Operator Theory
Advances and Applications
232

Michael Demuth
Werner Kirsch
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Mathematical Physics, Spectral Theory and Stochastic Analysis

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Contents

Preface

This volume contains survey articles on various aspects of operator theory and partial differential operators. These papers are meant as self-contained introductions to specific fields written by experts for non specialists. They are accessible for graduate students and young researchers but – we believe – they are also of interest to scientists already familiar with the respective area.

The topics covered range from differential operators on abstract manifolds to finite difference operators on a lattice modeling some aspects of impure superconductors. All of them share a view towards applications in physics.

The idea to collect these contributions arose during a conference organized by D. Mayer, I. Witt and one of the editors in Goslar (Germany) in September 2011. But, instead of collecting highly specialized articles on most recent research as for conference proceedings our focus was on introductory aspects and readability of up-to-date contributions.

During the Goslar conference mentioned we also had the opportunity to celebrate the 65th birthday of Michael Demuth. It gives me a great pleasure to wish him here once more many happy years to come.

October 2012

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A Survey on the Krein–von Neumann Extension, the Corresponding Abstract Buckling Problem, and Weyl-type Spectral Asymptotics for Perturbed Krein Laplacians in Nonsmooth Domains

Mark S. Ashbaugh, Fritz Gesztesy, Marius Mitrea,
Roman Shterenberg and Gerald Teschl

Dedicated with great pleasure to Michael Demuth on the occasion of his 65th birthday

Abstract. In the first (and abstract) part of this survey we prove the unitary equivalence of the inverse of the Krein–von Neumann extension (on the orthogonal complement of its kernel) of a densely defined, closed, strictly positive operator, $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$ in a Hilbert space $\mathcal{H}$ to an abstract buckling problem operator.

In the concrete case where $S = \overline{-\Delta|_{C_0^\infty(\Omega)}}$ in $L^2(\Omega; d^n x)$ for $\Omega \subset \mathbb{R}^n$ an open, bounded (and sufficiently regular) set, this recovers, as a particular case of a general result due to G. Grubb, that the eigenvalue problem for the Krein Laplacian S_K (i.e., the Krein–von Neumann extension of S),

$$S_K v = \lambda v, \quad \lambda \neq 0,$$

is in one-to-one correspondence with the problem of *the buckling of a clamped plate*,

$$(-\Delta)^2 u = \lambda(-\Delta)u \text{ in } \Omega, \quad \lambda \neq 0, \quad u \in H_0^2(\Omega),$$

where u and v are related via the pair of formulas

$$u = S_F^{-1}(-\Delta)v, \quad v = \lambda^{-1}(-\Delta)u,$$

with S_F the Friedrichs extension of S.

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This establishes the Krein extension as a natural object in elasticity theory (in analogy to the Friedrichs extension, which found natural applications in quantum mechanics, elasticity, etc.).

In the second, and principal part of this survey, we study spectral properties for $H_{K,\Omega}$, the Krein–von Neumann extension of the perturbed Laplacian $-\Delta + V$ (in short, the perturbed Krein Laplacian) defined on $C_0^\infty(\Omega)$, where V is measurable, bounded and nonnegative, in a bounded open set $\Omega \subset \mathbb{R}^n$ belonging to a class of nonsmooth domains which contains all convex domains, along with all domains of class $C^{1,r}$, $r > 1/2$. (Contrary to other uses of the notion of "domain", a domain in this survey denotes an open set without any connectivity hypotheses. In addition, by a "smooth domain" we mean a domain with a sufficiently smooth, typically, a C^∞-smooth, boundary.) In particular, in the aforementioned context we establish the Weyl asymptotic formula

$$\#\{j \in \mathbb{N} \,|\, \lambda_{K,\Omega,j} \leq \lambda\} = (2\pi)^{-n} v_n |\Omega| \, \lambda^{n/2} + O\big(\lambda^{(n-(1/2))/2}\big) \text{ as } \lambda \to \infty,$$

where $v_n = \pi^{n/2}/\Gamma((n/2)+1)$ denotes the volume of the unit ball in $\mathbb{R}^n$, $|\Omega$ denotes the volume of Ω, and $\lambda_{K,\Omega,j}$, $j \in \mathbb{N}$, are the non-zero eigenvalues of $H_{K,\Omega}$, listed in increasing order according to their multiplicities. We prove this formula by showing that the perturbed Krein Laplacian (i.e., the Krein–von Neumann extension of $-\Delta + V$ defined on $C_0^\infty(\Omega)$) is spectrally equivalent to the buckling of a clamped plate problem, and using an abstract result of Kozlov from the mid 1980's. Our work builds on that of Grubb in the early 1980's, who has considered similar issues for elliptic operators in smooth domains, and shows that the question posed by Alonso and Simon in 1980 pertaining to the validity of the above Weyl asymptotic formula continues to have an affirmative answer in this nonsmooth setting.

We also study certain exterior-type domains $\Omega = \mathbb{R}^n \backslash K$, $n \geq 3$, with $K \subset \mathbb{R}^n$ compact and vanishing Bessel capacity $B_{2,2}(K) = 0$, to prove equality of Friedrichs and Krein Laplacians in $L^2(\Omega; d^n x)$, that is, $-\Delta|_{C_0^\infty(\Omega)}$ has a unique nonnegative self-adjoint extension in $L^2(\Omega; d^n x)$.

