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J. Donald Monk

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Second Revised Edition

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Contents

Foreword

This book is the successor of **Cardinal functions on Boolean algebras** (Birkhäuser 1990) and **Cardinal invariants on Boolean algebras** (Birkhäuser 1996). It contains most of the material of these books, and adds the following:

(1) Indication of the progress made on the open problems formulated in the earlier versions, with detailed solutions in many cases.
(2) Inclusion of some new cardinal functions, mainly those associated with continuum cardinals.

The material on sheaves, Boolean products, and Boolean powers has been omitted, since these no longer play a role in our discussion of the cardinal invariants.

Although many problems in the earlier versions have been solved, many of them are still open. In this edition we repeat those unsolved problems, and add several new ones.

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0 Introduction

This book is concerned with the theory of certain natural functions k which assign to each infinite Boolean algebra A a cardinal number $k(A)$ or a set $k(A)$ of cardinal numbers. The purpose of the book is to survey this area of the theory of BAs, giving proofs for a large number of results, some of which are new, mentioning most of the known results, and formulating open problems. Some of the open problems are somewhat vague ("Characterize..." or something like that), but frequently these are even more important than the specific problems we state; so we have opted to enumerate problems of both sorts in order to focus attention on them.

The framework that we shall set forth and then follow in investigating cardinal functions seems to us to be important for several reasons. First of all, the functions themselves seem intrinsically interesting. Many of the questions which naturally arise can be easily answered on the basis of our current knowledge of the structure of Boolean algebras, but some of these answers require rather deep arguments of set theory, algebra, or topology. This provides another interest in their study: as a natural source of applications of set-theoretical, algebraic, or topological methods. Some of the unresolved questions are rather obscure and uninteresting, but some of them have a general interest. Altogether, the study of cardinal functions seems to bring a unity and depth to many isolated investigations in the theory of BAs.

There are several surveys of cardinal functions on Boolean algebras, or, more generally, on topological spaces: See Arhangelskiĭ [78], Comfort [71], van Douwen [89], Hodel [84], Juhász [75], Juhász [80], Juhász [84], Monk [84], Monk [90], and Monk [96]. We shall not assume any acquaintance with any of these. On the other hand, we shall frequently refer to results proved in Part I of the Handbook of Boolean Algebras, Koppelberg [89]. One additional bit of terminology: in a weak product $\prod^{\mathrm{w}}_{i\in I} A_i$, we call an element a of *type* 1 iff $\{i \in I : a_i \neq 0\}$, the 1-*support* of a, is finite, and of *type* 2 iff $\{i \in I : a_i \neq 1\}$, called the 2-*support* of a, is finite.

We have not attempted to give a complete history of the results mentioned in this book. The references can be consulted for a detailed background.

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Definition of the main cardinal functions considered

We defer until later the discussion of the existence of some of these functions; they do not all exist for every BA.

Cellularity. A subset X of a BA A is called *disjoint* if its members are pairwise disjoint. The *cellularity* of A, denoted by $\mathrm{c}(A)$, is

$$\sup\{|X| : X \text{ is a disjoint subset of } A\}.$$

Depth. $\mathrm{Depth}(A)$ is

$$\sup\{|X| : X \text{ is a subset of } A \text{ well ordered by the Boolean ordering}\}.$$

Topological density. The *density* of a topological space X, denoted by $\mathrm{d}(X)$, is the smallest cardinal κ such that X has a dense subspace of cardinality κ. The *topological density* of a BA A, also denoted by $\mathrm{d}(A)$, is the density of its Stone space $\mathrm{Ult}(A)$.

π-weight. A subset X of a BA A is *dense* in A if for all $a \in A^+$ there is an $x \in X^+$ such that $x \leq a$. The π-*weight* of a BA A, denoted by $\pi(A)$, is

$$\min\{|X| : X \text{ is dense in } A\}.$$

This could also be called the *algebraic density* of A. (Recall that for any subset X of a BA, X^+ is the collection of nonzero elements of X.)

Length. $\mathrm{Length}(A)$ is

$$\sup\{|X| : X \text{ is a chain in } A\}.$$

Irredundance. A subset X of a BA A is *irredundant* if for all $x \in X$, $x \notin \langle X\backslash\{x\}\rangle$. (Recall that $\langle Y\rangle$ is the subalgebra generated by Y.) The *irredundance* of A, denoted by $\mathrm{Irr}(A)$, is

$$\sup\{|X| : X \text{ is an irredundant subset of } A\}.$$

Cardinality. This is just $|A|$. Sometimes we denote it by $\mathrm{card}(A)$.

Independence. A subset X of A is called *independent* if X is a set of free generators for $\langle X\rangle$. Then the *independence* of A, denoted by $\mathrm{Ind}(A)$, is

$$\sup\{|X| : X \text{ is an independent subset of } A\}.$$

π-character. For any ultrafilter F on A, let $\pi\chi(F) = \min\{|X| : X \text{ is dense in } F\}$. Note here that it is not required that $X \subseteq F$. Then the π-*character* of A, denoted by $\pi\chi(A)$, is

$$\sup\{\pi\chi(F) : F \text{ is an ultrafilter of } A\}.$$

Tightness. For any ultrafilter F on A, let $\mathrm{t}(F) = \min\{\kappa :$ if Y is contained in $\mathrm{Ult}(A)$ and F is contained in $\bigcup Y$, then there is a subset Z of Y of power at most κ such that F is contained in $\bigcup Z\}$. Then the *tightness* of A, denoted by $\mathrm{t}(A)$, is

$$\sup\{\mathrm{t}(F) : F \text{ is an ultrafilter on } A\}.$$

Spread. The *spread* of A, denoted by $\mathrm{s}(A)$, is

$$\sup\{|D| : D \subseteq \mathrm{Ult}(A), \text{ and } D \text{ is discrete in the relative topology}\}.$$

Character. The *character* of A, denoted by $\chi(A)$, is

$$\min\{\kappa : \text{ every ultrafilter on } A \text{ can be generated by at most } \kappa \text{ elements}\}.$$

Hereditary Lindelöf degree. For any topological space X, the *Lindelöf degree* of X is the smallest cardinal $\mathrm{L}(X)$ such that every open cover of X has a subcover with at most $\mathrm{L}(X)$ elements. Then the *hereditary Lindelöf degree* of A, denoted by $\mathrm{hL}(A)$, is

$$\sup\{\mathrm{L}(X) : X \text{ is a subspace of } \mathrm{Ult}(A)\}.$$

Hereditary density. The *hereditary density* of A, $\mathrm{hd}(A)$, is

$$\sup\{\mathrm{d}(S) : S \text{ is a subspace of } \mathrm{Ult}(A)\}.$$

Incomparability. A subset X of A is *incomparable* if for any two distinct elements $x, y \in X$ we have $x \not\leq y$ and $y \not\leq x$. The *incomparability* of A, denoted by $\mathrm{Inc}(A)$, is

$$\sup\{|X| : X \text{ is an incomparable subset of } A\}.$$

Hereditary cofinality. This cardinal function, h-cof(A), is

$$\min\{\kappa : \text{ for all } X \subseteq A \text{ there is a } C \subseteq X \text{ with } |C| \leq \kappa \text{ and } C \text{ cofinal in } X\}.$$

Number of ultrafilters. Of course, this is the same as the cardinality of the Stone space of A, and is denoted by $|\mathrm{Ult}(A)|$.

Number of automorphisms. We denote by $\mathrm{Aut}(A)$ the set of all automorphisms of A. So this cardinal function is $|\mathrm{Aut}(A)|$.

Number of endomorphisms. We denote by $\mathrm{End}(A)$ the set of all endomorphisms of A, and hence this cardinal function is $|\mathrm{End}(A)|$.

Number of ideals of ***A***. We denote by $\mathrm{Id}(A)$ the set of all ideals of A, so here we have the cardinal function $|\mathrm{Id}(A)|$.

Number of subalgebras of ***A***. We denote by $\mathrm{Sub}(A)$ the set of all subalgebras of A; $|\mathrm{Sub}(A)|$ is this cardinal function.

Some classifications of cardinal functions

Some theorems which we shall present, especially some involving unions or ultraproducts, are true for several of our functions, with essentially the same proof. For this reason we introduce some rather ad hoc classifications of the functions. Some of the statements below are proved later in the book.

A cardinal function k is an *ordinary sup-function* with respect to P if P is a function assigning to every infinite BA A a subset $P(A)$ of $\mathscr{P}(A)$ so that the following conditions hold for any infinite BA A:

(1) $k(A) = \sup\{|X| : X \in P(A)\}$;
(2) If B is a subalgebra of A, then $P(B) \subseteq P(A)$ and $X \cap B \in P(B)$ for any $X \in P(A)$.
(3) For each infinite cardinal κ there is a BA C of size κ such that there is an $X \in P(C)$ with $|X| = \kappa$.

Table 0.1 lists some ordinary sup-functions.

Table 0.1: ordinary sup-functions

Function	The subset $P(A)$
$\mathrm{c}(A)$	$\{X : X$ is disjoint$\}$
$\mathrm{Depth}(A)$	$\{X : X$ is well ordered by the Boolean ordering of $A\}$
$\mathrm{Length}(A)$	$\{X : X$ is linearly ordered by the Boolean ordering of $A\}$
$\mathrm{Irr}(A)$	$\{X : X$ is irredundant$\}$
$\mathrm{Ind}(A)$	$\{X : X$ is independent$\}$
$\mathrm{s}(A)$	$\{X : X$ is ideal-independent$\}$
$\mathrm{Inc}(A)$	$\{X : X$ is incomparable$\}$

Given any ordinary sup-function k with respect to a function P and any infinite cardinal κ, we say that A satisfies the $\kappa - k$*-chain condition* provided that $|X| < \kappa$ for all $X \in P(A)$.

