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Contract Theory in Continuous-Time Models

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To my parents Antun and Vjera
To my wife Ying and my son Albert

Preface

Why We Wrote This Book In recent years there has been a significant increase in interest in continuous-time Principal–Agent models and their applications. Even though the approach is technical in nature, it often leads to elegant solutions with clear economic predictions. Our monograph sets out to survey some of the literature in a systematic way, using a general theoretical framework. The framework we find natural and general enough to include most of the existing results is the use of the so-called Stochastic Maximum Principle, in models driven by Brownian Motion. It is basically the Stochastic Calculus of Variations, used to find first order conditions for optimality. This leads to the characterization of optimal contracts through a system of Forward-Backward Stochastic Differential Equations (FBSDE's). Even though there is no general existence theory for the FBSDE's that appear in this context, in a number of special cases they can be solved explicitly, thus leading to the analytic form of optimal contracts, and enabling derivation of many qualitative economic conclusions. When assuming Markovian models, we can also identify sufficient conditions via the standard approach of using Hamilton–Jacobi–Bellman Partial Differential Equations (HJB PDE's).

Who Is It For This book is aimed at researchers and graduate students in Economic Theory, Mathematical Economics and Finance, and Mathematics. It provides a general methodological framework, which, hopefully, can be used to develop further advances, both in applications and in theory. It also presents, in its last part, a primer on BSDE's and FBSDE's. We have used the material from the book when teaching PhD courses in contract theory at Caltech and at the University of Zagreb.

Prerequisites A solid knowledge of Stochastic Calculus and the theory of SDE's is required, although the reader not interested in the proofs will need more of an intuitive understanding of the related mathematical concepts, than a familiarity with the technical details of the mathematical theory. A knowledge of Microeconomics is also helpful, although nothing more than a basic understanding of utility functions is required.

Structure of the Book We have divided the book into an introduction, three main middle parts, and the last part. The introduction describes the three main settings: risk sharing, hidden actions and hidden types. It also presents a simple example of each. Then, each middle part presents a general theory for the three settings, with a variety of special cases and applications. The last part presents the basics of the BSDE's theory and the FBSDE's theory.

Web Page for This Book sites.google.com/site/contracttheorycvitaniczhang/. This is a link to the book web page that will be regularly updated with material related to the book, such as corrections of typos.

Acknowledgements Our foremost gratitude goes to our families for the understanding and overall support they provided during the times we spent working on our joint research leading to this book, and for the work on the book itself. We are grateful for the support from the staff of Springer, especially Catriona Byrne, Marina Reizakis and Annika Elting. A number of colleagues and students have made useful comments and suggestions, and pointed out errors in the working manuscript, including Jin Ma, Ajay Subramanian, Xuhu Wan, Xunyu Zhou, Hualei Chang and Nikola Sandrić, and anonymous reviewers.

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Of course, we are solely responsible for any remaining errors, and the opinions, findings and conclusions or suggestions in this book do not necessarily reflect anyone's opinions but the authors'.

Final Word We hope that you will find the subject of this book interesting in its economic content, and elegant in its mathematical execution. We would be grateful to the careful reader who could inform us of any remaining typos and errors noticed, or any other comments, by sending an e-mail to our current e-mail addresses. Enjoy!

Los Angeles, USA
April 2012

Jakša Cvitanić
Jianfeng Zhang

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Part I
Introduction

Chapter 1
Principal–Agent Problem

Abstract A Principal–Agent problem is a problem of optimal contracting between two parties, one of which, namely the agent, may be able to influence the value of the outcome process with his actions. What kind of contract is optimal typically depends on whether those actions are observable/contractable or not, and on whether there are characteristics of the agent that are not known to the principal. There are three main types of these problems: (i) the first best case, or risk sharing, in which both parties have the same information; (ii) the second best case, or moral hazard, in which the action of the agent is hidden or not contractable; (iii) the third best case or adverse selection, in which the type of the agent is hidden.