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Keywords. Lipschitz domains, Krein Laplacian, eigenvalues, spectral analysis, Weyl asymptotics, buckling problem.

Contents

1. Introduction

In connection with the first and abstract part of this survey, the connection between the Krein–von Neumann extension and an abstract buckling problem, suppose that S is a densely defined, symmetric, closed operator with nonzero deficiency indices in a separable complex Hilbert space $\mathcal{H}$ that satisfies

$$S \geq \varepsilon I_{\mathcal{H}} \text{ for some } \varepsilon > 0, \tag{1.1}$$

and denote by S_K and S_F the Krein–von Neumann and Friedrichs extensions of S, respectively (with $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$).

Then an abstract version of Proposition 1 in Grubb [97], describing an intimate connection between the nonzero eigenvalues of the Krein–von Neumann extension of an appropriate minimal elliptic differential operator of order $2m$,

$m \in \mathbb{N}$, and nonzero eigenvalues of a suitable higher-order buckling problem (cf. Example 3.5), to be proved in Lemma 3.1, can be summarized as follows:

$$\text{There exists } 0 \neq v \in \operatorname{dom}(S_K) \text{ satisfying } S_K v = \lambda v, \quad \lambda \neq 0, \tag{1.2}$$

if and only if

$$\text{there exists a } 0 \neq u \in \operatorname{dom}(S^*S) \text{ such that } S^*Su = \lambda Su, \tag{1.3}$$

and the solutions v of (1.2) are in one-to-one correspondence with the solutions u of (1.3) given by the pair of formulas

$$u = (S_F)^{-1} S_K v, \quad v = \lambda^{-1} Su. \tag{1.4}$$

Next, we will go a step further and describe a unitary equivalence result going beyond the connection between the eigenvalue problems (1.2) and (1.3): Given S, we introduce the following sesquilinear forms in $\mathcal{H}$,

$$a(u,v) = (Su, Sv)_{\mathcal{H}}, \quad u,v \in \operatorname{dom}(a) = \operatorname{dom}(S), \tag{1.5}$$

$$b(u,v) = (u, Sv)_{\mathcal{H}}, \quad u,v \in \operatorname{dom}(b) = \operatorname{dom}(S). \tag{1.6}$$

Then S being densely defined and closed, implies that the sesquilinear form a is also densely defined and closed, and thus one can introduce the Hilbert space

$$\mathcal{W} = (\operatorname{dom}(S), (\cdot,\cdot)_{\mathcal{W}}) \tag{1.7}$$

with associated scalar product

$$(u,v)_{\mathcal{W}} = a(u,v) = (Su, Sv)_{\mathcal{H}}, \quad u,v \in \operatorname{dom}(S). \tag{1.8}$$

Suppressing for simplicity the continuous embedding operator of $\mathcal{W}$ into $\mathcal{H}$, we now introduce the following operator T in $\mathcal{W}$ by

$$(w_1, Tw_2)_{\mathcal{W}} = a(w_1, Tw_2) = b(w_1, w_2) = (w_1, Sw_2)_{\mathcal{H}}, \quad w_1, w_2 \in \mathcal{W}. \tag{1.9}$$

One can prove that T is self-adjoint, nonnegative, and bounded and we will call T the *abstract buckling problem operator* associated with the Krein–von Neumann extension S_K of S.

Next, introducing the Hilbert space $\widehat{\mathcal{H}}$ by

$$\widehat{\mathcal{H}} = [\ker(S^*)]^{\perp} = \big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]\mathcal{H} = \big[I_{\mathcal{H}} - P_{\ker(S_K)}\big]\mathcal{H} = [\ker(S_K)]^{\perp}, \tag{1.10}$$

where $P_{\mathcal{M}}$ denotes the orthogonal projection onto the subspace $\mathcal{M} \subset \mathcal{H}$, we introduce the operator

$$\widehat{S} : \begin{cases} \mathcal{W} \to \widehat{\mathcal{H}}, \\ w \mapsto Sw, \end{cases} \tag{1.11}$$

and note that $\widehat{S} \in \mathcal{B}(\mathcal{W}, \widehat{\mathcal{H}})$ maps $\mathcal{W}$ unitarily onto $\widehat{\mathcal{H}}$.