A cardinal function k is an *ultra-sup function with respect to* P if P is a function assigning to each infinite BA a subset $P(A)$ of $\mathscr{P}(A)$ such that the following conditions hold:

(1) $k(A) = \sup\{|X| : X \in P(A)\}$.
(2) If $\langle A_i : i \in I\rangle$ is a sequence of BAs, F is an ultrafilter on I, and $X_i \in P(A_i)$ for all $i \in I$, then $\{f/F : f(i) \in X_i \text{ for all } i \in I\} \in P\left(\prod_{i\in I} A_i/F\right)$.

All of the above ordinary sup-functions except Depth are also ultra-sup functions.

For the next classification, extend the first-order language for BAs by adding two unary relation symbols $\mathbf{F}$ and $\mathbf{P}$. Then we say that k is a *sup-min* function if there are sentences $\varphi(\mathbf{F},\mathbf{P})$ and $\psi(\mathbf{F})$ in this extended language such that:

(1) $k(A) = \sup\{\min\{|P| : (A,F,P) \models \varphi\} : A$ is infinite and $(A,F) \models \psi\}$. In particular, for any BA A there exist $F, P \subseteq A$ such that $(A,F,P) \models \varphi$.

(2) φ has the form $\forall x \in \mathbf{P}(x \neq 0 \wedge \varphi'(\mathbf{F},x)) \wedge \forall x \in \mathbf{F}\,\exists y \in \mathbf{P}\varphi''(\mathbf{F},x,y)$.

(3) $(A,F) \models \psi(\mathbf{F}) \rightarrow \exists x(x \neq 0 \wedge \varphi'(\mathbf{F},x))$.

Some sup-min functions are listed in Table 0.2, where $\mu(\mathbf{F})$ is the formula saying that $\mathbf{F}$ is an ultrafilter.

Table 0.2: sup-min functions

Function	$\psi(\mathbf{F})$	$\varphi(\mathbf{F},\mathbf{P})$
π	$\forall x \mathbf{F} x$	$\forall x \in \mathbf{P}(x \neq 0) \wedge \forall x \in \mathbf{F}\,\exists y \in \mathbf{P}(x \neq 0 \rightarrow y \leq x)$
$\pi\chi$	$\mu(\mathbf{F})$	$\forall x \in \mathbf{P}(x \neq 0) \wedge \forall x \in \mathbf{F}\,\exists y \in \mathbf{P}(y \leq x)$
χ	$\mu(\mathbf{F})$	$\forall x \in \mathbf{P}(x \neq 0 \wedge x \in \mathbf{F}) \wedge \forall x \in \mathbf{F}\,\exists y \in \mathbf{P}(y \leq x)$
h-cof	$x = x$	$\forall x \in \mathbf{P}(x \neq 0 \wedge x \in \mathbf{F}) \wedge \forall x \in \mathbf{F}\,\exists y \in \mathbf{P}(y \geq x)$

A cardinal function k is an *order-independence* function if there exists a sentence φ in the language of $(\omega, <, \omega, \omega)$ such that the following two conditions hold:

(1) For any infinite BA A we have $k(A) = \sup\{\lambda$: there exists a sequence $\langle a_\alpha : \alpha < \lambda\rangle$ of elements of A such that for all finite $G, H \subseteq \lambda$ such that $(\lambda, <, G, H) \models \varphi$ we have $\prod_{\alpha \in G} a_\alpha \cdot \prod_{\alpha \in H} -a_\alpha \neq 0\}$.

(2) If λ is an infinite cardinal, $(\lambda, <, G, H) \models \varphi$, $G', H' \subseteq \lambda$, and f is a one-to-one function from $G \cup H$ onto $G' \cup H'$ such that for all $\alpha, \beta \in G \cup H$, if $\alpha < \beta$ then $f(\alpha) < f(\beta)$, then $(\lambda, <, G', H') \models \varphi$.

Some order-independence functions are listed in Table 0.3.

Table 0.3: order-independence functions

Function	φ
t	$\forall x \in \mathbf{G}\,\forall y \in \mathbf{H}\,(x < y)$
hd	$\exists x \in \mathbf{G}\,\forall y \in \mathbf{G}\,(x = y) \wedge \forall x \in \mathbf{G}\,\forall y \in \mathbf{H}\,(x < y)$
hL	$\exists x \in \mathbf{H}\,\forall y \in \mathbf{H}\,(x = y) \wedge \forall x \in \mathbf{G}\,\forall y \in \mathbf{H}\,(x < y)$

Algebraic properties of a single function

Now we go into more detail on the properties of a single function which we shall investigate. From the point of view of general algebra, the main questions are: what happens to the cardinal function k under the passage to subalgebras, homomorphic images, products, and free products? There are natural problems too about more special operations on algebras in general, or on Boolean algebras in particular: what happens to k under weak products, amalgamated free products, unions of well-ordered chains of subalgebras, ultraproducts, dense subalgebras, subdirect products, moderate products, one-point gluing, Alexsandroff duplication, and the exponential? The mentioned operations which are not discussed in the Handbook will be explained in Chapter 1. There are also several special kinds of subalgebras where one can ask what happens to the functions when passing to such a special subalgebra. Many of these special subalgebras are discussed in Heindorf, Shapiro [94]. For ease of reference, we list here ones which we consider to be worthwhile to investigate in this context:

$A \leq_{\mathrm{reg}} B$: A is a regular subalgebra of B. (Handbook, page 21.)

$A \leq_{\mathrm{rc}} B$; A is relatively complete in B. (Handbook, page 123.)

$A \leq_{\pi} B$: A is a dense subalgebra of B.

$A \leq_s B$: B is a simple extension of A. (See Chapter 2.)

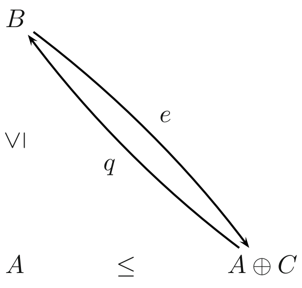
$A \leq_m B$: B is a minimal extension of A. (See Chapter 2.)

$A \leq_{\mathrm{mg}} B$: B is minimally generated over A. (See Chapter 2.)

$A \leq_{\mathrm{free}} B$: B is a free extension of A. This means that $B = A \oplus F$ for some free BA F.

$A \leq_{\sigma} B$: A is σ-embedded in B. This means that $A \leq B$, and for every $b \in B$, the ideal $\{a \in A : a \leq b\}$ of A is countably generated.

$A \leq_{\mathrm{proj}} B$: A is projectively embedded in B. This means that there is a free BA C and homomorphisms $e : B \to A \oplus C$ and $q : A \oplus C \to B$ such that $q \circ e = \mathrm{Id}_B$ and $e \restriction A = q \restriction A = \mathrm{Id}_A$. See Koppelberg [89b], page 752. This is illustrated by the following diagram:



$A \leq_{\mathrm{u}} B$: every ultrafilter on A has at least two different extensions to ultrafilters on B.

One may notice that several of the above functions, such as depth and spread, are defined as supremums of the cardinalities of sets satisfying some property P. So, a natural question is whether such sups are *attained*, that is, with depth as an example, whether for every BA A there always is a subset X well ordered by the Boolean ordering, with $|X| = \text{Depth}(A)$. Of course, this is only a question in case $\text{Depth}(A)$ is a limit cardinal. For such functions k defined by sups, we can define a closely related function k'; $k'(A)$ is the least cardinal such that there is no subset of A with the property P. So $k'(A) = (k(A))^+$ if k is attained, and $k'(A) = k(A)$ otherwise.

Derived functions

From a given cardinal function one can define several others; part of our work is to see what these new cardinal functions look like; frequently it turns out that they coincide with others of our basic functions, but sometimes we arrive at a new function in this way:

$$k_{\mathrm{H}+}(A) = \sup\{k(B) : B \text{ is a homomorphic image of } A\}.$$
$$k_{\mathrm{H}-}(A) = \inf\{k(B) : B \text{ is an infinite homomorphic image of } A\}.$$
$$k_{\mathrm{S}+}(A) = \sup\{k(B) : B \text{ is a subalgebra of } A\}.$$
$$k_{\mathrm{S}-}(A) = \inf\{k(B) : B \text{ is an infinite subalgebra of } A\}.$$
$$k_{\mathrm{h}+}(A) = \sup\{k(Y) : Y \text{ is a subspace of } \mathrm{Ult}A\}.$$
$$k_{\mathrm{h}-}(A) = \inf\{k(Y) : Y \text{ is an infinite subspace of } \mathrm{Ult}A\}.$$
$${}_{\mathrm{d}}k_{\mathrm{S}+}(A) = \sup\{k(B) : B \text{ is a dense subalgebra of } A\}.$$
$${}_{\mathrm{d}}k_{\mathrm{S}-}(A) = \inf\{k(B) : B \text{ is a dense subalgebra of } A\}.$$

Note that $k_{\mathrm{h}+}(A)$ and $k_{\mathrm{h}-}(A)$ make sense only if k is a function which naturally applies to topological spaces in general as well as BAs. Any infinite Boolean space has a denumerable discrete subspace, and frequently $k_{\mathrm{h}-}$ will take its value on such a subspace. Also note with respect to ${}_{\mathrm{d}}k_{\mathrm{S}+}(A)$ and ${}_{\mathrm{d}}k_{\mathrm{S}-}(A)$ that one could consider other kinds of subalgebras, as in the previous list of them.