1.1 Problem Formulation

The main topic of this volume is mathematical modeling and analysis of contracting between two parties, **Principal** and **Agent**, in an uncertain environment. As a typical example of a **Principal–Agent problem**, henceforth the **PA problem**, we can think of the principal as an investor (or a group of investors), and of the agent as a portfolio manager who manages the investors' money. Another interesting example from Finance is that of a company (as the principal) and its chief executive (as the agent). As may be guessed, the principal offers a contract to the agent who has to perform a certain task on the principal's behalf (in our model, it's only one type of task).

We will sometimes call the principal P and the agent A, and we will also call the principal "she" and the agent "he".

The economic problem is for the principal to construct a contract in such a way that: (i) the agent will accept the contract; this is called an **individual rationality (IR) constraint**, or a **participation constraint**; (ii) the principal will get the most out of the agent's performance, in terms of expected utility. How this should be done in an optimal way, depends crucially on the amount of information that is available to P and to A. There are three classical cases studied in the literature, and which we also focus on in this volume: **Risk Sharing** (RS) with symmetric information, **Hidden Action** (HA) and **Hidden Type** (HT).

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Risk Sharing The case of Risk Sharing, also called the **first best**, is the case in which P and A have the same information. They have to agree how to share the risk between themselves. It is typically assumed that the principal has all the *bargaining power*, in the sense that she offers the contract and also dictates the agent's actions, which the agent has to follow, or otherwise, the principal will penalize him with a severe penalty. Mathematically, the problem becomes a stochastic control problem for a single individual—the principal, who chooses both the contract and the actions, under the IR constraint. Alternatively, it can also be interpreted as a maximization of their joint welfare by a social planner. More precisely, but still in informal notation, if we denote by c the choice of contract and by a the choice of action, and by U_A and U_P the corresponding utility functions, the problem becomes

$$\max_{c,a}\big\{E\big[U_P(c,a)\big]+\lambda E\big[U_A(c,a)\big]\big\} \tag{1.1}$$

where $\lambda > 0$ is a Lagrange multiplier for the IR constraint, or a parameter which determines the level of risk sharing. The allocations that are obtained in this way are Pareto optimal.

Hidden Action This is the case in which actions of A are not observable by P. Because of this, there will typically be a loss in expected utility for P, and she will only be able to attain the **second best** reward. Many realistic examples do present cases of P not being able to deduce A's actions, either because it may be too costly to monitor A, or quite impossible. For example, it may be costly to monitor which stocks a portfolio manager picks and how much he invests in each, and it may be quite impossible to deduce how much effort he has put into collecting information for selecting those stocks.

It should be mentioned that the problem is of the same type even if the actions are observed, but cannot be contracted upon—the contract payoff cannot depend directly on A's actions.

Due to unobservable or non-contractable actions, P cannot choose directly the actions she would like A to perform. Instead, giving a contract c, she has to be aware which action $a = a(c)$ will be optimal for the agent to choose. Thus, this becomes a problem of *incentives*, in which P indirectly influences A to pick certain actions, by offering an appropriate contract. Because A can undertake actions that are not in the best interest of the principal, this case also goes under the name of **moral hazard**.

Mathematically, we first have to solve the *agent's problem* for a given fixed contract c:

$$V_A(c) := \max_a E\big[U_A(c,a)\big]. \tag{1.2}$$

Assuming there is one and only one optimal action $a(c)$ solving this problem, we then have to solve the *principal's problem*:

$$V_P := \max_c\big\{E\big[U_P\big(c,a(c)\big)\big]+\lambda E\big[U_A\big(c,a(c)\big)\big]\big\}. \tag{1.3}$$

Problem (1.2) can be very hard given that c can be chosen in quite an arbitrary way. A standard approach which makes this easier is to assume that the agent does not control the outcome of the task directly by his actions, but that he chooses the distribution of the outcome by choosing specific actions. More precisely, this will be modeled by having A choose probability distributions P^a under which the above expected values will be taken.

Hidden Type In many applications it is reasonable to assume that P does not know some key characteristics of A. For example, she may not know how capable an executive is, in terms of how much return he can produce per unit of effort. Or, P may not know what A's risk aversion is. Or how rich A is. An even more fundamental example is of a buyer (agent) and a seller (principal), in which the buyer may be of a type who cares more or cares less about the quality of the product (wine, for example). Those hidden characteristics, or types, may significantly alter A's behavior, given a certain contract.