Finally, defining the *reduced Krein–von Neumann operator* $\widehat{S}_K$ in $\widehat{\mathcal{H}}$ by

$$\widehat{S}_K := S_K|_{[\ker(S_K)]^{\perp}} \text{ in } \widehat{\mathcal{H}}, \tag{1.12}$$

we can state the principal unitary equivalence result to be proved in Theorem 3.4:

The inverse of the reduced Krein–von Neumann operator $\widehat{S}_K$ in $\widehat{\mathcal{H}}$ and the abstract buckling problem operator T in $\mathcal{W}$ are unitarily equivalent,

$$\big(\widehat{S}_K\big)^{-1} = \widehat{S}T(\widehat{S})^{-1}. \tag{1.13}$$

In addition,

$$\big(\widehat{S}_K\big)^{-1} = U_S\big[|S|^{-1}S|S|^{-1}\big](U_S)^{-1}. \tag{1.14}$$

Here we used the polar decomposition of S,

$$S = U_S|S|, \text{ with } |S| = (S^*S)^{1/2} \geq \varepsilon I_{\mathcal{H}},\ \varepsilon > 0, \text{ and } U_S \in \mathcal{B}\big(\mathcal{H}, \widehat{\mathcal{H}}\big) \text{ unitary}, \tag{1.15}$$

and one observes that the operator $|S|^{-1}S|S|^{-1} \in \mathcal{B}(\mathcal{H})$ in (1.14) is self-adjoint in $\mathcal{H}$.

As discussed at the end of Section 4, one can readily rewrite the abstract linear pencil buckling eigenvalue problem (1.3), $S^*Su = \lambda Su$, $\lambda \neq 0$, in the form of the standard eigenvalue problem $|S|^{-1}S|S|^{-1}w = \lambda^{-1}w$, $\lambda \neq 0$, $w = |S|u$, and hence establish the connection between (1.2), (1.3) and (1.13), (1.14).

As mentioned in the abstract, the concrete case where S is given by $S = \overline{-\Delta|_{C_0^\infty(\Omega)}}$ in $L^2(\Omega; d^nx)$, then yields the spectral equivalence between the inverse of the reduced Krein–von Neumann extension $\widehat{S}_K$ of S and the problem of the buckling of a clamped plate. More generally, Grubb [97] actually treated the case where S is generated by an appropriate elliptic differential expression of order $2m$, $m \in \mathbb{N}$, and also introduced the higher-order analog of the buckling problem; we briefly summarize this in Example 3.5.

The results of this connection between an abstract buckling problem and the Krein–von Neumann extension in Section 3 originally appeared in [30].

Turning to the second and principal part of this survey, the Weyl-type spectral asymptotics for perturbed Krein Laplacians, let $-\Delta_{D,\Omega}$ be the Dirichlet Laplacian associated with an open set $\Omega \subset \mathbb{R}^n$, and denote by $N_{D,\Omega}(\lambda)$ the corresponding spectral distribution function (i.e., the number of eigenvalues of $-\Delta_{D,\Omega}$ not exceeding λ). The study of the asymptotic behavior of $N_{D,\Omega}(\lambda)$ as $\lambda \to \infty$ has been initiated by Weyl in 1911–1913 (cf. [189], [188], and the references in [190]), in response to a question posed in 1908 by the physicist Lorentz, pertaining to the equipartition of energy in statistical mechanics. When $n = 2$ and Ω is a bounded domain with a piecewise smooth boundary, Weyl has shown that

$$N_{D,\Omega}(\lambda) = \frac{|\Omega|}{4\pi}\lambda + o(\lambda) \text{ as } \lambda \to \infty, \tag{1.16}$$

along with the three-dimensional analogue of (1.16). (We recall our convention to denote the volume of $\Omega \subset \mathbb{R}^n$ by $|\Omega|$.) In particular, this allowed him to complete a partial proof of Rayleigh, going back to 1903. This ground-breaking work has stimulated a great deal of activity in the intervening years, in which a large number of authors have provided sharper estimates for the remainder, and considered more general elliptic operators equipped with a variety of boundary conditions.

For a general elliptic differential operator $\mathcal{A}$ of order $2m$ ($m \in \mathbb{N}$), with smooth coefficients, acting on a smooth subdomain Ω of an n-dimensional smooth manifold, spectral asymptotics of the form

$$N_{D,\Omega}(\mathcal{A};\lambda) = (2\pi)^{-n}\biggl(\int_\Omega dx \int_{a^0(x,\xi)<1} d\xi\biggr)\lambda^{n/(2m)} + O\bigl(\lambda^{(n-1)/(2m)}\bigr) \text{ as } \lambda \to \infty, \tag{1.17}$$

where $a^0(x,\xi)$ denotes the principal symbol of $\mathcal{A}$, have then been subsequently established in increasing generality (a nice exposition can be found in [6]). At the same time, it has been realized that, as the smoothness of the domain Ω (by which we mean smoothness of the boundary of Ω) and the coefficients of $\mathcal{A}$ deteriorate, the degree of detail with which the remainder can be described decreases accordingly. Indeed, the smoothness of the boundary of the underlying domain Ω affects both the nature of the remainder in (1.17), as well as the types of differential operators and boundary conditions for which such an asymptotic formula holds. Understanding this correlation then became a central theme of research. For example, in the case of the Laplacian in an arbitrary bounded, open subset Ω of $\mathbb{R}^n$, Birman and Solomyak have shown in [40] (see also [41], [42], [43], [44]) that the following Weyl asymptotic formula holds

$$N_{D,\Omega}(\lambda) = (2\pi)^{-n} v_n |\Omega|\, \lambda^{n/2} + o\bigl(\lambda^{n/2}\bigr) \text{ as } \lambda \to \infty, \tag{1.18}$$