Given a function defined in terms of ultrafilters, like character above, there is usually an associated function l assigning a cardinal number to each ultrafilter on A. Then one can introduce two cardinal functions on A itself:

$$l_{\sup}(A) = \sup\{l(F) : F \text{ is an ultrafilter on } A\}.$$
$$l_{\inf}(A) = \inf\{l(F) : F \text{ is a non-principal ultrafilter on } A\}.$$

Another kind of derived function applies to cases where the function is defined as the sup of cardinalities of sets X with a property P, where P is such that maximal families with the property P exist (usually seen by Zorn's lemma). For such a function k, we define

$$k_{\mathrm{mm}}(A) = \min\{|X| : X \text{ is an infinite maximal family satisfying } P\};$$
$$k_{\mathrm{spect}}(A) = \{|X| : X \text{ is an infinite maximal family satisfying } P\}.$$

We also consider the following two spectrum functions, which assign to each BA a set of cardinal numbers:

$$\begin{aligned} k_{\mathrm{Hs}}(A) &= \{k(B) : B \text{ is an infinite homomorphic image of } A\} \\ &\quad \text{(the \textit{homomorphic spectrum} of } A) \\ k_{\mathrm{Ss}}(A) &= \{k(B) : B \text{ is an infinite subalgebra of } A\} \\ &\quad \text{(the \textit{subalgebra spectrum} of } A) \end{aligned}$$

It is also possible to define a *caliber* notion for many of our functions, in analogy to the well-known caliber notion for cellularity. Given a property P associated with a cardinal function, a BA A is said to have κ, λ, P-*caliber* if among any set of λ elements of A there are κ elements with property P. The property P is not necessarily one used to define the function; thus for cellularity P is the finite intersection property, while for independence it is, indeed, independence.

Comparing two functions

Given two cardinal functions k and l, one can try to determine whether $k(A) \leq l(A)$ for every BA A or $l(A) \leq k(A)$ for every BA A. Given that one of these cases arises, it is natural to consider whether the difference can be arbitrarily large (as with cellularity and spread, for example), or if it is subject to restrictions (as with depth and length). If no general relationship is known, a counterexample is needed, and again one can try to find a counterexample with an arbitrarily large difference between the two functions. Of course, the known inequalities between our functions help in order to limit the number of cases that need to be considered for constructing such counterexamples; here the diagrams at the end of the book are sometimes useful. For example, knowing that $\pi\chi$ can be greater than c, we also know that χ can be greater than c.

Other considerations

In addition to the above systematic goals in discussing cardinal functions, there are some more ideas which we shall not explore in such detail. One can compare several cardinal functions, instead of just two at a time. Several deep theorems of this sort are known, and we shall mention a few of them. There is also a large number of relationships between cardinal functions which involve cardinal arithmetic; for example, $\mathrm{Length}(A) \leq 2^{\mathrm{Depth}(A)}$ for any BA A. We mention a few of these as we go along.

One can compare two cardinal functions while considering algebraic operations; for example, comparing functions k, l with respect to the formation of subalgebras. We shall investigate just two of the many possibilities here:

$$k_{\mathrm{Sr}}(A) = \{(\kappa,\lambda): \text{ there is an infinite subalgebra } B \text{ of } A \text{ such that } |B| = \lambda \text{ and } k(B) = \kappa\};$$
$$k_{\mathrm{Hr}}(A) = \{(\kappa,\lambda): \text{ there is an infinite homomorphic image } B \text{ of } A \text{ such that } |B| = \lambda \text{ and } k(B) = \kappa\}.$$

These are called, respectively, the *subalgebra k relation* and the *homomorphic k relation.*

For each function k, it would be nice to be able to characterize the possible relations k_{Sr} and k_{Hr} in purely cardinal number terms.

Another general idea applies to several functions that are defined somehow in terms of finite sets; the idea is to take bounded versions of them. For example, independence has bounded versions: for any positive integer n, a subset X of a BA A is called *n-independent* if for every subset Y of X with at most n elements and every $\varepsilon \in {}^Y 2$ we have $\prod_{y\in Y} y^{\varepsilon y} \neq 0$. (Here $x^1 = x$, $x^0 = -x$ for any x.) And then we define $\mathrm{Ind}_n(A) = \sup\{|X| : X \text{ is } n\text{-independent}\}$. It is interesting to investigate this notion and its relationship to actual independence; and similar things can be done for various other functions.

Special classes of Boolean algebras

We are interested in all of the above ideas not only for the class of all BAs, but also for various important subclasses: complete BAs, interval algebras, tree algebras, and superatomic algebras, which are discussed in the Handbook. To a lesser extent we give facts about cardinal functions for other subclasses like all atomic BAs, atomless BAs, initial chain algebras, minimally generated algebras, pseudo-tree algebras, semigroup algebras, and tail algebras. In Chapter 2 we describe some properties of the special classes mentioned which are not discussed in the Handbook, partly to establish notation.

1 Special Operations on Boolean Algebras

We give the basic definitions and facts about several operations on Boolean algebras which were not discussed in the Handbook.

We begin with some elementary but useful results concerning products.

Proposition 1.1. *C is a homomorphic image of $A \times B$* iff *C is isomorphic to $A' \times B'$ for some homomorphic images A' and B' of A and B respectively.*

Proof. We may assume that A and B are non-trivial.

$\Leftarrow$: obvious. $\Rightarrow$: suppose that f is a homomorphism from $A \times B$ onto C. It suffices to show that $C \restriction f(1,0)$ is a homomorphic image of A. Let I be a maximal ideal in A, and for any $a \in A$ let $g(a) = f(a, a/I) \cdot f(1,0)$. Clearly g is a homomorphism from A into $C \restriction f(1,0)$. To show that it is onto, let $x \in C \restriction f(1,0)$. Say $f(a,b) = x$. Then

$$g(a) = f(a, a/I) \cdot f(1,0) = f(a,b) \cdot f(1,0) = x. \qquad \square$$

Proposition 1.2. *Let $\langle A_i : i \in I \rangle$ be a system of non-trivial BAs, with I infinite. Then C is a homomorphic image of $\prod^{\mathrm{w}}_{i \in I} A_i$* iff *there is a system $\langle B_i : i \in I \rangle$ of BAs such that $\forall i \in I[B_i$ is a homomorphic image of $A_i]$, and $C \cong \prod^{\mathrm{w}}_{i \in I} B_i$.*

Proof. For brevity let $D = \prod^{\mathrm{w}}_{i \in I} A_i$.

For $\Leftarrow$, suppose that $f_i : A_i \to B_i$ is a surjective homomorphism for each $i \in I$. Define $g : D \to \prod^{\mathrm{w}}_{i \in I} B_i$ by setting, for each $a \in D$, $(g(a))_i = f_i(a_i)$. Clearly g is a homomorphism from D into $\prod^{\mathrm{w}}_{i \in I} B_i$. To show that it is surjective, let $b \in \prod^{\mathrm{w}}_{i \in I} B_i$. For each $i \in I$ choose $a_i \in A_i$ such that $f_i(a_i) = b_i$, with $a_i = 0$ if $b_i = 0$ and $a_i = 1$ if $b_i = 1$. Clearly $a \in D$ and $g(a) = b$. Thus $\prod^{\mathrm{w}}_{i \in I} B_i \cong D/\ker(g)$.

For $\Rightarrow$, suppose that K is a proper ideal in D. For each $i \in I$ let $L_i = \{b_i : b \in K\}$. Clearly L_i is an ideal in A_i.

Case 1. There is an $a \in K$ of type 2. Let F be the 2-support of a. Define $f([b]_K) = \langle [b_i]_{L_i} : i \in F \rangle$ for each $b \in D$. Clearly f is well defined, and is a homomorphism from D/K into $\prod_{i \in F} A_i/L_i$. It is one-one, for suppose that $b_i \in L_i$ for all $i \in F$. For each $i \in F$ choose $c^i \in K$ such that $b_i = c^i_i$. Then $b \le a + \sum_{i \in F} c^i \in K$, and so

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$b \in K$. It maps onto $\prod_{i\in F} A_i/L_i$; for suppose that $x_i \in A_i$ for each $i \in F$. Extend to a function $x \in D$. Then $f([x]_K) = \langle [x_i]_{L_i} : i \in F\rangle$, as desired. Let $B_i = A_i/L_i$ for all $i \in F$, and let B_i be trivial for all $i \in I\backslash F$.

Case 2. Every $a \in K$ is of type 1. For each $i \in I$ define $\chi^i \in D$ by setting, for each $j \in I$,

$$\chi^i(j) = \begin{cases} 1 & \text{if } \; i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $J = \{i \in I : \chi^i \notin K\}$. If $J = \emptyset$, then K is a maximal ideal in D, and $|D/K| = 2$, giving the desired result, letting one B_i be a two element homomorphic image of A_i, the other B_is trivial.