It is typically assumed in the HT case, as we also do in this book, that P will offer a **menu of contracts**, one for each type, from which A can choose. Under certain conditions, a so-called **revelation principle** holds, which says that it is sufficient to consider contracts which are **truth-telling**: the agent will reveal his true type by choosing the contract $c(\theta)$ which was meant for his type θ. In particular, the main assumption needed for the revelation principle is that of *full commitment*: once agreed on the contract, the parties cannot change their mind in the future, even if both are willing to renegotiate. This is an assumption that we make throughout.

If the hidden type case is combined with hidden actions, then, generally, the principal gets only her **third best** reward. Since A can pretend to be of a different type than he really is, which can adversely affect P's utility, the hidden type case is also called a case of **adverse selection**. An example is the case of a health insurance company (principal) and an individual (agent) who seeks health insurance, but only if he already has medical problems, and the insurance company may not know about it.

Mathematically, we again first have to solve the agent's problem when he chooses a contract $c(\theta')$ and he is of type θ:

$$V_A\big(c(\theta'),\theta\big) := \max_a E^\theta\big[U_A\big(c(\theta'),a,\theta\big)\big]. \tag{1.4}$$

We assume that the principal's belief about the distribution of types is given by a distribution function $F(\theta)$. Denote by $\mathcal{T}$ the set of truth-telling menus of contracts $c(\theta)$. Assuming there is one and only one optimal action $a(c(\theta'),\theta)$ solving the agent's problem for each pair $(c(\theta'),\theta)$, and denoting $a(c(\theta)) := a(c(\theta),\theta)$ (the action taken when A reveals the truth) we then have to solve the principal's problem

$$V_P := \max_{c\in\mathcal{T}} \int \big\{E^\theta\big[U_P\big(c(\theta),a\big(c(\theta)\big)\big)\big] + \lambda(\theta)E^\theta\big[U_A\big(c(\theta),a\big(c(\theta)\big)\big)\big]\big\}dF(\theta). \tag{1.5}$$

Note that the principal faces now an additional, *truth-telling constraint*, that is, $c \in \mathcal{T}$, which can be written as

$$\max_{\theta'} V_A\big(c(\theta'),\theta\big) = V_A\big(c(\theta),\theta\big). \tag{1.6}$$

1.2 Further Reading

There are a number of books that have the PA problem as one of the main topics. We mention here Laffont and Martimort (2001), Salanie (2005), and Bolton and Dewatripont (2005), which all contain the general theory in discrete-time, more advanced topics and many applications.

References

Bolton, P., Dewatripont, M.: Contract Theory. MIT Press, Cambridge (2005)
Laffont, J.J., Martimort, D.: The Theory of Incentives: The Principal–Agent Model. Princeton University Press, Princeton (2001)
Salanie, B.: The Economics of Contracts: A Primer, 2nd edn. MIT Press, Cambridge (2005)

Chapter 2
Single-Period Examples

Abstract In this chapter we consider simple examples in one-period models, whose continuous versions will be studied later in the book. Principal–Agent problems in single-period models become more tractable if exponential utility functions are assumed. However, even then, there are cases in which tractability requires considering only linear contracts. Optimal contracts which cannot contract upon the agent's actions are more sensitive to the output than those that can. When the agents' type is unknown to the principal, the agents of "higher" type may have to be paid more to make them reveal their type.

2.1 Risk Sharing

Assume that the contract payment occurs once, at the final time $T = 1$, and we denote it C_1. The principal draws utility from the final value of an *output process* X, given by

$$X_1 = X_0 + a + B_1 \tag{2.1}$$

where B_1 is a fixed random variable. The constant a is the action of the agent.