where v_n denotes the volume of the unit ball in $\mathbb{R}^n$, and $|\Omega|$ stands for the n-dimensional Euclidean volume of Ω. (Actually, (1.18) extends to unbounded Ω with finite volume $|\Omega|$, but this will not be addressed in this survey.) On the other hand, it is known that (1.18) may fail for the Neumann Laplacian $-\Delta_{N,\Omega}$. Furthermore, if $\alpha \in (0,1)$ then Netrusov and Safarov have proved that

$$\Omega \in \mathrm{Lip}_\alpha \text{ implies } N_{D,\Omega}(\lambda) = (2\pi)^{-n} v_n |\Omega|\, \lambda^{n/2} + O\bigl(\lambda^{(n-\alpha)/2}\bigr) \text{ as } \lambda \to \infty, \tag{1.19}$$

where Lip_α is the class of bounded domains whose boundaries can be locally described by means of graphs of functions satisfying a Hölder condition of order α; this result is sharp. See [149] where this intriguing result (along with others, similar in spirit) has been obtained. Surprising connections between Weyl's asymptotic formula and geometric measure theory have been explored in [57], [109], [128] for fractal domains. Collectively, this body of work shows that the nature of the Weyl asymptotic formula is intimately related not only to the geometrical properties of the domain (as well as the type of boundary conditions), but also to the smoothness properties of its boundary (the monographs by Ivrii [112] and Safarov and Vassiliev [167] contain a wealth of information on this circle of ideas).

These considerations are by no means limited to the Laplacian; see [58] for the case of the Stokes operator, and [39], [45] for the case the Maxwell system in nonsmooth domains. However, even in the case of the Laplace operator, besides $-\Delta_{D,\Omega}$ and $-\Delta_{N,\Omega}$ there is a multitude of other concrete extensions of the Laplacian $-\Delta$ on $C_0^\infty(\Omega)$ as a nonnegative, self-adjoint operator in $L^2(\Omega; d^n x)$.

The smallest (in the operator theoretic order sense) such realization has been introduced, in an abstract setting, by M. Krein [124]. Later it was realized that in the case where the symmetric operator, whose self-adjoint extensions are sought, has a strictly positive lower bound, Krein's construction coincides with one that von Neumann had discussed in his seminal paper [183] in 1929.

For the purpose of this introduction we now briefly recall the construction of the Krein–von Neumann extension of appropriate $L^2(\Omega; d^n x)$-realizations of the differential operator $\mathcal{A}$ of order $2m$, $m \in \mathbb{N}$,

$$\mathcal{A} = \sum_{0 \le |\alpha| \le 2m} a_\alpha(\cdot) D^\alpha, \tag{1.20}$$

$$D^\alpha = (-i\partial/\partial x_1)^{\alpha_1} \cdots (-i\partial/\partial x_n)^{\alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n, \tag{1.21}$$

$$a_\alpha(\cdot) \in C^\infty(\overline{\Omega}), \quad C^\infty(\overline{\Omega}) = \bigcap_{k \in \mathbb{N}_0} C^k(\overline{\Omega}), \tag{1.22}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded C^∞ domain. Introducing the particular $L^2(\Omega; d^n x)$-realization $A_{c,\Omega}$ of $\mathcal{A}$ defined by

$$A_{c,\Omega} u = \mathcal{A} u, \quad u \in \operatorname{dom}(A_{c,\Omega}) := C_0^\infty(\Omega), \tag{1.23}$$

we assume the coefficients a_α in $\mathcal{A}$ are chosen such that $A_{c,\Omega}$ is symmetric,

$$(u, A_{c,\Omega} v)_{L^2(\Omega; d^n x)} = (A_{c,\Omega} u, v)_{L^2(\Omega; d^n x)}, \quad u, v \in C_0^\infty(\Omega), \tag{1.24}$$

has a (strictly) positive lower bound, that is, there exists $\kappa_0 > 0$ such that

$$(u, A_{c,\Omega} u)_{L^2(\Omega; d^n x)} \ge \kappa_0 \, \|u\|^2_{L^2(\Omega; d^n x)}, \quad u \in C_0^\infty(\Omega), \tag{1.25}$$

and is strongly elliptic, that is, there exists $\kappa_1 > 0$ such that

$$a^0(x, \xi) := \operatorname{Re}\Bigg(\sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha \Bigg) \ge \kappa_1 \, |\xi|^{2m}, \quad x \in \overline{\Omega}, \ \xi \in \mathbb{R}^n. \tag{1.26}$$

Next, let $A_{\min,\Omega}$ and $A_{\max,\Omega}$ be the $L^2(\Omega; d^n x)$-realizations of $\mathcal{A}$ with domains (cf. [6], [100])

$$\operatorname{dom}(A_{\min,\Omega}) := H_0^{2m}(\Omega), \tag{1.27}$$

$$\operatorname{dom}(A_{\max,\Omega}) := \big\{ u \in L^2(\Omega; d^n x) \,\big|\, \mathcal{A} u \in L^2(\Omega; d^n x) \big\}. \tag{1.28}$$