Suppose that $\emptyset \neq J$ and J is finite. Let M be the maximal ideal of D consisting of all elements of type 1. For each $b \in D$ define $f([b]_K) = (\langle [b_i]_{L_i} : i \in J\rangle, [b]_M)$. Clearly f is a well-defined homomorphism from D/K into $(\prod_{i\in J}(A_i/L_i)) \times 2$. f is one-one: suppose that $b_i \in L_i$ for all $i \in J$ and $b \in M$. For each $i \in J$ choose $c^i \in K$ such that $b_i = c^i_i$. Let F be the 1-support of b. Define $d \in D$ by

$$d(i) = \begin{cases} c^i_i & \text{if } \; i \in J, \\ 1 & \text{if } \; i \in F\backslash J, \\ 0 & \text{otherwise.} \end{cases}$$

Then $d \in K$ and $b \leq d$, so $b \in K$. Also, f maps onto $(\prod_{i\in J} A_i/L_i) \times 2$. For, suppose that $c \in \prod_{i\in J} A_i$ and $\varepsilon \in 2$. If $\varepsilon = 1$, extend c to $b \in D$ by defining $b(i) = 1$ for all $i \in I\backslash J$. Clearly $f([b]_K) = (\langle [c_i]_{L_i} : i \in J\rangle, 1)$. If $\varepsilon = 0$, extend c to $b \in D$ by defining $b(i) = 0$ for all $i \in I\backslash J$. Clearly $f([b]_K) = (\langle [c_i]_{L_i} : i \in J\rangle, 0)$. Now we can let $B_i = A/L_i$ for all $i \in J$, B_i a two-element homomorphic image of A_i for some $i \in I\backslash J$, and all other B_is trivial.

Suppose that J is infinite. For each $b \in D$ define $f([b]_K) = \langle [b_i]_{L_i} : i \in J\rangle$. Clearly f is a well-defined homomorphism from D/K into $\prod^{\mathrm{w}}_{i\in J} A_i/L_i$. f is one-one: suppose that $b_i \in L_i$ for all $i \in J$. For each $i \in J$ choose $c^i \in K$ such that $b_i = c^i_i$. Since J is infinite, $1 \notin L_i$ for each $i \in J$, and $b_i \in L_i$ for each $i \in J$, it follows that b is of type 1. Let F be the 1-support of b. Define $d \in D$ by

$$d(i) = \begin{cases} c^i_i & \text{if } \; i \in J, \\ 1 & \text{if } \; i \in F\backslash J, \\ 0 & \text{otherwise.} \end{cases}$$

Then $d \in K$ and $b \leq d$, so $b \in K$. Also, f maps onto $\prod^{\mathrm{w}}_{i\in J} A_i/L_i$. For, suppose that $c \in \prod^{\mathrm{w}}_{i\in J} A_i$. If c is of type 2, extend c to $b \in D$ by defining $b(i) = 1$ for all $i \in I\backslash J$. Clearly $f([b]_K) = \langle [c_i]_{L_i} : i \in J\rangle$. If c is of type 1, extend c to $b \in D$ by defining $b(i) = 0$ for all $i \in I\backslash J$. Clearly $f([b]_K) = \langle [c_i]_{L_i} : i \in J\rangle$. Now we can define $B_i = A_i/L_i$ for all $i \in J$, with the other B_is trivial. □

Concerning arbitrary products, we have the following simple result.

Proposition 1.3. *Let $\langle A_k : k \in K\rangle$ be a system of BAs, and $\langle I_k : k \in K\rangle$ a system such that I_k is an ideal in A_k for all $k \in K$. Let $J = \{a \in \prod_{k\in K} A_k : \forall k \in K[a_k \in I_k]\}$. Then J is an ideal in $\prod_{k\in K} A_k$, and $(\prod_{k\in K} A_k)/J \cong \prod_{k\in K}(A_k/I_k)$.*

Proof. Clearly J is an ideal in $\prod_{k\in K} A_k$. Now for each $a \in \prod_{k\in K} A_k$ and each $k \in K$ let $(f(a))_k = [a_k]_{I_k}$. Clearly f is a homomorphism from $\prod_{k\in K} A_k$ onto $\prod_{k\in K}(A_k/I_k)$ with kernel J, so the result follows by the homomorphism theorem. □

However, the property of Proposition 1.2 does not extend to arbitrary products. In the following example we use the notation Finco(I) for the BA of finite and cofinite subsets of I.

Example 1.4. For each $k \in \omega$ let $A_k = \text{Finco}(\omega)$. Then ${}^\omega 2$ is a subalgebra of $\prod_{k\in\omega} A_k$. Let f be a homomorphism from ${}^\omega 2$ onto $\mathscr{P}(\omega)/\text{fin}$, and extend f to a homomorphism g from $\prod_{k\in\omega} A_k$ into the completion of $\mathscr{P}(\omega)/\text{fin}$. Then there is an ideal J on $\prod_{k\in\omega} A_k$ such that $(\prod_{k\in\omega} A_k)/J$ is isomorphic to $\text{rng}(g)$. Note that $\text{rng}(g)$ is atomless. But if $\langle I_k : k \in \omega\rangle$ is any sequence of ideals on Finco(ω), then $\prod_{k\in\omega}(A_k/I_k)$ is atomic. Thus $(\prod_{k\in\omega} A_k)/J$ is not isomorphic to $\prod_{k\in\omega}(A_k/I_k)$.

Moderate products

This operation, due to Weese [80] and Gurevich [82] independently, is extensively studied in Heindorf [92], to whom the name is due. Suppose that $\langle A_i : i \in I\rangle$ is a system of BAs; we assume that A_i is a field of subsets of some set J_i, and that the J_i's are pairwise disjoint. Furthermore, let B be an algebra of subsets of I containing all of the finite subsets of I. For each $b \in B$ let $\bar{b} = \bigcup_{i\in b} J_i$. Set $K = \bigcup_{i\in I} J_i$. For each $b \in B$, each finite $F \subset I$, and each $a \in \prod_{i\in F} A_i$, the set

$$\bar{b} \cup \bigcup_{i\in F} a_i$$

will be denoted by $h(b, F, a)$. If $F \cap b = \emptyset$ and $0 \subset a_i \subset J_i$ for every $i \in F$, then we call (b, F, a) *normal.*

Proposition 1.5. *Assume the above notation.*

(i) *For any $b \in B$, $F \in [I]^{<\omega}$, and $a \in \prod_{i\in F} A_i$ we have $h(b, F, a) = h(b', F', a')$, where $b' = b \cup \{i \in F : a_i = J_i\}$, $F' = \{i \in F\backslash b : \emptyset \subset a_i \subset J_i\}$, and $a' = a \restriction F'$; moreover, (b', F', a') is normal.*

(ii) *If (b, F, a) is normal, then $K\backslash h(b, F, a) = h(I\backslash(b\cup F), F, a')$, where $a'_i = J_i\backslash a_i$ for all $i \in F$; and $(I\backslash(b \cup F), F, a')$ is normal.*

(iii) *$h(b, F, a) \cap h(b', F', a') = h(b'', F'', a'')$ for normal (b, F, a) and (b', F', a'), where $b'' = b \cap b'$, $F'' = (F' \cap b) \cup (F \cap b') \cup \{i \in F \cap F' : a_i \cap a'_i \neq \emptyset\}$, and* *for any* $i \in F''$,

$$a''_i = \begin{cases} a'_i & \textit{if } \; i \in F' \cap b, \\ a_i & \textit{if } \; i \in F \cap b', \\ a_i \cap a'_i & \textit{if } \; i \in F \cap F' \textit{ and } a_i \cap a'_i \neq \emptyset; \end{cases}$$

and (b'', F'', a'') *is normal.*

(iv) *If* (b, F, a) *and* (b', F', a') *are normal, then* $h(b, F, a) \subseteq h(b', F', a')$ *iff* $b \subseteq b'$, $F' \cap b = \emptyset$, $F \subseteq b' \cup F'$, *and* $\forall i \in F \cap F'[a_i \subseteq a'_i]$.

(v) *If* (b, F, a) *and* (b', F', a') *are normal and* $h(b, F, a) = h(b', F', a')$, *then* $b = b'$, $F = F'$, *and* $a = a'$.

Proof. It is straightforward to check (i)–(iii). For (iv), suppose that (b, F, a) and (b', F', a') are normal.

First suppose that $h(b, F, a) \subseteq h(b', F', a')$. Clearly $b \subseteq b'$ and $F' \cap b = \emptyset$. Next, suppose that $i \in F \backslash b'$. Then $a_i \subseteq h(b', F', a')$, so $i \in F'$. Now suppose that $i \in F \cap F'$. Then $a_i \subseteq h(b, F, i)$ and $i \notin b'$, so $a_i \subseteq a'_i$.

Second, suppose that $b \subseteq b'$, $F' \cap b = \emptyset$, $F \subseteq b' \cup F'$, and $\forall i \in F \cap F'[a_i \subseteq a'_i]$. Take any $x \in h(b, F, a)$. If $x \in b$, then $x \in b'$ and hence $x \in h(b', F', a')$. Suppose that $i \in F$ and $x \in a_i$. If $i \in b'$, then $x \in h(b', F', a')$. Suppose that $i \notin b'$. So $i \in F \backslash b'$, and so $i \in F'$. Since then $i \in F \cap F'$, we have $a_i \subseteq a'_i$, and so $x \in h(b', F', a')$.

(v) follows from (iv). □

The BA of all sets $h(b, F, a)$ is the *moderate product of the* A_i*'s over* B, and is denoted by $\prod^B_{i \in I} A_i$.

Theorem 1.6. *Suppose that* $\langle A_i : i \in I \rangle$ *is a system of BAs; each* A_i *is a field of subsets of some set* J_i*; the* J_i*'s are pairwise disjoint, and* $\mathrm{Finco}(I) \leq B \leq \mathscr{P}(I)$.