With full information, the principal maximizes the following case of (1.1), with $g(a)$ denoting a *cost function*:

$$E\big[U_P(X_1 - C_1) + \lambda U_A\big(C_1 - g(a)\big)\big]. \tag{2.2}$$

Setting the derivative with respect to C_1 inside the expectation equal to zero, we get the first order condition

$$\frac{U_P'(X_1 - C_1)}{U_A'(C_1 - g(a))} = \lambda. \tag{2.3}$$

This is the so-called *Borch rule* for risk-sharing, a classical result that says that *the ratio of marginal utilities of P and A is constant at the risk-sharing optimum.*

We assume now that the utility functions are exponential and the cost of action is quadratic:

$$U_A\big(C_1 - g(a)\big) = -\frac{1}{\gamma_A} e^{-\gamma_A[C_1 - ka^2/2]}, \tag{2.4}$$

J. Cvitanić, J. Zhang, *Contract Theory in Continuous-Time Models*, Springer Finance, DOI 10.1007/978-3-642-14200-0_2,

$$U_P(X_1 - C_1) = -\frac{1}{\gamma_P} e^{-\gamma_P[X_1 - C_1]}. \tag{2.5}$$

Denote

$$\rho := \frac{1}{\gamma_A + \gamma_P}. \tag{2.6}$$

We can compute the optimal C_1 from (2.3), and get

$$C_1 = \rho\big[\gamma_P X_1 + \gamma_A k a^2/2 + \log\lambda\big]. \tag{2.7}$$

This is a typical result: *for exponential utility functions the optimal contract is linear* in the output process. We see that the *sensitivity* of the contract with respect to the output is given by $\frac{\gamma_P}{\gamma_A+\gamma_P} \le 1$, and it gets smaller as the agent's risk aversion gets larger relative to the principal's. A very risk-averse agent should not be exposed much to the uncertainty of the output. In the limit when P is risk-neutral, or A is infinitely risk-averse, that is, $\gamma_P = 0$ or $\gamma_A = \infty$, the agent is paid a fixed cash payment. On the other hand, when A is risk-neutral, that is, $\gamma_A = 0$, the sensitivity is equal to its maximum value of one, and what happens is that *at the end of the period the principal sells the whole firm to the risk-neutral agent in exchange for cash payment*. The risk is completely taken over by the risk-neutral agent.

If we now take a derivative of the objective function with respect to a, and use the first order condition (2.3) for C_1, a simple computation gives us

$$a = 1/k,$$

which is the optimal action. We see another typical feature of exponential utilities: *the optimal action does not depend on the value of the output*. In fact, here, when there is also full information, it does not depend on risk aversions either, and this feature will extend to more general risk-sharing models and other utility functions.

Note that the optimal contract C_1, as given in (2.7), explicitly depends on the action a. Thus, this is not going to be a feasible contract when the action is not observable. Moreover, if, in the hidden action case, the principal replaced a in (2.7) with $1/k$, and offered such a contract, it can be verified that the agent would not choose $1/k$ as the optimal action, and the contract would not attain the first best utility for the principal. We discuss hidden action next.

2.2 Hidden Action

Even though the above example is very simple, it is hard to deal with examples like this in the case of hidden action. We will see that it is actually easier to get more general results in continuous-time models. For example, we will here derive the contract which is optimal among linear contracts, but we will show later that in a continuous-time model the same linear contract is in fact optimal even if we allow general (not just linear) contracts.

Regardless of whether we have a discrete-time or a continuous-time model, for HA models we suppose that the agent can choose the distribution of X_1 by his action, in a way which is unobservable or non-contractable by the principal. More precisely, let us change somewhat the above model by assuming that under some fixed probability $P = P^0$,

$$X_1 = X_0 + \sigma B_1$$

where X_0 is a constant and B_1 is a random variable that has a standard normal distribution. For simplicity of notation set $X_0 = 0$. Given action a we assume that the probability P changes to P^a, under which the distribution of B_1 is normal with mean a/σ and variance one. Thus, under P^a, X_1 has mean a. We see that by choosing action a the agent influences only the distribution and not directly the outcome value of X_1.