Throughout this manuscript, $H^s(\Omega)$ denotes the L^2-based Sobolev space of order $s \in \mathbb{R}$ in Ω, and $H_0^s(\Omega)$ is the subspace of $H^s(\mathbb{R}^n)$ consisting of distributions supported in $\overline{\Omega}$ (for $s > \frac{1}{2}$, $\big(s - \frac{1}{2}\big) \notin \mathbb{N}$, the space $H_0^s(\Omega)$ can be alternatively described as the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$). Given that the domain Ω is smooth, elliptic regularity implies

$$(A_{\min,\Omega})^* = A_{\max,\Omega} \ \text{ and } \ \overline{A_{c,\Omega}} = A_{\min,\Omega}. \tag{1.29}$$

Functional analytic considerations (cf. the discussion in Section 2) dictate that the Krein–von Neumann (sometimes also called the "soft") extension $A_{K,\Omega}$ of $A_{c,\Omega}$

on $C_0^\infty(\Omega)$ is the $L^2(\Omega; d^n x)$-realization of $A_{c,\Omega}$ with domain (cf. (2.10) derived abstractly by Krein)

$$\mathrm{dom}(A_{K,\Omega}) = \mathrm{dom}\left(\overline{A_{c,\Omega}}\right) \dot{+} \ker\left((A_{c,\Omega})^*\right). \tag{1.30}$$

Above and elsewhere, $X \dot{+} Y$ denotes the direct sum of two subspaces, X and Y, of a larger space Z, with the property that $X \cap Y = \{0\}$. Thus, granted (1.29), we have

$$\begin{aligned} \mathrm{dom}(A_{K,\Omega}) &= \mathrm{dom}(A_{\min,\Omega}) \dot{+} \ker(A_{\max,\Omega}) \\ &= H_0^{2m}(\Omega) \dot{+} \left\{u \in L^2(\Omega; d^n x) \,\middle|\, Au = 0 \text{ in } \Omega\right\}. \end{aligned} \tag{1.31}$$

In summary, for domains with smooth boundaries, $A_{K,\Omega}$ is the self-adjoint realization of $A_{c,\Omega}$ with domain given by (1.31).

Denote by $\gamma_D^m u := \left(\gamma_N^j u\right)_{0 \le j \le m-1}$ the Dirichlet trace operator of order $m \in \mathbb{N}$ (where ν denotes the outward unit normal to Ω and $\gamma_N u := \partial_\nu u$ stands for the normal derivative, or Neumann trace), and let $A_{D,\Omega}$ be the Dirichlet (sometimes also called the "hard") realization of $A_{c,\Omega}$ in $L^2(\Omega; d^n x)$ with domain

$$\mathrm{dom}(A_{D,\Omega}) := \left\{u \in H^{2m}(\Omega) \,\middle|\, \gamma_D^m u = 0\right\}. \tag{1.32}$$

Then $A_{K,\Omega}$, $A_{D,\Omega}$ are "extremal" in the following sense: Any nonnegative self-adjoint extension $\widetilde{A}$ in $L^2(\Omega; d^n x)$ of $A_{c,\Omega}$ (cf. (1.23)), necessarily satisfies

$$A_{K,\Omega} \le \widetilde{A} \le A_{D,\Omega} \tag{1.33}$$

in the sense of quadratic forms (cf. the discussion surrounding (2.4)).

Returning to the case where $A_{c,\Omega} = -\Delta|_{C_0^\infty(\Omega)}$, for a bounded domain Ω with a C^∞-smooth boundary, $\partial\Omega$, the corresponding Krein–von Neumann extension admits the following description

$$\begin{aligned} & -\Delta_{K,\Omega} u := -\Delta u, \\ & u \in \mathrm{dom}(-\Delta_{K,\Omega}) := \{v \in \mathrm{dom}(-\Delta_{\max,\Omega}) \,|\, \gamma_N v + M_{D,N,\Omega}(\gamma_D v) = 0\}, \end{aligned} \tag{1.34}$$

where $M_{D,N,\Omega}$ is (up to a minus sign) an energy-dependent Dirichlet-to-Neumann map, or Weyl–Titchmarsh operator for the Laplacian. Compared with (1.31), the description (1.34) has the advantage of making explicit the boundary condition implicit in the definition of membership to $\mathrm{dom}(-\Delta_{K,\Omega})$. Nonetheless, as opposed to the classical Dirichlet and Neumann boundary condition, this turns out to be *nonlocal* in nature, as it involves $M_{D,N,\Omega}$ which, when Ω is smooth, is a boundary pseudodifferential operator of order 1. Thus, informally speaking, (1.34) is the realization of the Laplacian with the boundary condition

$$\partial_\nu u = \partial_\nu H(u) \ \text{ on } \ \partial\Omega, \tag{1.35}$$

where, given a reasonable function w in Ω, $H(w)$ is the harmonic extension of the Dirichlet boundary trace $\gamma_D^0 w$ to Ω (cf. (4.15)).

While at first sight the nonlocal boundary condition $\gamma_N v + M_{D,N,\Omega}(\gamma_D v) = 0$ in (1.34) for the Krein Laplacian $-\Delta_{K,\Omega}$ may seem familiar from the abstract approach to self-adjoint extensions of semibounded symmetric operators within the theory of boundary value spaces, there are some crucial distinctions in the concrete case of Laplacians on (nonsmooth) domains which will be delineated at the end of Section 6.