(i) *For every* $i \in I$ *we have* $J_i \in \prod^B_{i \in I} A_i$ *and* $A_i = \left(\prod^B_{i \in I} A_i\right) \restriction J_i$.

(ii) B *is isomorphic to a subalgebra of* $\prod^B_{i \in I} A_i$*; in fact,* $\langle \overline{b} : b \in B \rangle$ *is an isomorphic embedding.*

(iii) *If* $\mathrm{Finco}(I) \leq B \leq C \leq \mathscr{P}(I)$, *then* $\prod^B_{i \in I} A_i \leq \prod^C_{i \in I} A_i$.

(iv) $\prod^B_{i \in I} A_i$ *can be embedded in* $\prod_{i \in I} A_i$.

(v) $\prod^{\mathrm{w}}_{i \in I} A_i \cong \prod^{\mathrm{Finco}(I)}_{i \in I} A_i$.

(vi) *If* I *is finite, then* $B = \mathscr{P}(I)$, *and* $\prod^B_{i \in I} A_i \cong \prod_{i \in I} A_i$.

(vii) *For each* $i \in I$ *and each* $x \in \prod^B_{i \in I} A_i$ *let* $f(x) = x \cap J_i$. *Then* f *is a homomorphism from* $\prod^B_{i \in I} A_i$ *onto* A_i.

(viii) *Let* g *be the natural isomorphism of* $\mathscr{P}(I)$ *onto* ${}^I2$. *Then*

$$\prod^B_{i \in I} A_i \cong \left\langle \prod^{\mathrm{w}}_{i \in I} A_i \cup g[B] \right\rangle_{\prod_{i \in I} A_i}.$$

(ix) *If* $b \in [I]^{<\omega}$, *then* $\prod_{i\in b} A_i \cong \left(\prod_{i\in I}^{B} A_i\right) \restriction \overline{b}$.

(x) *Suppose that* $U \subseteq \prod_{i\in I}^{B} A_i$. *Then* U *is an ultrafilter on* $\prod_{i\in I}^{B} A_i$ iff *one of the following holds:*

  (a) *There exist an* $i \in I$ *and an ultrafilter* V *on* A_i *such that* $U = \{h(b,F,a) : (b,F,a)$ *is normal, and* $i \in b$ *or* $(i \notin b$ *and* $i \in F$ *and* $a_i \in V)\}$.

  (b) *There is a nonprincipal ultrafilter* W *on* B *such that* $U = \{h(b,F,a) : (b,F,a)$ *is normal and* $b \in W\}$.

Proof. (i)–(iii) are clear by Proposition 1.5. For (iv), define $(f(x))_i = x \cap J_i$ for any $x \in \prod_{i\in I}^{B} A_i$ and each $i \in I$. We show that f is the desired embedding. For $\cdot$, let normal $(b,F,a), (b',F',a')$ be given, and let (b'',F'',a'') be as in Proposition 1.5(iii). If $i \in b''$, then

$$[h(b,F,a) \cap h(b',F',a')] \cap J_i = J_i = [h(b,F,a) \cap J_i] \cap [h(b',F',a') \cap J_i].$$

If $i \in F' \cap b$, then

$$[h(b,F,a) \cap h(b',F',a')] \cap J_i = J_i \cap a'_i = [h(b,F,a) \cap J_i] \cap [h(b',F',a') \cap J_i].$$

If $i \in F \cap b'$, then

$$[h(b,F,a) \cap h(b',F',a')] \cap J_i = a_i \cap J_i = [h(b,F,a) \cap J_i] \cap [h(b',F',a') \cap J_i].$$

Finally, if $i \in F \cap F'$ and $a_i \cap a'_i \neq \emptyset$, then

$$[h(b,F,a) \cap h(b',F',a')] \cap J_i = a_i \cap a'_i = [h(b,F,a) \cap J_i] \cap [h(b',F',a') \cap J_i].$$

For $-$, recall Proposition 1.5(ii). If $i \in b$, then

$$[K \backslash h(b,F,a)] \cap J_i = \emptyset = J_i \backslash J_i = J_i \backslash [h(b,F,a) \cap J_i].$$

If $i \in F$, then

$$[K \backslash h(b,F,a)] \cap J_i = J_i \backslash a_i = J_i \backslash [h(b,F,a) \cap J_i].$$

Finally, if $i \in I \backslash (b \cup F)$, then

$$[K \backslash h(b,F,a)] \cap J_i = J_i = J_i \backslash \emptyset = J_i \backslash [h(b,F,a) \cap J_i].$$

Clearly f is one-one; this finishes the proof of (iv).

In case $B = \mathrm{Finco}(I)$, this mapping is easily seen to be onto $\prod_{i\in I}^{\mathrm{w}} A_i$, proving (v).

(vi) clearly follows from (v).

For (vii), clearly f is a homomorphism from $\prod_{i\in I}^{B} A_i$ into A_i. For each $x \in A_i$ we have $x \in \prod_{i\in I}^{B} A_i$ and $f(x) = x$. So f maps onto A_i.

For (viii), we use the function f defined in the proof of (iv). Note that if $b \in B$, then $f(\overline{b}) = g(b)$, and if F is a finite subset of I and $a \in \prod_{i\in I} A_i$, then $f(h(\emptyset, F, a) = k$, where

$$k(i) = \begin{cases} a_i & \text{if } i \in F, \\ \emptyset & \text{otherwise.} \end{cases}$$

If (b, F, a) is normal, then $h(b, F, a) = \overline{b} \cup h(\emptyset, F, a)$. Hence $\{\overline{b} : b \in B\} \cup \{h(\emptyset, F, a) : F \in [I]^{<\omega}, a \in \prod_{i\in I} A_i\}$ generates $\prod_{i\in I}^{B} A_i$. Clearly also the image of this set under f generates the right side of the equation in (viii). Hence (viii) follows.

For (ix), define $f(x) = \bigcup_{i\in b} x_i$ for any $x \in \prod_{i\in b} A_i$; clearly this is the desired isomorphism.

Finally, we consider (x). First suppose that U is an ultrafilter on $\prod_{i\in I}^{B} A_i$.

Case 1. There is an $i \in I$ such that $h(\{i\}, \emptyset, \emptyset) \in U$. Let $V = \{x \in A_i^+ : h(\emptyset, \{i\}, \{(i, x)\}) \in U$. Suppose that $x \in V$ and $x \subseteq y \subseteq J_i$. Then

$$h(\emptyset, \{i\}, \{(i, x)\}) \subseteq h(\emptyset, \{i\}, \{(i, y)\}),$$

so $h(\emptyset, \{i\}, \{(i, y)\}) \in U$. It follows that $y \in V$. Next suppose that $x, y \in V$. Now $h(\emptyset, \{i\}, \{(i, x)\}) \cap h(\emptyset, \{i\}, \{(i, y)\}) = h(\emptyset, \{i\}, \{(i, x \cap y)\})$. Thus $h(\emptyset, \{i\}, \{(i, x \cap y)\}) \in U$, hence $x \cap y \in V$. So V is a filter. Now let $x \in A_i$, and suppose that $x \notin V$. Thus $K \backslash h(\emptyset, \{i\}, \{(i, x)\}) \in U$. Now

$$K \backslash h(\emptyset, \{i\}, \{(i, x)\}) = h(I \backslash \{i\}, \{i\}, \{(i, J_i \backslash x)\}),$$

and

$$h(\{i\}, \emptyset, \emptyset) \cap h(I \backslash \{i\}, \{i\}, \{(i, J_i \backslash x)\}) = h(\emptyset, \{i\}, \{(i, J_i \backslash x)\}),$$

so $h(\emptyset, \{i\}, \{(i, J_i \backslash x)\}) \in U$. It follows that $J_i \backslash x \in V$. So V is an ultrafilter on A_i.

Now suppose that $h(b, F, a) \in U$ with (b, F, a) normal. Suppose that $i \notin b$. Now $h(b, F, a) \cap h(\{i\}, \emptyset, \emptyset) \in U$ and hence this set is nonempty, so it follows that $i \in F$, and hence

$$h(b, F, a) \cap h(\{i\}, \emptyset, \emptyset) = h(\emptyset, \{i\}, \{(i, a_i)\});$$

hence $a_i \in V$.

Conversely, suppose that (b, F, a) is normal. Suppose first that $i \in b$. Then we have $h(\{i\}, \emptyset, \emptyset) \subseteq h(b, F, a)$, so $h(b, F, a) \in U$. Second, suppose that $i \notin b$, $i \in F$, and $a_i \in V$. Then $h(\emptyset, \{i\}, \{(i, a_i)\}) \in U$. Then $h(\emptyset, \{i\}, \{(i, a_i)\}) \subseteq h(b, F, a)$, so $h(b, F, a) \in U$.

Thus we have shown that Case 1 implies (a).

Case 2. There is no $i \in I$ such that $h(\{i\}, \emptyset, \emptyset) \in U$. Let $W = \{b \in B : h(b, \emptyset, \emptyset) \in U\}$. Clearly W is a nonprincipal ultrafilter on B. Suppose that $h(b, F, a) \in U$ with (b, F, a) normal. For each $i \in F$ we have $K \backslash h(\{i\}, \emptyset, \emptyset) = h(I \backslash \{i\}, \emptyset, \emptyset) \in U$, and hence

$$h(b, F, a) \cap \bigcap_{i\in F} h(I \backslash \{i\}, \emptyset, \emptyset) = h(b, \emptyset, \emptyset)$$

is in U. So $b \in W$.

