0)$ is completely continuous.*

Proof Since $F'(x_0) \in L(X, Y)$, it is sufficient to prove that $F'(x_0)(B_1(0))$ is relatively compact. Recall that $F(x_0 + h) = Fx_0 + F'(x_0)h + \omega(x_0; h)$ with $|\omega(x_0; h)| \leq \varepsilon\delta$ for $|h| \leq \delta = \delta(x_0, \varepsilon)$. Therefore,

$$\delta F'(x_0)\big(B_1(0)\big) = F'(x_0)\big(B_\delta(0)\big) \subset -Fx_0 + F\big(B_\delta(0)\big) + \delta B_\varepsilon(0),$$

and this implies $\delta \cdot \alpha(F'(x_0)(B_1(0))) \leq 2\varepsilon\delta$, i.e., $\alpha(F'(x_0)(B_1(0))) = 0$ since $\varepsilon > 0$ has been arbitrary. □

Proposition 1.5.4 *Let X, Y be Banach spaces, $A \subset X$ closed bounded and $F \in \mathcal{K}(A, Y)$. Then F has an extension $\tilde{F} \in \mathcal{K}(X, Y)$ such that $\tilde{F}(X) \subset \operatorname{conv} F(A)$.*

1.5.3 The Leray–Schauder Degree

Let X be a real Banach space, $\Omega \subset X$ open bounded, $F \in \mathcal{K}(\Omega)$ and $y \notin (I - F)(\partial\Omega)$. On these admissible triplets $(I - F, \Omega, y)$ we want to define a $\mathbb{Z}$-valued function D that satisfies the three basic conditions corresponding to (D1)–(D3) of the Brouwer degree, namely

(D1) $D(I, \Omega, y) = 1$ for $y \in \Omega$;

(D2) $D(I - F, \Omega, y) = D(I - F, \Omega_1, y) + D(I - F, \Omega_2, y)$ whenever Ω_1 and Ω_2 are disjoint open subsets of Ω such that $y \notin (I - F)(\bar{\Omega}\backslash(\Omega_1 \cup \Omega_2))$;

(D3) $D(I - H(t, \cdot), \Omega, y(t))$ is independent of $t \in [0, 1]$ whenever $H : [0, 1] \times \bar{\Omega} \to X$ is compact, $y : [0, 1] \to X$ is continuous and $y(t) \notin (I - H(t, \cdot))(\partial\Omega)$ on $[0, 1]$.

Since $G = I - F$ is proper and $y \notin G(\partial\Omega)$, we have $\varrho = \varrho(y, G(\partial\Omega)) > 0$, and if we choose $F_1 \in \mathcal{F}(\bar{\Omega})$ such that $\sup\{|F_1x - Fx| : x \in \bar{\Omega}\} < \varrho$, then $H(t, x) = Fx + t(F_1x - Fx)$ satisfies (D3) with $y(t) \equiv y$, and therefore $D(I - F, \Omega, y) = D(I - F_1, \Omega, y)$.

Next, since $F_1(\bar{\Omega})$ is contained in a finite-dimensional subspace, we may choose a subspace X_1 with $\dim X_1 < \infty$ such that $y \in X_1$ and $F(\bar{\Omega}) \subset X_1$.

Then $x - F_1x = y$ for some $x \in \Omega$ implies that x is already in $\Omega \cap X_1$ and this suggests that $D(I - F_1, \Omega, y)$ should already be determined by the Brouwer degree of $(I - F_1)|_{\overline{\Omega \cap X_1}}$ with respect to $\Omega \cap X_1$ and y. Notice, in particular, that $\Omega \cap X_1 = \emptyset$ implies $0 = D(I - F_1, \Omega, y) = D(I - F, \Omega, y)$, by (D2).