Even with that modification, the agent's problem is still hard in this single period model for arbitrary contracts. In fact, Mirrlees (1999) shows that, in general, we cannot expect the existence of an optimal contract in such a setting. For this reason, in this example we restrict ourselves only to the contracts which are linear in X_1, or, equivalently, in B_1:

$$C_1 = k_0 + k_1 B_1.$$

Denoting by E^a the expectation operator under probability P^a, the agent's problem (1.2) then is to minimize

$$E^a\left[e^{-\gamma_A(k_0+k_1B_1-ka^2/2)}\right] = e^{-\gamma_A(k_0-ka^2/2+k_1a/\sigma-\frac{1}{2}k_1^2\gamma_A)}$$

where we used the fact that

$$E^a\left[e^{cB_1}\right] = e^{ca/\sigma+\frac{1}{2}c^2}. \tag{2.8}$$

We see that the optimal action a is

$$a = \frac{k_1}{k\sigma}. \tag{2.9}$$

That is, it is proportional to the sensitivity k_1 of the contract to the output process, and inversely proportional to the penalty parameter and the uncertainty parameter.

We now use a method which will also prove useful in the continuous-time case. We suppose that the principal decides to give expected utility of R_0 to the agent. This means that, using $C_1 = k_0 + \sigma ka B_1$, the fact that the mean of B_1 under P^a is a/σ, and using (2.8) and (2.9),

$$R_0 = -\frac{1}{\gamma_A}E^a\left[e^{-\gamma_A(C_1-ka^2/2)}\right] = -\frac{1}{\gamma_A}e^{-\gamma_A(k_0+ka^2/2-\frac{1}{2}\gamma_A\sigma^2k^2a^2)}. \tag{2.10}$$

Computing $e^{-\gamma_A k_0}$ from this and using $C_1 = k_0 + \sigma ka B_1$ again, we can write

$$-\frac{1}{\gamma_A}e^{-\gamma_A C_1} = R_0 e^{-\gamma_A(-ka^2/2+\frac{1}{2}\gamma_A\sigma^2k^2a^2+\sigma ka B_1)}. \tag{2.11}$$

This is *a representation of the contract payoff in terms of the agent's promised utility R_0 and the source of uncertainty B_1*, which will be crucial later on, too. Using

$e^{\gamma_P C_1} = (e^{-\gamma_A C_1})^{-\gamma_P/\gamma_A}$, $X_1 = \sigma B_1$ and (2.11), we can write the principal's expected utility as

$$E^a\big[U_P(X_1 - C_1)\big] = -\frac{1}{\gamma_P}(-\gamma_A R_0)^{-\gamma_P/\gamma_A} E^a\big[e^{-\gamma_P(\sigma B_1 + ka^2/2 - \frac{1}{2}\gamma_A\sigma^2k^2a^2 - \sigma kaB_1)}\big]$$

which can be computed as

$$-\frac{1}{\gamma_P}(-\gamma_A R_0)^{-\gamma_P/\gamma_A} e^{-\gamma_P(-\frac{1}{2}\gamma_P\sigma^2(ka-1)^2 - a(ka-1) + ka^2/2 - \frac{1}{2}\gamma_A\sigma^2k^2a^2)}.$$

Setting the derivative thereof with respect to a to zero, we get the optimal a as

$$a = \frac{1/(k\sigma^2) + \gamma_P}{1/\sigma^2 + k(\gamma_A + \gamma_P)}. \tag{2.12}$$

The sensitivity of the contract is $k_1/\sigma = ka$, that is, we have

$$C_1 = \tilde{k}_0 + kaX_1 = \tilde{k}_0 + \frac{1/(k\sigma^2) + \gamma_P}{1/(k\sigma^2) + \gamma_A + \gamma_P} X_1$$

for some constant $\tilde{k}_0$. Recall that in the risk-sharing, first best case the optimal action is $a = 1/k$ and the sensitivity is $\gamma_P/(\gamma_A + \gamma_P)$, thus both are independent of the level of uncertainty σ, and the action is even independent of the risk aversions. Here, the action and the sensitivity depend on the risk aversions. As the level of risk σ goes to zero, the action approaches the first best action, because then the action becomes, in the limit, fully observable.

It is easy to check that the sensitivity of the above HA contract is decreasing in the level of uncertainty σ and always higher than the sensitivity of the RS contract—when the action is unobservable the principal is forced to try to induce more effort by offering higher incentives, but less so when the risk is higher. In the limit when σ goes to infinity, the two sensitivities become equal.