Conversely, suppose that (b, F, a) is normal and $b \in W$. Since $h(b, \emptyset, \emptyset) \subseteq h(b, F, a)$, we have $h(b, F, a) \in U$.

This finishes the proof of $\Rightarrow$.

For $\Leftarrow$, first suppose that (a) holds; we want to show that U is an ultrafilter on $\prod^B_{i\in I} A_i$. Suppose that $h(b, F, a) \in U$, $h(b, F, a) \subseteq h(b', F', a')$, with (b, F, a) and (b', F', a') normal.

Case 1. $i \in b$. Then also $i \in b'$, so $h(b', F', a') \in U$.

Case 2. $i \notin b$, $i \in F$, and $a_i \in V$. If $i \in b'$, then $h(b', F', a') \in U$. Suppose that $i \notin b'$. Then $i \in F'$, so $i \in F \cap F'$ and hence $a_i \subseteq a'_i$. Hence $a'_i \in V$ and so $h(b', F', a') \in U$.

Next, suppose that $h(b, F, a), h(b', F', a') \in U$, with $(b, F, a), (b', F', a')$ normal. Then $h(b, F, a) \cap h(b', F', a') = h(b'', F'', a'')$ as in 1.5(iii). We consider some subcases.

Subcase 2.1. $i \in b \cap b'$. Then $h(b'', F'', a'') \in U$.

Subcase 2.2. $i \in b\backslash b'$. Then $i \in F'$ and $a'_i \in V$. Now $i \in F' \cap b \subseteq F''$ and $a''_i = a'_i$, so $h(b'', F'', a'') \in U$.

Subcase 2.3. $i \in b'\backslash b$. Similar to Subcase 2.2.

Subcase 2.4. $i \notin b \cup b'$. Then $i \in F$, $a_i \in V$, $i \in F'$, $a'_i \in V$. So $i \in F \cap F'$ and $a_i \cap a'_i \in V$, hence $a_i \cap a'_i \neq \emptyset$. So $h(b'', F'', a'') \in U$.

Thus U is a filter. Clearly $\emptyset \notin U$. Finally, suppose that a normal (b, F, a) is given. Suppose that $i \notin b$ and ($i \notin F$, or $i \in F$ and $a_i \notin V$). If $i \notin F$, then $i \in I\backslash(b \cup F)$ and hence $K\backslash h(b, F, a) \in U$. Suppose that $i \in F$ and $a_i \notin V$. Then $J_i\backslash a_i \in V$, and hence $K\backslash h(b, F, a) = h(I\backslash(b \cup F), F, a') \in U$, where $a'_j = J_j\backslash a_j$ for all $j \in F$.

Second, suppose that (b) holds. Clearly U is a proper filter. To show that it is an ultrafilter, suppose that (b, F, a) is normal and $h(b, F, a) \notin U$. Now $K\backslash h(b, F, a) = h(I\backslash(b \cup F), F, a')$ with $a'_i = J_i\backslash a_i$ for all $i \in F$. Since $h(b, F, a) \notin U$, we have $b \notin W$, hence $I\backslash b \in W$. Since W is nonprincipal, we have $\{i\} \notin W$ for all $i \in F$, hence $(I\backslash\{i\}) \in W$. Hence $I\backslash(b \cup F) = (I\backslash b) \cap \bigcap_{i\in F}(I\backslash\{i\}) \in W$, and so $(K\backslash h(b, F, a)) \in U$. So U is an ultrafilter. $\square$

Theorem 1.6(viii) suggests an alternative formulation of the notion of moderate products. Let I be an infinite set, $\langle A_i : i \in I\rangle$ a system of BAs, and $\mathrm{Finco}(I) \leq B \leq \mathscr{P}(I)$. Let g be the natural isomorphism from $\mathscr{P}(I)$ onto ${}^I2$. Then we can take the moderate product to be

$$\left\langle \prod^{\mathrm{w}}_{i\in I} A_i \cup g[B] \right\rangle_{\prod_{i\in I} A_i}.$$

Note that here one does not need to assume that each A_i is an algebra of sets.

Theorem 1.7. *Assume the hypotheses of Theorem* 1.6, *and suppose that* $L \stackrel{\text{def}}{=} \{i \in I : |A_i| > 2\}$ *is infinite. Then* $\prod_{i\in I}^B A_i$ *is not complete.*

Proof. For each $i \in L$ choose $a_i \in A_i$ such that $0 \subset a_i \subset J_i$. Suppose that $\sum_{i\in L} a_i$ exists in $\prod_{i\in I}^B A_i$; say it is equal to $h(b, F, c)$, where we may assume that (b, F, c) is normal. Fix $i \in L\backslash F$. Then $a_i \subseteq h(b, F, c)$ implies that $i \in b$. But then $h(b\backslash\{i\}, F', c')$ is still an upper bound, where $F' = F \cup \{i\}$ and c' extends c with $c'_i = a_i$. Since $h(b\backslash\{i\}, F', c') \subset h(b, F, c)$, this is a contradiction. □

It is clear that if each A_i is atomless, then so is $\prod_{i\in I}^B A_i$; similarly for each A_i atomic. It is somewhat less trivial to check that the moderate product preserves superatomicity:

Theorem 1.8. *If each* A_i *is superatomic and also* B *is superatomic, then* $\prod_{i\in I}^B A_i$ *is superatomic.*

Proof. For brevity write $C = \prod_{i\in I}^B A_i$. It suffices to show that if f is a homomorphism from C onto a nontrivial BA D, then D has an atom. We consider two cases.

Case 1. $f(J_i) \neq 0$ for some $i \in I$. Let $f' = f \restriction A_i$. Clearly f' is a homomorphism from A_i onto $D \restriction f(J_i)$. Let $f'(u_i)$ be an atom of $D \restriction f(J_i)$; this is possible since A_i is superatomic. Clearly $f'(u_i)$ is also an atom of D.

Case 2. $f(J_i) = 0$ for all $i \in I$. Note that $h(b, F, a) = h(b, \emptyset, \emptyset) \cup h(\emptyset, F, a)$ for any b, F, a. Hence $f(h(b, F, a)) = f(h(b, \emptyset, \emptyset))$. For each $b \in B$, let $k(b) = f(h(b, \emptyset, \emptyset)$. Clearly k is a homomorphism from B onto D, so D has an atom since B is superatomic. □

An important use of moderate products in connection with homomorphisms has been given by Koszmider [99]. We give details on his construction.

Proposition 1.9. *If* K *is an ideal in* $\prod_{i\in I}^B A_i$ *and* $i \in I$, *then* $\{x \cap J_i : x \in K\}$ *is an ideal in* A_i.

Proof. Clearly $\{x \cap J_i : x \in K\}$ is closed under unions. Now suppose that $y \in A_i$ and $y \leq x \cap J_i$ with $x \in K$. Clearly $y \in \prod_{i\in I}^B A_i$ and $y = y \cap J_i$, so y is in the set too. □

Note in fact that this ideal is merely $K \cap A_i$.

Proposition 1.10. *Suppose that* C *is an infinite homomorphic image of* $\prod_{i\in I}^B A_i$, *with* I *infinite. Then there is an infinite homomorphic image* D *of* C *which is also a homomorphic image of* B *or of some* A_i.

Proof. Let K be the kernel of a homomorphism of $\prod_{i\in I}^B A_i$ onto C. So $\prod_{i\in I}^B A_i/K$ is isomorphic to C and hence is infinite. For each $i \in I$ let $L_i = K \cap A_i$. So by Proposition 1.9, each L_i is an ideal in A_i. We now consider several cases.

Case 1. $N \stackrel{\text{def}}{=} \{i \in I : J_i \notin K\}$ is finite, and $\forall i \in I[A_i/L_i$ is finite]. Let $M = \langle K \cup \{J_i : i \in I, J_i \notin K\}\rangle^{\text{id}}$.

(1) $\prod_{i\in I}^B A_i/M$ is infinite.

In fact, define $f : \prod_{i\in I}^B A_i/K \to \prod_{i\in I}^B A_i/M \times \prod_{i\in N}(A_i/L_i)$ by

$$f([a]_K) = ([a]_M, \langle [a_i]_{L_i} : i \in N\rangle).$$

First of all, f is well defined. For, if $a \in K$, then $a \in M$, and $a_i \in L_i$ for each $i \in N$. Moreover, f is an injection. For, if $a \in M$ and $a_i \in L_i$ for each $i \in N$, then there is a $b \in K$ such that $a \subseteq b \cup \bigcup_{i\in N} J_i$, so

$$\begin{aligned} a &= \left[a \setminus \bigcup_{i\in N} J_i\right] \cup \left[a \cap \bigcup_{i\in N} J_i\right] \\ &= \left[a \setminus \bigcup_{i\in N} J_i\right] \cup \bigcup_{i\in N} a_i \\ &\subseteq b \cup \bigcup_{i\in N} a_i \\ &\in K. \end{aligned}$$

Thus, indeed, f is an injection. Hence (1) follows.

Now define $g(b) = [h(b,\emptyset,\emptyset)]_M$ for any $b \in B$. Then g is a homomorphism of B into $\prod_{i\in I}^B A_i/M$, by Theorem 1.6(ii). For any element $h(b,F,a)$ of $\prod_{i\in I}^B A_i$ we have $[h(b,F,a)]_M = [h(b,\emptyset,\emptyset)]_M$, since $h(b,F,a) \triangle h(b,\emptyset,\emptyset) = \bigcup_{i\in F} a_i$, and for all $i \in F$, if $i \notin N$ then $a_i \subseteq J_i \in K$, while if $i \in N$, then still $a_i \subseteq J_i \in M$. Thus g maps onto $\prod_{i\in I}^B A_i/M$, and so $\prod_{i\in I}^B A_i/M$ is as desired.