For rough domains, matters are more delicate as the nature of the boundary trace operators and the standard elliptic regularity theory are both fundamentally affected. Following work in [89], here we shall consider the class of *quasi-convex domains*. The latter is the subclass of bounded, Lipschitz domains in $\mathbb{R}^n$ characterized by the demand that

(i) there exists a sequence of relatively compact, C^2-subdomains exhausting the original domain, and whose second fundamental forms are bounded from below in a uniform fashion (for a precise formulation see Definition 5.3),

or

(ii) near every boundary point there exists a suitably small $\delta > 0$, such that the boundary is given by the graph of a function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ (suitably rotated and translated) which is Lipschitz and whose derivative satisfy the pointwise $H^{1/2}$-multiplier condition

$$\begin{aligned} &\sum_{k=1}^{n-1} \|f_k \, \partial_k \varphi_j\|_{H^{1/2}(\mathbb{R}^{n-1})} \\ &\quad \leq \delta \sum_{k=1}^{n-1} \|f_k\|_{H^{1/2}(\mathbb{R}^{n-1})}, \quad f_1, \dots, f_{n-1} \in H^{1/2}(\mathbb{R}^{n-1}). \end{aligned} \tag{1.36}$$

See Hypothesis 5.7 for a precise formulation. In particular, (1.36) is automatically satisfied when $\omega(\nabla\varphi, t)$, the modulus of continuity of $\nabla\varphi$ at scale t, satisfies the square-Dini condition (compare to [140], [141], where this type of domain was introduced and studied),

$$\int_0^1 \Big(\frac{\omega(\nabla\varphi; t)}{t^{1/2}}\Big)^2 \frac{dt}{t} < \infty. \tag{1.37}$$

In turn, (1.37) is automatically satisfied if the Lipschitz function φ is of class $C^{1,r}$ for some $r > 1/2$. As a result, examples of quasi-convex domains include:

(i) All bounded (geometrically) convex domains.
(ii) All bounded Lipschitz domains satisfying a uniform exterior ball condition (which, informally speaking, means that a ball of fixed radius can be "rolled" along the boundary).
(iii) All open sets which are the image of a domain as in $(i), (ii)$ above under a $C^{1,1}$-diffeomorphism.
(iv) All bounded domains of class $C^{1,r}$ for some $r > 1/2$.

We note that being quasi-convex is a local property of the boundary. The philosophy behind this concept is that Lipschitz-type singularities are allowed in the boundary as long as they are directed outwardly (see Figure 1 on p. 43). The key feature of this class of domains is the fact that the classical elliptic regularity property

$$\operatorname{dom}(-\Delta_{D,\Omega}) \subset H^2(\Omega), \quad \operatorname{dom}(-\Delta_{N,\Omega}) \subset H^2(\Omega) \tag{1.38}$$

remains valid. In this vein, it is worth recalling that the presence of a single re-entrant corner for the domain Ω invalidates (1.38). All our results in this survey are actually valid for the class of bounded Lipschitz domains for which (1.38) holds. Condition (1.38) is, however, a regularity assumption on the boundary of the Lipschitz domain Ω and the class of quasi-convex domains is the largest one for which we know (1.38) to hold. Under the hypothesis of quasi-convexity, it has been shown in [89] that the Krein Laplacian $-\Delta_{K,\Omega}$ (i.e., the Krein–von Neumann extension of the Laplacian $-\Delta$ defined on $C_0^\infty(\Omega)$) in (1.34) is a well-defined self-adjoint operator which agrees with the operator constructed using the recipe in (1.31).

The main issue of this survey is the study of the spectral properties of $H_{K,\Omega}$, the Krein–von Neumann extension of the perturbed Laplacian

$$-\Delta + V \text{ on } C_0^\infty(\Omega), \tag{1.39}$$

in the case where both the potential V and the domain Ω are nonsmooth. As regards the former, we shall assume that $0 \leq V \in L^\infty(\Omega; d^n x)$, and we shall assume that $\Omega \subset \mathbb{R}^n$ is a quasi-convex domain (more on this shortly). In particular, we wish to clarify the extent to which a Weyl asymptotic formula continues to hold for this operator. For us, this undertaking was originally inspired by the discussion by Alonso and Simon in [14]. At the end of that paper, the authors comment to the effect that *"It seems to us that the Krein extension of $-\Delta$, i.e., $-\Delta$ with the boundary condition* (1.35)*, is a natural object and therefore worthy of further study. For example: Are the asymptotics of its nonzero eigenvalues given by Weyl's formula?"* Subsequently we have learned that when Ω is C^∞-smooth this has been shown to be the case by Grubb in [97]. More specifically, in that paper Grubb has proved that if $N_{K,\Omega}(\mathcal{A};\lambda)$ denotes the number of nonzero eigenvalues of $A_{K,\Omega}$ (defined as in (1.31)) not exceeding λ, then

$$\Omega \in C^\infty \text{ implies } N_{K,\Omega}(\mathcal{A};\lambda) = C_{A,n}\lambda^{n/(2m)} + O\big(\lambda^{(n-\theta)/(2m)}\big) \text{ as } \lambda \to \infty, \tag{1.40}$$

where, with $a^0(x,\xi)$ as in (1.26),

$$C_{A,n} := (2\pi)^{-n} \int_\Omega d^n x \int_{a^0(x,\xi)<1} d^n \xi \tag{1.41}$$

and

$$\theta := \max\left\{\frac{1}{2} - \varepsilon\,, \frac{2m}{2m+n-1}\right\}, \text{ with } \varepsilon > 0 \text{ arbitrary.} \tag{1.42}$$