To make this precise, notice first that there exists a continuous projection P_1 from X onto X_1. Then $X = X_1 \oplus X_2$, where $X_2 = P_2(x)$, $P_2 = I - P_1$, and X_2 is closed since P_2 is continuous. Let $\Omega_1 = \Omega \cap X_1 \neq \emptyset$ and $\tilde{F}_1 : X_1 \to X_1$ be any continuous extension of $F_1|_{\bar{\Omega}_1}$. Then we obtain $D(I - F_1, \Omega; y) = D(I - \tilde{F}_1 P_1, \Omega, y)$, by means of (D3) applied to $H(t, x) = tF_1x + (1 - t)\tilde{F}_1 P_1x$ and $y(t) \equiv y$. But all solutions in Ω of $x - \tilde{F}_1 P_1x = y$ belong to Ω_1 and therefore (D2) tells us that we may replace Ω by any bounded open set which contains Ω_1, for example by $\Omega_1 + B_1(0)$, where $B_1(0)$ is the unit ball of X_2. Hence, we have

$$D(I - F, \Omega, y) = D(I - F_1, \Omega, y) = D\big(I - \tilde{F}_1 P_1, \Omega_1 + B_1(0), y\big)$$
$$= D\big(I - F_1 P_1, \Omega_1 + B_1(0), y\big).$$

Now, you will guess how we have to proceed. Given any open bounded set $\Omega_1 \subset X_1$, $f \in \bar{\Omega}_1 \to X_1$ continuous and $y \in X_1 \backslash f(\partial\Omega_1)$, we define

$$d_0(f, \Omega_1, y) = D\big(I - (I - f)P_1, \Omega_1 + B_1(0), y\big)$$
$$= D\big(f P_1 + P_2, \Omega_1 + B_1(0), y\big).$$

Then (D1)–(D3) imply that d_0 satisfies (d1)–(d3), and therefore d_0 is the Brouwer degree for X_1. In particular, choosing $f = (I - F)|_{\overline{\Omega \cap X_1}}$, we obtain

$$D(I - F_1, \Omega, y) = d_0\big((I - F)|_{\overline{\Omega \cap X_1}}, \overline{\Omega \cap X_1}, y\big).$$

Thus, there is at most on function D. But the construction of D is now a simple exercise in using Theorem 1.4.3. In fact, if F_2 and X_2 satisfy the same conditions as F_1 and X_1, we let X_0 be the span of X_1 and X_2 and $\Omega_0 = \Omega \cap X_0$. Then Theorem 1.4.3 implies

$$d\big((I - F_i)|_{\bar{\Omega}_0}, \Omega_0, y\big) = d\big((I - F_i)|_{\bar{\Omega}_i}, \Omega_i, y\big) \quad \text{for } i = 1, 2$$

and since $x - h(t, x) \neq y$ on $[0, 1] \times \partial\Omega_0$ for $h(t, x) = tF_1 x + (1 - t)F_2 x$, (d3) implies $d((I - F_1)|_{\bar{\Omega}_0}, \Omega_0, y) = d((I - F_2)|_{\bar{\Omega}_0}, \Omega_0, y)$. Therefore, we define $D(I - F, \Omega, y)$ by $d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y)$ for any pair F_1 and X_1 of the type mentioned above. Let us write down this result as

Theorem 1.5.1 *Let X be a real Banach space and*

$$M = \big\{(I - F, \Omega, y) : \Omega \subset X \textit{ open bounded}, \ F \in \mathcal{K}(\bar{\Omega}) \textit{ and } y \notin (I - F)(\partial\Omega)\big\}.$$

Then there exists exactly a function $D : M \to \mathbb{Z}$, the Leray–Schauder degree, satisfying (D1)–(D3). *The integer $D(I - F, \Omega, y)$ is given by $d((I - F_1)|_{\Omega_1}, \Omega_1, y)$, where F_1 is any map in $\mathcal{F}(\bar{\Omega})$ such that* $\sup_{\bar{\Omega}} |F_1 x - F x| \leq \varrho(y, (I - F)(\partial\Omega))$, $\Omega_1 = \Omega \cap X_1$, *and X_1 is any subsequence of X such that* $\dim X_1 < \infty$, $y \in X_1$ *and* $F_1(\bar{\Omega}) \subset X_1$, *and d is the Brouwer degree of X_1.*

Further properties of the Leray–Schauder degree

Theorem 1.5.2 *Besides* (D1)–(D3), *the Leray–Schauder degree has the following properties*:

(D4) $D(I - F, \Omega, y) \neq 0$ *implies* $(I - F)^{-1}(y) \neq \emptyset$;

(D5) *$D(I-G,\Omega,y)=D(I-F,\Omega,y)$ for $G\in\mathcal{K}(\bar{\Omega})\cap B_r(F)$ and $D(I-F,\Omega,\cdot)$ is constant on $B_r(y)$, where $r=\varrho(y,(I-F)(\partial\Omega))$. Even more: $D(I-F,\Omega,\cdot)$ is constant on every connected component of $X\backslash(I-F)(\partial\Omega)$;*

(D6) *$D(I-G,\Omega,y)=D(I-F,\Omega,y)$ whenever $G|_{\partial\Omega}=F|_{\partial\Omega}$;*

(D7) *$D(I-F,\Omega,y)=D(I-F,\Omega_1,y)$ for every open subset Ω_1 of Ω such that $y\notin(I-F)(\bar{\Omega}\backslash\Omega_1)$.*

We have a product formula

Theorem 1.5.3 *Let $\Omega\subset X$ be open bounded, $F_0\in\mathcal{K}(\bar{\Omega})$ and $F=I-F_0$, $G_0: X\to X$ completely continuous and $G=I-G_0$, $y\notin GF(\partial\Omega)$ and $(K_\lambda)_{\lambda\in\Lambda}$ the connected components of $X\backslash F(\partial\Omega)$. Then*

$$D(GF,\Omega,y)=\sum_{\lambda\in\Lambda}D(F,\Omega,K_\lambda)D(G,K_\lambda,y)$$

where only finitely many terms are non-zero and $D(F,\Omega,K_\lambda)$ is $D(F,\Omega,z)$ for any $z\in K_\lambda$.

1.6 Fredholm Operators

Definition 1.6.1 Suppose that X,Y are Banach spaces, $L\in\mathcal{L}(X,Y)$ (linear bounded maps) is called a Fredholm operator, if

(1) Range L is closed;
(2) $\dim \operatorname{Ker} L<\infty$;
(3) $\operatorname{Coker} L=Y/\operatorname{Range} L$ has finite dimension.

We denote $\mathcal{F}(X,Y)$ all Fredholm operators from X to Y. Especially as $Y=X$, we denote $\mathcal{F}(X)$.

Definition 1.6.2 Assume $L\in\mathcal{F}(X,Y)$, let

$$\operatorname{ind}(L)\Delta q \dim\operatorname{Ker} L-\dim\operatorname{Coker} L,$$

it is called the index of L.

Example 1.6.1 If $F: X\to X$ is linear compact, then $T=I-F\in\mathcal{F}(X)$, and $\operatorname{ind}(T)=0$.

For the Leray–Shauder degree theory extending to Fredholm operators of index 0, please see [50].

Theorem 1.6.1 (Gohberg and Krein, see [165]) *The set $\mathcal{F}(X,Y)$ of Fredholm operators is open in the space of all bounded operators $\mathcal{L}(X,Y)$ in the norm topology. Furthermore the index is continuous on $\mathcal{F}(X,Y)$.*

Definition 1.6.3 Suppose that X, Y are C^1 Banach manifolds and $f : X \to Y$ is a C^1 map. A point $x \in X$ is called a regular point of f if $Df(x) : T_x(X) \to T_{f(x)}(Y)$ is surjective, and is singular if not regular. The images of the singular points under f are called the singular values or critical values, their complement the regular values.

Note that if $y \in Y$ is not in the image of f it is automatically a regular value.

Definition 1.6.4 Assume $U \subset X$, a map $f \in C^1(U, Y)$ is called Fredholm map if for each $x \in U$, the derivative $Df(x) : T_x(U) \to T_{f(x)}(Y)$ is a Fredholm operator. The index of f is defined to be the index of $Df(x)$ for some x. By Theorem 1.6.1, if U is connected, then ind $f'(x)$ (or $Df(x)$) does not depend on x, it is denoted by $\operatorname{ind}(f)$.