Case 2. N is infinite, and $\forall i \in I[A_i/L_i$ is finite]. For each $i \in N$ let M_i be a maximal ideal in A_i such that $L_i \subseteq M_i$. Let P be the ideal generated by $K \cup \bigcup_{i\in N} M_i$. For each $i \in N$ we have $J_i \notin P$, and so $\prod_{i\in I}^B A_i/P$ is infinite. Define $g(b) = [h(b,\emptyset,\emptyset)]_P$ for all $b \in B$. Then g is a homomorphism of B into $\prod_{i\in I}^B A_i/M$, by Theorem 1.6(ii). We claim that it is a surjection. Let $h(b,F,a)$ be any element of $\prod_{i\in I}^B A_i$, with (b,F,a) normal, and let $c = b \cup \{i \in F : a_i \notin M_i\}$. Then

$$\begin{aligned} h(b,F,a) \triangle h(c,\emptyset,\emptyset) &= (h(b,F,a) \setminus h(c,\emptyset,\emptyset)) \cup (h(c,\emptyset,\emptyset) \setminus h(b,F,a)) \\ &= \bigcup_{\substack{i\in F \\ a_i \in M_i}} a_i \cup \bigcup_{\substack{i\in F \\ a_i \notin M_i}} (J_i \setminus a_i). \end{aligned}$$

Now if $i \in F$ and $a_i \in M_i$ then $a_i \in P$, while if $i \in F$ and $a_i \notin M_i$, then $(J_i \setminus a_i) \in M_i \subseteq P$. Hence g maps onto $\prod_{i\in I}^B A_i/P$, as desired.

Case 3. There is an $i_0 \in I$ such that A_{i_0}/L_{i_0} is infinite. Let M be the ideal generated by $K \cup \left\{\bigcup_{i\neq i_0} J_i\right\}$, and define

$$g([h(b,F,a)]_M) = [h(b,F,a) \cap J_{i_0}]_{L_{i_0}}.$$

If $h(b,F,a) \in M$, then there is a $c \in K$ such that $h(b,F,a) \subseteq c \cup \bigcup_{i \neq i_0} J_i$, hence $h(b,F,a) \cap J_{i_0} \subseteq c$, and so $h(b,F,a) \cap J_{i_0} \in (K \cap A_{i_0}) = L_{i_0}$. So g is well defined. It clearly maps onto A_{i_0}/L_{i_0}, as desired. $\square$

Proposition 1.11. *Suppose that X is a subset of $\prod_{i\in I}^B A_i$ with $|X| = \kappa$ uncountable and regular, $\kappa > |I|$. Then there exist $Y \in [X]^\kappa$ and a finite $G \subseteq I$ such that $\langle Y\rangle$ is isomorphic to a subalgebra of $(B \restriction (I\backslash G)) \times \prod_{i\in G} A_i$.*

Proof. Let $\langle F_\xi : \xi < \kappa\rangle \in {}^\kappa([I]^{<\omega})$, $b \in {}^\kappa B$ and a with domain κ be such that $a_\xi \in \prod_{i\in F_\xi} A_i$ for all $\xi < \kappa$, (b_ξ, F_ξ, a_ξ) is normal, and

$$X = \{h(b_\xi, F_\xi, a_\xi) : \xi < \kappa\}.$$

Choose $Y \in [\kappa]^\kappa$ and $G \in [I]^{<\omega}$ such that $F_\xi = G$ for all $\xi \in Y$. Let

$$W = \left\{ h(c,G,d) : c \in B,\ c \cap G = \emptyset,\ d \in \prod_{i\in G} A_i \right\}.$$

This is a subalgebra of $\prod_{i\in I}^B A_i$ by the above computation rules, and $Y \subseteq W$. Now define

$$f(h(c,G,d)) = (h(c,\emptyset,\emptyset), d)$$

for each $h(c,G,d) \in W$. We claim that f is an isomorphism from W into $(B \restriction (I\backslash G)) \times \prod_{i\in G} A_i$. In fact, f clearly preserves $\cdot$. For $-$,

$$\begin{aligned} f(-h(c,G,d)) &= f(h(I\backslash(c\cup G), G, -d) \\ &= (h(I\backslash(c\cup G), \emptyset, \emptyset), -d) \\ &= (h(I\backslash G, \emptyset, \emptyset)\cdot h(I\backslash c, \emptyset, \emptyset), -d) \\ &= (h(I\backslash G, \emptyset, \emptyset)\cdot -h(c, \emptyset, \emptyset), -d) \\ &= -f(h(c,G,d)). \end{aligned}$$ $\square$

Let L be the set of all countable limit ordinals, and L_2 the set of all countable limits of elements of L. Let $\langle c_\alpha^n : \alpha \in L_2, n \in \omega\rangle$ be such that for all $\alpha \in L_2$, $\langle c_\alpha^n : n \in \omega\rangle$ is strictly increasing, cofinal in α, with $c_\alpha^0 = 0$ and $c_\alpha^{i+1} \in L$ for all $i \in \omega$.

Let A and B be subalgebras of $\mathscr{P}(\omega)$, with $[\omega]^{<\omega} \subseteq B$. We construct $\langle C_\alpha : \alpha \in L\rangle$ by recursion. Each C_α will be a subalgebra of $\mathscr{P}(\alpha)$. For each $\beta \in L$ let f_β be a bijection from ω to $[\beta, \beta+\omega)$.

Define $C_\omega = A$. If C_α has been defined, with $\alpha \in L$, let

$$C_{\alpha+\omega} = \langle C_\alpha \cup \{f_\alpha[a] : a \in A\}\rangle_{\mathscr{P}(\alpha+\omega)}.$$

If $\beta \in L_2$ and C_α has been defined for all $\alpha < \beta$ such that $\alpha \in L$, let $C_\beta = \prod_{i\in\omega}^B D_i^\beta$, with

$$D_i^\beta = C_{c_\beta^{i+1}} \restriction [c_\beta^i, c_\beta^{i+1}).$$

Here in general $G \restriction h = \{g \cdot h : g \in G\}$ for G a subalgebra of H and $h \in H$, without assuming that $h \in G$. Next, for each $\alpha \in L$ let $E_\alpha = C_\alpha \cup \{a : \omega_1 \backslash a \in C_\alpha\}$. Thus E_α is a subalgebra of $\mathscr{P}(\omega_1)$. Finally, let

$$\prod_c^B A = \left\langle \bigcup_{\alpha \in L} C_\alpha \right\rangle_{\mathscr{P}(\omega_1)}.$$

Lemma 1.12. *Assume the above notation. Then*

(i) *If $\alpha, \beta \in L$ and $\alpha < \beta$, then $C_\alpha \subseteq C_\beta$.*
(ii) *If $\alpha, \beta \in L$ and $\alpha \leq \beta$, then $\alpha \in C_\beta$.*
(iii) *If $\alpha_1, \alpha_2, \beta \in L$ and $\alpha_1 < \alpha_2 \leq \beta$, then $[\alpha_1, \alpha_2) \in C_\beta$.*
(iv) *If $\alpha, \beta \in L$, $\alpha < \beta$, define $f_{\beta\alpha}(a) = a \cap \alpha$ for each $a \in C_\beta$. Then $f_{\beta\alpha}$ is a homomorphism from C_β onto C_α which is the identity on C_α.*
(v) *If $\alpha, \beta \in L$ and $\alpha < \beta$, then $E_\alpha \subseteq E_\beta$, and there is a homomorphism from E_β onto E_α which is the identity on E_α.*
(vi) $\prod_c^B A = \bigcup_{\alpha \in L} E_\alpha$.
(vii) *For any $\alpha \in L$, $C_{\alpha+\omega} \cong C_\alpha \times A$.*
(viii) $L \subseteq \prod_c^B A$.

Proof. We prove (i)–(iv) simultaneously by induction on β. The case $\beta = \omega$ holds vacuously. Now assume inductively that $\beta > \omega$.

Case 1. $\beta = \gamma + \omega$ for some $\gamma \in L$. For (i), suppose that $\alpha \in L$ and $\alpha < \beta$. Then $C_\alpha \subseteq C_\gamma$ by the inductive hypothesis, and $C_\gamma \subseteq C_\beta$ by construction.

For (ii), suppose that $\alpha \leq \beta$ with $\alpha \in L$. If $\alpha < \beta$, then $\alpha \in C_\alpha \subseteq C_\beta$ by the inductive hypothesis. Obviously $\beta \in C_\beta$.

For (iii), we consider two subcases.

Subcase 1.1. $\alpha_2 < \beta$. Then $[\alpha_1, \alpha_2) \in C_\gamma$ by the inductive hypothesis, and $C_\gamma \subseteq C_\beta$ by construction.

Subcase 1.2. $\alpha_2 = \beta$. Then $[\alpha_1, \gamma) \in C_\gamma$ by the inductive hypothesis, and $[\gamma, \beta) = f_\gamma[\omega] \in C_\beta$ by construction. So $[\alpha_1, \beta) = [\alpha_1, \gamma) \cup [\gamma, \beta) \in C_\beta$.

For (iv), we have $a \cap \alpha \in C_\alpha$ for each a in the generating set of C_β, and so (iv) holds.