See also [143], [144], and most recently, [102], where the authors derive a sharpening of the remainder in (1.40) to any $\theta < 1$. To show (1.40)–(1.42), Grubb has reduced the eigenvalue problem

$$\mathcal{A}u = \lambda u, \quad u \in \operatorname{dom}(A_{K,\Omega}),\ \lambda > 0, \tag{1.43}$$

to the higher-order, elliptic system

$$\begin{cases} \mathcal{A}^2 v = \lambda\,\mathcal{A} v \ \text{ in } \ \Omega, \\ \gamma_D^{2m} v = 0 \ \text{ on } \ \partial\Omega, \\ v \in C^\infty(\overline{\Omega}). \end{cases} \tag{1.44}$$

Then the strategy is to use known asymptotics for the spectral distribution function of regular elliptic boundary problems, along with perturbation results due to Birman, Solomyak, and Grubb (see the literature cited in [97] for precise references). It should be noted that the fact that the boundary of Ω and the coefficients of $\mathcal{A}$ are smooth plays an important role in Grubb's proof. First, this is used to ensure that (1.29) holds which, in turn, allows for the concrete representation (1.31) (a formula which in effect lies at the start of the entire theory, as Grubb adopts this as the *definition* of the domains of the Krein–von Neumann extension). In addition, at a more technical level, Lemma 3 in [97] is justified by making appeal to the theory of pseudo-differential operators on $\partial\Omega$, assumed to be an $(n-1)$-dimensional C^∞ manifold. In our case, that is, when dealing with the Krein–von Neumann extension of the perturbed Laplacian (1.39), we establish the following theorem:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a quasi-convex domain, assume that $0 \le V \in L^\infty(\Omega; d^n x)$, and denote by $H_{K,\Omega}$ the Krein–von Neumann extension of the perturbed Laplacian* (1.39)*. Then there exists a sequence of numbers*

$$0 < \lambda_{K,\Omega,1} \le \lambda_{K,\Omega,2} \le \cdots \le \lambda_{K,\Omega,j} \le \lambda_{K,\Omega,j+1} \le \cdots \tag{1.45}$$

converging to infinity, with the following properties.

(i) *The spectrum of $H_{K,\Omega}$ is given by*

$$\sigma(H_{K,\Omega}) = \{0\} \cup \{\lambda_{K,\Omega,j}\}_{j\in\mathbb{N}}, \tag{1.46}$$

and each number $\lambda_{K,\Omega,j}$, $j \in \mathbb{N}$, is an eigenvalue for $H_{K,\Omega}$ of finite multiplicity.

(ii) *There exists a countable family of orthonormal eigenfunctions for $H_{K,\Omega}$ which span the orthogonal complement of the kernel of this operator. More precisely, there exists a collection of functions $\{w_j\}_{j\in\mathbb{N}}$ with the following properties:*

$$w_j \in \mathrm{dom}(H_{K,\Omega}) \ \text{ and } \ H_{K,\Omega} w_j = \lambda_{K,\Omega,j} w_j, \ j \in \mathbb{N}, \tag{1.47}$$

$$(w_j, w_k)_{L^2(\Omega; d^n x)} = \delta_{j,k}, \ j,k \in \mathbb{N}, \tag{1.48}$$

$$L^2(\Omega; d^n x) = \ker(H_{K,\Omega}) \oplus \overline{\mathrm{lin.\,span}\{w_j\}_{j\in\mathbb{N}}}, \quad (\textit{orthogonal direct sum}). \tag{1.49}$$

If V is Lipschitz then $w_j \in H^{1/2}(\Omega)$ for every j and, in fact, $w_j \in C^\infty(\overline{\Omega})$ for every j if Ω is C^∞ and $V \in C^\infty(\overline{\Omega})$.