Theorem 1.6.2 (Sard–Smale, see [165]) *Assume X is a separable Banach space and Y is a Banach space. Let $f : X \to Y$ be a C^q Fredholm map with $q > \max\{\operatorname{ind}(f), 0\}$. Then the regular values of f are almost all of Y (or the set of critical values is of first category).*

Corollary 1.6.1 (See [165]) *X is a separable Banach space and Y is a Banach space. Let $f : X \to Y$ be a C^1 Fredholm map of negative index, its image contains no interior points.*

Corollary 1.6.2 (See [165]) *X is a separable Banach space and Y is a Banach space. Let $f : X \to Y$ be a C^q Fredholm map with $q > \max\{\operatorname{ind}(f), 0\}$, then for almost all $y \in Y$, $f^{-1}(y)$ is a submanifold of X whose dimension is equal to index of f or is empty.*

Definition 1.6.5 A map is proper if the inverse image of a compact set is compact.

Theorem 1.6.3 (See [165]) *A Fredholm map is locally proper. In other words, if $f : X \to Y$ is Fredholm and $x \in X$, there exists a neighborhood N of x such that f restricted to N is proper.*

1.7 Fixed Point Index

Remember that a subset $K \neq \emptyset$ of X is called a retract of X if there is a continuous map $R : X \to K$, a retraction, such that $Rx = x$ on K. Recall also that every closed convex subset is a retract and that every retract is closed but not necessarily convex; remember that $\partial B_1(0)$ is a retract of X if $\dim X = \infty$.

Whenever we are concerned with subsets of a retract K, it is understood that all topological notions are understood with respect to the topology induced by $|\cdot|$ on K.

Now, let $\Omega \subset K$ be open and $F : \bar{\Omega} \to K$ compact and such that $\operatorname{Fix}(F) \cap \partial\Omega = \emptyset$, where $\operatorname{Fix}(F) = \{x \in \bar{\Omega} \mid F(x) = x\}$. If $R : X \to K$ is retraction, then

$D(I - FR, R^{-1}(\Omega, 0))$ is defined, and it follows immediately from the homotopy invariance and the excision property (D3) and (D7) that this integer is the same for all retractions of X onto K. Conventionally, this number is called the fixed point index over Ω with respect to K for the compact F, $i(F, \Omega, K)$ for short. The map $i : M \to \mathbb{Z}$ with

$$M = \big\{(F, \Omega, K) : K \subset X \text{ retract}, \ \Omega \subset K \text{ open}, F : \bar{\Omega} \to K \text{ compact},$$
$$\text{Fix}(F) \cap \partial\Omega = \emptyset\big\},$$

inherits the properties of Leray–Schauder degree D.

Let X be a Banach space, $K \subset X$ a cone and $F : K \to X$. Since one often knows $F(0) = 0$ but fixed points in $K\backslash\{0\}$ are of interest, the simplest abstract approach is to consider a shell $\{x \in K : 0 < \varrho \le \|x\| \le r\}$ and to impose conditions at the lower and upper boundary sufficient for F to have a fixed point in the shell. In the sequel, we let $K_r = K \cap B_r(0)$ and we shall write $i(F, \Omega)$ for $i(F, \Omega, K)$ whenever the index is defined. Let us start with

Theorem 1.7.1 *Let X be a Banach space, $K \subset X$ a cone and $F : \bar{K}_r \to K$ is γ-condensing. Suppose that*

(a) *$Fx \neq \lambda x$ for $\|x\| = r$ and $\lambda > 1$;*
(b) *there exist a smaller radius $\varrho \in (0, r)$ and an $e \in K\backslash\{0\}$ such that $x - Fx \neq \lambda e$ for $\|x\| = \varrho$ and $\lambda > 0$.*

Then F has a fixed point in $\{x \in K : \varrho \le \|x\| \le r\}$.

As a trivial consequence we have the following corollary on 'compression of conical shells':

Corollary 1.7.1 *Suppose that $F : \bar{K}_r \to K$ is γ-condensing and such that*

(a) *$Fx \ngeq x$ on $\|x\| = r$.*
(b) *$Fx \nleq x$ on $\|x\| = \varrho$, for some $\varrho \in (0, r)$.*

Then F has a fixed point in $\{x \in K : \varrho < \|x\| < r\}$.