For fixed σ, the induced action now depends on risk aversions. For the risk-neutral agent, the action is again the first best, and the principal transfers the whole firm to the agent.

However, as the agent's risk aversion increases (relative to the principal's), in the HA case the principal can optimally induce only lower and lower effort from the agent, paying him with lower and lower sensitivity to the output. On the other hand, given A's risk aversion, as P becomes more risk-averse she offers a higher percentage of the output to the agent.

As mentioned at the beginning of this section, we will show later that the above contract is actually optimal among all contracts, linear or not, when we allow continuous actions by the agent.

2.3 Hidden Type

We now add to the above HA model a parameter θ, unknown to the principal, which characterizes the agent. More precisely, for agent of type θ, we assume that, given action a, the mean of the normal random variable B_1 is $(\theta + a)/\sigma$ (the variance is still equal to one). The interpretation is that θ is the "return" that A can produce with no effort, due to his individual-specific skills.

We again restrict ourselves only to the contracts which are linear in X_1, and P offers a menu of contracts depending on type θ, from which A can choose:

$$C_1(\theta) = k_0(\theta) + k_1(\theta)B_1.$$

Denoting by $E^{a,\theta}$ the expectation operator under probability P^a and type θ, the agent's problem (1.4) is

$$\begin{aligned}-\gamma_A V_A\big(\theta, \theta'\big) &:= \min_a E^{a,\theta}\big[e^{-\gamma_A(k_0(\theta')+k_1(\theta')B_1-ka^2/2)}\big]\\ &= \min_a e^{-\gamma_A(k_0(\theta')-ka^2/2+k_1(\theta')(a+\theta)/\sigma-\frac{1}{2}k_1^2(\theta')\gamma_A)}.\end{aligned}$$

We see that the optimal action $a = a(\theta')$ is

$$a\big(\theta'\big) = \frac{k_1(\theta')}{k\sigma} \tag{2.13}$$

and

$$-\gamma_A V_A\big(\theta, \theta'\big) = e^{-\gamma_A(k_0(\theta')+\frac{1}{2}k_1^2(\theta')(\frac{1}{k\sigma^2}-\gamma_A)+\frac{\theta}{\sigma}k_1(\theta'))}. \tag{2.14}$$

Denote with ∂/∂_θ the derivative with respect to the first argument, and with $\partial/\partial_{\theta'}$ the derivative with respect to the second argument. In order for the contract to be truth-telling, $\max_{\theta'} V_A(\theta, \theta')$ has to be attained at $\theta' = \theta$, which leads to the first order condition

$$0 = \frac{\partial}{\partial\theta'}V_A(\theta, \theta).$$

We denote by $R(\theta)$ the expected utility of the agent of type θ, given that he was offered a truth-telling contract. In other words, we have

$$R(\theta) = V_A(\theta, \theta).$$

Note that then we have, under the above first order condition,

$$R'(\theta) = \frac{d}{d\theta}V_A(\theta, \theta) = \frac{\partial}{\partial\theta}V_A(\theta, \theta) + \frac{\partial}{\partial\theta'}V_A(\theta, \theta) = \frac{\partial}{\partial\theta}V_A(\theta, \theta).$$

Using this and taking the latter derivative in (2.14), we get the following consequence of the first order condition (with a slight abuse of notation introduced by the second term):

$$k_1(\theta) = k_1\big(R(\theta), R'(\theta)\big) = -\frac{1}{\gamma_A}\sigma\frac{R'(\theta)}{R(\theta)}. \tag{2.15}$$

Using (2.14) with $\theta = \theta'$, we obtain

$$-\gamma_A R(\theta) = e^{-\gamma_A(k_0(\theta)+\frac{1}{2}k_1^2(\theta)(\frac{1}{k\sigma^2}-\gamma_A)+\frac{\theta}{\sigma}k_1(\theta))}. \tag{2.16}$$

Computing $e^{-\gamma_A k_0(\theta)}$ from this, and using $C_1 = k_0 + k_1 B_1$, we can write

$$-\frac{1}{\gamma_A} e^{-\gamma_A C_1(\theta)} = R(\theta) e^{-\gamma_A(-\frac{1}{2}k_1^2(\theta)(\frac{1}{k\sigma^2}-\gamma_A)-\frac{\theta}{\sigma}k_1(\theta)+k_1(\theta)B_1)}. \tag{2.17}$$