Case 2. $\beta \in L_2$. First we take (ii). Clearly $\beta \in C_\beta$ by construction. Now suppose that $\alpha < \beta$. Choose $i \in \omega$ such that $c_\beta^i \leq \alpha < c_\beta^{i+1}$. Then by the inductive hypothesis, $\alpha \in C_{c_\beta^{i+1}}$, so $[c_\beta^i, \alpha) \in C_{c_\beta^{i+1}} \restriction [c_\beta^i, c_\beta^{i+1}) = D_i^\beta$. Clearly $[0, c_\beta^i) \in C_\beta$. Hence $\alpha \in C_\beta$.

Now assume the hypotheses of (iii). By (ii), $[\alpha_1, \alpha_2) = \alpha_2 \backslash \alpha_1 \in C_\beta$. Thus (iii) holds.

Now assume the hypotheses of (i), and suppose that $a \in C_\alpha$. In particular, $a \subseteq \alpha$. Choose i_0 such that $c_\beta^{i_0} \leq \alpha < c_\beta^{i_0+1}$. If $i + 1 \leq i_0$, then $a \cap c_\beta^{i+1} \in C_{c_\beta^{i+1}}$ by the inductive hypothesis on (iv), so $a \cap [c^i_\beta, c^{i+1}_\beta) \in D^\beta_i$. Also, $[c^{i_0}_\beta, \alpha) \in C_{c^{i_0+1}_\beta}$ by the inductive hypothesis on (iii), and $a \in C_{c^{i_0+1}_\beta}$ by the inductive hypothesis on (i). Hence $a \cap [c^{i_0}_\beta, \alpha) = a \cap [c^{i_0}_\beta, \alpha) \cap [c^{i_0}_\beta, c^{i_0+1}_\beta) \in D^\beta_{i_0}$. Thus a is a sum of elements of D^β_i for $i+1 \leq i_0$ and of an element of $D^\beta_{i_0}$, so $a \in C_\beta$, as desired in (i).

For (iv), suppose that $\alpha < \beta$, and suppose that $a \in C_\beta$. Again take i_0 such that $c^{i_0}_\beta \leq \alpha < c^{i_0+1}_\beta$.

(1) If $i+1 \leq i_0$, then $[c^i_\beta, c^{i+1}_\beta) \cap a \in C_\alpha$.

In fact, clearly $[c^i_\beta, c^{i+1}_\beta) \cap a \in D^\beta_i$. By (iii) for c^{i+1}_β we have $[c^i_\beta, c^{i+1}_\beta) \in C_{c^{i+1}_\beta}$, so $D^\beta_i \subseteq C_{c^{i+1}_\beta}$, and hence $[c^i_\beta, c^{i+1}_\beta) \cap a \in C_{c^{i+1}_\beta} \subseteq C_\alpha$ by the inductive hypothesis on (i) for α. Thus (1) holds.

(2) $[c^{i_0}_\beta, \alpha) \cap a \in C_\alpha$.

For, again clearly $[c^{i_0}_\beta, c^{i_0+1}_\beta) \cap a \in D^\beta_{i_0}$, so as above, $[c^{i_0}_\beta, c^{i_0+1}_\beta) \cap a \in C_{c^{i_0+1}_\beta}$. Then by the inductive hypothesis on (iv) for $c^{i_0+1}_\beta$,

$$[c^{i_0}_\beta, \alpha) \cap a = [c^{i_0}_\beta, c^{i_0+1}_\beta) \cap a \cap \alpha \in C_\alpha,$$

giving (2). Now by (1) and (2), $a \cap \alpha = \bigcup_{i+1 \leq i_0} (a \cap [c^i_\beta, c^{i+1}_\beta)) \cup ([c^{i_0}_\beta, \alpha) \cap a) \in C_\alpha$.

This finishes the proof of (i)–(iv). For (v), from (i) it is clear that $E_\alpha \subseteq E_\beta$. Now for each $a \in E_\beta$, let

$$g(a) = \begin{cases} a \cap \alpha & \text{if } a \in C_\beta, \\ (\omega_1 \backslash \alpha) \cup a & \text{otherwise.} \end{cases}$$

Now g maps into E_α. For, if $a \in C_\beta$, then $g(a) = a \cap \alpha \in C_\alpha \subseteq E_\alpha$ by (iv). If $\omega_1 \backslash a \in C_\beta$, then

$$\omega_1 \backslash g(a) = \omega_1 \backslash ((\omega_1 \backslash \alpha) \cup a) = \omega_1 \cap \alpha \cap (\omega_1 \backslash a) = \alpha \cap (\omega_1 \backslash a),$$

and this is in C_α by (iv) again.

We check that g preserves $\cup$. Suppose that $a, b \in E_\beta$. If $a, b \in C_\beta$, then $a \cup b \in C_\beta$, and $g(a \cup b) = (a \cup b) \cap \alpha = (a \cap \alpha) \cup (b \cap \alpha) = g(a) \cup g(b)$. If $a \in C_\beta$ and $b \notin C_\beta$, then $\omega_1 \backslash b \in C_\beta \subseteq \beta$, hence $\omega_1 \backslash \beta \subseteq b$, and so $a \cup b \notin C_\beta$; it follows that

$$\begin{aligned} g(a) \cup g(b) &= (a \cap \alpha) \cup (\omega_1 \backslash \alpha) \cup b \\ &= (\omega_1 \backslash \alpha) \cup a \cup b \\ &= g(a \cup b). \end{aligned}$$

Similarly if $a \notin C_\beta$ and $b \in C_\beta$. Finally, if $a, b \notin C_\beta$, then

$$g(a) \cup g(b) = (\omega_1 \backslash \alpha) \cup a \cup b = g(a \cup b).$$

So g preserves $\cup$. For complement, if $a \in C_\beta$, then

$$g(\omega_1 \backslash a) = (\omega_1 \backslash \alpha) \cup (\omega_1 \backslash a) = \omega_1 \backslash (a \cap \alpha) = \omega_1 \backslash g(a),$$

and if $a \notin C_\beta$, then

$$\omega_1 \backslash g(a) = \omega_1 \backslash ((\omega_1 \backslash \alpha) \cup a) = \alpha \backslash a = g(\omega_1 \backslash a).$$

So g is a homomorphism. If $a \in C_\alpha$, clearly $g(a) = a$. If $\omega_1 \backslash a \in C_\alpha$, then $g(a) = (\omega_1 \backslash \alpha) \cup a = a$ since $(\omega_1 \backslash \alpha) \subseteq a$. Thus (v) holds.

For (vi), note that each E_α is clearly a subalgebra of $\langle \bigcup_{\alpha \in L} C_\alpha \rangle$, and so $\bigcup_{\alpha \in L} E_\alpha \subseteq \langle \bigcup_{\alpha \in L} C_\alpha \rangle$. Since $\bigcup_{\alpha \in L} E_\alpha$ is a subalgebra of $\mathscr{P}(\omega_1)$ containing $\bigcup_{\alpha \in L} C_\alpha$, (vi) follows.

Turning to (vii), for any $(a, b) \in C_\alpha \times A$ we define $g(a, b) = a \cup f_\alpha[b]$. Clearly g preserves $+$. For $-$, we have

$$\begin{aligned}
-g(a,b) &= -(a \cup f_\alpha[b]) \\
&= (\alpha + \omega) \backslash (a \cup f_\alpha[b]) \\
&= ((\alpha + \omega) \backslash a) \cap ((\alpha + \omega) \backslash f_\alpha[b]) \\
&= ((\alpha \backslash a) \cup [\alpha, \alpha + \omega)) \cap (\alpha \cup [\alpha, \alpha + \omega) \backslash f_\alpha[b]) \\
&= (\alpha \backslash a) \cup ([\alpha, \alpha + \omega) \backslash f_\alpha[b]) \\
&= (\alpha \backslash a) \cup f_\alpha[-b] \\
&= g(-a, -b) \\
&= g(-(a, b)).
\end{aligned}$$

Thus g is a homomorphism. Its range clearly contains $C_\alpha \cup \{f_\alpha[a] : a \in A\}$ and is contained in $\langle C_\alpha \cup \{f_\alpha[a] : a \in A\} \rangle$. It is clearly one-one. So g is the desired isomorphism.

Finally, (viii) is immediate from (ii). □

Proposition 1.13. *Assume the notation above.*

(i) *If $x \in E_\alpha$, then $x \cap \alpha \in C_\alpha$.*
(ii) *C_α is a maximal ideal in E_α.*
(iii) *For every ideal K in E_α, the set $K \cap C_\alpha$ is an ideal in C_α.*
(iv) *For every ideal K in E_α and every $x \in E_\alpha$, let $f([x]_K) = [x \cap \alpha]_{C_\alpha \cap K}$. Then f is well defined, and is a homomorphism from E_α / K onto $C_\alpha / (K \cap C_\alpha)$. Moreover, it is one-one on C_α / K.*
(v) *For every ideal K in E_α, if E_α / K is infinite, then so is $C_\alpha / (K \cap C_\alpha)$.*

Proof. (i): Assume that $x \in E_\alpha$. If $x \in C_\alpha$, then the conclusion is obvious. Suppose that $x \notin C_\alpha$. Then $(\omega_1 \backslash x) \in C_\alpha$, and so the set $\alpha \backslash (\omega_1 \backslash x)$ is also in C_α. This set is equal to $x \cap \alpha$. So (i) holds.

(ii): Obviously C_α is closed under $\cup$. If $x \in C_\alpha$ and $y \subseteq x$, with $y \in E_\alpha$, then by (i), $y \in C_\alpha$. Finally, C_α is obviously maximal.