Theorem 1.7.2 *Let $0 < \varrho < r$, $F : \bar{K}_r \to K$ compact and such that*

(a) *$Fx \neq \lambda x$ on $\|x\| = r$ and $\lambda > 1$.*
(b) *$Fx \neq \lambda x$ on $\|x\| = \varrho$ and $\lambda < 1$.*
(c) $\inf\{|Fx| : \|x\| = \varrho\} > 0$.

Then F has a fixed point in $\bar{K}_r\backslash K_\varrho$.

Remark 1.7.1 Note that this section is included in Deimling [81].

1.8 Banach's Contract Theorem, Implicit Functions Theorem

Theorem 1.8.1 *Let X be a Banach space, $D \subset X$ closed and $F : D \to D$ a strict contraction, i.e., $\|F(x) - F(y)\| \le k\|x - y\|$ for some $0 < k < 1$ and all $x, y \in D$. Then F has a unique fixed point x^*. For any $x_0 \in D$, let $x_{n+1} = F(x_n) = F^n x_0$, then $x_n \to x^*$, and $\|x_n - x^*\| \le (1-k)^{-1}k^n\|F(x_0) - x_0\|$.*

Theorem 1.8.2 (Implicit function theorem) *Let X, Y, Z be Banach spaces, $U \subset X$ and $V \subset Y$ neighborhoods of x_0 and y_0, respectively, $F : U \times V \to Z$ continuous and continuously differentiable with respect to y. Suppose also that $F(x_0, y_0) = 0$ and $F_y^{-1}(x_0, y_0) \in L(Z, Y)$. Then there exist balls $\bar{B}_r(x_0) \subset U$, $\bar{B}_\delta(y_0) \subset V$ and exactly one map $T : B_r(x_0) \to B_\delta(y_0)$ such that $Tx_0 = y_0$ and $F(x, Tx) = 0$ on $B_r(x_0)$. This map T is continuous.*

Moreover, T is as smooth as F, possibly on a smaller ball $B_\varrho(x_0) \subset B_r(x_0)$, i.e., $F \in C^m(U \times V)$ implies that $T \in C^m(B_\varrho(x_0))$.

Theorem 1.8.3 (Inverse function theorem) *Let X, Y be Banach space, U_0 a neighborhood of x_0, $G : U_0 \to Y$ continuously differentiable and $G'(x_0)^{-1} \in L(Y, X)$. Then G is a local homeomorphism, i.e., there is a neighborhood $U \subset U_0$ of x_0 such that $G|_U$ is a homeomorphism onto the neighborhood $G(U)$ of $y_0 = Gx_0$. Furthermore, there is a possibly small neighborhood $V \subset U$ such that $G|_U^{-1} \in C^1(G(V))$ and*

$$\big(G|_U^{-1}\big)'(G_X) = G'(x)^{-1} \quad \text{on } V.$$

Actually $G|_U^{-1}$ is as smooth as G, i.e., $G|_U^{-1} \in C^m(G(V))$ if $G \in C^m(U_0)$, also for $m = \infty$.

1.9 Krein–Rutman Theorem

Let E be a real Banach space. We denote by $L(E) := L(E, E)$ the Banach space of all continuous linear operators in E. Let $P^* = \{f \in E^* | f(x) \ge 0, \forall x \in P\}$, if $\overline{P - P} = E$ (i.e., P is total), then $P^* \subset E^*$ is a cone. Then for every $T \in L(E)$, the limit

$$r(T) := \lim_{n \to \infty} \left\|T^k\right\|^{1/k}$$

exists and is called the spectral radius of T.

Recall that a linear operator $T \in L(E)$ is called compact if the image of the unit ball is relatively compact in E. An eigenvalue λ of a linear operator T is called simple if $\dim(\bigcup_{k=1}^{\infty} \ker(\lambda - T)^k) = 1$.

Theorem 1.9.1 (Krein and Rutman, see [8]) *Let (E, P) be an OBS with total positive cone. Suppose that $T \in L(E)$ is compact and has a positive spectral radius $r(T)$. Then $r(T)$ is an eigenvalue of T and of the dual operator T^* with eigenvector in P and in P^*, respectively.*