Using $e^{\gamma_P C_1} = (e^{-\gamma_A C_1})^{-\gamma_P/\gamma_A}$, $X_1 = \sigma B_1$ and (2.17), we can write the principal's expected utility as

$$\begin{aligned}&E^{a,\theta}\big[U_P\big(X_1 - C_1(\theta)\big)\big]\\&= -\frac{1}{\gamma_P}\big(-\gamma_A R(\theta)\big)^{-\gamma_P/\gamma_A} E^{a,\theta}\big[e^{-\gamma_P(\sigma B_1+\frac{1}{2}k_1^2(\theta)(\frac{1}{k\sigma^2}-\gamma_A)+\frac{\theta}{\sigma}k_1(\theta)-k_1(\theta)B_1)}\big].\end{aligned}$$

Assume henceforth that the first order condition (2.15) is also sufficient for truth-telling (which has to be verified later when a solution is obtained). Then, the principal's utility can be computed as, abbreviating $k_1 = k_1(R(\theta), R'(\theta))$,

$$\begin{aligned}&v_P\big(R(\theta), R'(\theta), \theta\big)\\&:= -\frac{1}{\gamma_P}\big(-\gamma_A R(\theta)\big)^{-\gamma_P/\gamma_A} e^{-\gamma_P(-\frac{1}{2}\gamma_P(\sigma-k_1)^2+\frac{1}{\sigma}(\frac{k_1}{k\sigma}+\theta)(\sigma-k_1)+\frac{1}{2}k_1^2(\frac{1}{k\sigma^2}-\gamma_A)+\frac{\theta}{\sigma}k_1)}.\end{aligned} \tag{2.18}$$

Suppose now that the principal has a prior distribution $F(\theta)$ on the interval $[\theta_L, \theta_H]$ for θ. Also suppose that the agent of type θ needs to be given expected utility of at least $R_0(\theta)$. Then, since we have already taken into account the truth-telling constraint by expressing k_1 in terms of R, R', her problem (1.5) becomes

$$\max_{R(\theta)\geq R_0(\theta)} \int_{\theta_L}^{\theta_U} v_P\big(R(\theta), R'(\theta), \theta\big) dF(\theta).$$

This is a calculus of variations problem, which is quite hard in general. We simplify further by assuming the risk-neutral principal,

$$U_P(x) = x.$$

The results are obtained either by repeating the above arguments, or by formally replacing $\frac{1}{\gamma_P}(1-e^{-\gamma_P x})$ by x (the limit when $\gamma_P = 0$), and noticing that maximizing with utility $\frac{1}{\gamma_P}(1-e^{-\gamma_P x})$ is the same as maximizing with utility $-\frac{1}{\gamma_P}e^{-\gamma_P x}$. We get

$$\begin{aligned}&v_P\big(R(\theta), R'(\theta), \theta\big)\\&= \frac{1}{\sigma}(\sigma - k_1)(a+\theta) + \frac{1}{\gamma_A}\log\big(-\gamma_A R(\theta)\big) + \frac{1}{2}k_1^2\bigg(\frac{1}{k\sigma^2} - \gamma_A\bigg) + \frac{\theta}{\sigma}k_1\\&= \frac{1}{\gamma_A}\log\big(-\gamma_A R(\theta)\big) - \frac{1}{2}k_1^2\bigg(\frac{1}{k\sigma^2} + \gamma_A\bigg) + \frac{k_1}{k\sigma} + \theta.\end{aligned} \tag{2.19}$$

Simplifying further, we assume that $F(\theta)$ is the uniform distribution on $[\theta_L, \theta_H]$. Introduce a *certainty equivalent*[1] of the agent's utility

$$\tilde{R}(\theta) = -\frac{1}{\gamma_A} \log\bigl(-\gamma_A R(\theta)\bigr)$$

so that, by (2.15),

$$k_1(\theta) = \sigma \tilde{R}'(\theta).$$

Then, the principal's problem is equivalent to

$$\min_{R(\theta) \geq R_0(\theta)} \int_{\theta_L}^{\theta_U} \left[\tilde{R}(\theta) + \frac{1}{2} \bigl(\tilde{R}'\bigr)^2(\theta) \bigl(\sigma^2 \gamma_A + 1/k\bigr) - \tilde{R}'(\theta)/k \right] d\theta. \tag{2.20}$$

This can be solved using standard calculus of variations techniques, as we prove later in an analogous continuous-time model. We state here the results without the proofs. Denote

$$\beta = \frac{1/\sigma^2}{1/(k\sigma^2) + \gamma_A}.$$

We have the following

Theorem 2.3.1 *Assume the above setup and that $R_0(\theta) \equiv R_0$. Then, the principal's problem* (2.20) *has a unique solution as follows. Denote $\theta^* := \max\{\theta_H - 1/k, \theta_L\}$. The optimal choice of agent's certainty equivalent $\tilde{R}$ by the principal is given by*

$$\tilde{R}(\theta) = \begin{cases} \tilde{R}_0, & \theta_L \leq \theta < \theta^*; \\ \tilde{R}_0 + \beta\theta^2/2 + \beta(1/k - \theta_H)\theta - \beta(\theta^*)^2/2 - \beta(1/k - \theta_H)\theta^*, \\ \quad \theta^* \leq \theta \leq \theta_H. \end{cases} \tag{2.21}$$

The optimal agent's effort is given by

$$a(\theta) = \tilde{R}'(\theta)/k = \begin{cases} 0, & \theta_L \leq \theta < \theta^*; \\ \frac{\beta}{k}(1/k + \theta - \theta_H), & \theta^* \leq \theta \leq \theta_H. \end{cases} \tag{2.22}$$

The optimal contract is of the form

$$C_1(\theta) = \begin{cases} k_0(\theta), & \theta_L \leq \theta < \theta^*; \\ k_0(\theta) + \beta(1/k + \theta - \theta_H)(X_1 - X_0), & \theta^* \leq \theta \leq \theta_H. \end{cases} \tag{2.23}$$

We see that if the interval of possible type values is large, more precisely, if $\theta_H - \theta_L > 1/k$, a range of lower type agents gets no "rent" above the reservation value R_0, the corresponding contract is not incentive as it does not depend on X_1, and the effort is zero. The higher type agents get certainty equivalent $\tilde{R}(\theta)$ which is quadratically increasing in their type θ. This monotonicity is typical for hidden

[1] Given a utility function U, certainty equivalent CE of a random variable X is a real number such that $U(CE) = E[U(X)]$.

action problems: higher type agents may have to be paid an "*informational rent*" above the reservation value R_0 so that they would not pretend to be of lower type and try to shirk.

As the volatility σ, or A's risk aversion γ_A get larger, the contracts for the high type agents get closer to the non-incentive contract for the low type agents, as it gets harder to provide incentives anyway. On the other extreme, as $\sigma^2\gamma_A$ tends to zero, the incentives and the rent for the high type agents get higher. If the agents are risk-neutral or $\sigma = 0$, the contract for the highest type agent $\theta = \theta_H$ is to sell the whole firm to him.

In the special case when the agent is also risk-neutral, we will show later in a continuous-time setting that the above contract is optimal among all contracts, linear or not.

2.4 Further Reading

Early papers discussing risk sharing are Borch (1962) and Wilson (1968). The hidden action setting with exponential utilities is thoroughly analyzed in Holmström and Milgrom (1987), which is also the first paper that considers the continuous-time setting. The hidden type example is a single-period version of the model in Cvitanić and Zhang (2007).

References

Borch, K.: Equilibrium in a reinsurance market. Econometrica **30**, 424–444 (1962)

Cvitanić, J., Zhang, J.: Optimal compensation with adverse selection and dynamic actions. Math. Financ. Econ. **1**, 21–55 (2007)

Holmström, B., Milgrom, P.: Aggregation and linearity in the provision of intertemporal incentives. Econometrica **55**, 303–328 (1987)

Wilson, R.: The theory of syndicates. Econometrica **36**, 119–132 (1968)

Part II
First Best: Risk Sharing Under Full Information