(iii) *The following min-max principle holds:*

$$\lambda_{K,\Omega,j} = \min_{\substack{W_j \text{ subspace of } H_0^2(\Omega) \\ \dim(W_j)=j}} \left(\max_{0\neq u\in W_j} \left(\frac{\|(-\Delta+V)u\|^2_{L^2(\Omega;d^nx)}}{\|\nabla u\|^2_{(L^2(\Omega;d^nx))^n} + \|V^{1/2}u\|^2_{L^2(\Omega;d^nx)}} \right)\right), \quad j\in\mathbb{N}. \tag{1.50}$$

(iv) *If*

$$0 < \lambda_{D,\Omega,1} \leq \lambda_{D,\Omega,2} \leq \cdots \leq \lambda_{D,\Omega,j} \leq \lambda_{D,\Omega,j+1} \leq \cdots \tag{1.51}$$

are the eigenvalues of the perturbed Dirichlet Laplacian $-\Delta_{D,\Omega}$ *(i.e., the Friedrichs extension of* (1.39) *in* $L^2(\Omega;d^nx)$*), listed according to their multiplicities, then*

$$0 < \lambda_{D,\Omega,j} \leq \lambda_{K,\Omega,j}, \quad j\in\mathbb{N}, \tag{1.52}$$

Consequently introducing the spectral distribution functions

$$N_{X,\Omega}(\lambda) := \#\{j\in\mathbb{N}\,|\,\lambda_{X,\Omega,j}\leq\lambda\}, \quad X\in\{D,K\}, \tag{1.53}$$

one has

$$N_{K,\Omega}(\lambda) \leq N_{D,\Omega}(\lambda). \tag{1.54}$$

(v) *Corresponding to the case* $V\equiv 0$*, the first nonzero eigenvalue* $\lambda^{(0)}_{K,\Omega,1}$ *of* $-\Delta_{K,\Omega}$ *satisfies*

$$\lambda^{(0)}_{D,\Omega,2} \leq \lambda^{(0)}_{K,\Omega,1} \ \text{ and } \ \lambda^{(0)}_{K,\Omega,2} \leq \frac{n^2+8n+20}{(n+2)^2}\lambda^{(0)}_{K,\Omega,1}. \tag{1.55}$$

In addition,

$$\sum_{j=1}^{n} \lambda^{(0)}_{K,\Omega,j+1} < (n+4)\lambda^{(0)}_{K,\Omega,1} - \frac{4}{n+4}(\lambda^{(0)}_{K,\Omega,2}-\lambda^{(0)}_{K,\Omega,1}) \leqslant (n+4)\lambda^{(0)}_{K,\Omega,1}, \tag{1.56}$$

and

$$\sum_{j=1}^{k} \big(\lambda^{(0)}_{K,\Omega,k+1}-\lambda^{(0)}_{K,\Omega,j}\big)^2 \leq \frac{4(n+2)}{n^2}\sum_{j=1}^{k}\big(\lambda^{(0)}_{K,\Omega,k+1}-\lambda^{(0)}_{K,\Omega,j}\big)\lambda^{(0)}_{K,\Omega,j} \quad k\in\mathbb{N}. \tag{1.57}$$

Moreover, if Ω *is a bounded, convex domain in* $\mathbb{R}^n$*, then the first two Dirichlet eigenvalues and the first nonzero eigenvalue of the Krein Laplacian in* Ω *satisfy*

$$\lambda^{(0)}_{D,\Omega,2} \leq \lambda^{(0)}_{K,\Omega,1} \leq 4\,\lambda^{(0)}_{D,\Omega,1}. \tag{1.58}$$

(vi) *The following Weyl asymptotic formula holds:*

$$N_{K,\Omega}(\lambda) = (2\pi)^{-n}v_n|\Omega|\,\lambda^{n/2} + O\big(\lambda^{(n-(1/2))/2}\big) \ \text{ as } \ \lambda\to\infty, \tag{1.59}$$

where, as before, v_n *denotes the volume of the unit ball in* $\mathbb{R}^n$*, and* $|\Omega|$ *stands for the n-dimensional Euclidean volume of* Ω*.*

This theorem answers the question posed by Alonso and Simon in [14] (which corresponds to $V \equiv 0$), and further extends the work by Grubb in [97] in the sense that we allow nonsmooth domains and coefficients. To prove this result, we adopt Grubb's strategy and show that the eigenvalue problem

$$(-\Delta + V)u = \lambda\, u, \quad u \in \mathrm{dom}(H_{K,\Omega}),\ \lambda > 0, \tag{1.60}$$

is equivalent to the following fourth-order problem

$$\begin{cases} (-\Delta + V)^2 w = \lambda\, (-\Delta + V)w \ \text{ in } \ \Omega, \\ \gamma_D w = \gamma_N w = 0 \ \text{ on } \ \partial\Omega, \\ w \in \mathrm{dom}(-\Delta_{\max}). \end{cases} \tag{1.61}$$

This is closely related to the so-called problem of the *buckling of a clamped plate*,

$$\begin{cases} -\Delta^2 w = \lambda\, \Delta w \ \text{ in } \ \Omega, \\ \gamma_D w = \gamma_N w = 0 \ \text{ on } \ \partial\Omega, \\ w \in \mathrm{dom}(-\Delta_{\max}), \end{cases} \tag{1.62}$$

to which (1.61) reduces when $V \equiv 0$. From a physical point of view, the nature of the later boundary value problem can be described as follows. In the two-dimensional setting, the bifurcation problem for a clamped, homogeneous plate in the shape of Ω, with uniform lateral compression on its edges has the eigenvalues λ of the problem (1.61) as its critical points. In particular, the first eigenvalue of (1.61) is proportional to the load compression at which the plate buckles.

One of the upshots of our work in this context is establishing a definite connection between the Krein–von Neumann extension of the Laplacian and the buckling problem (1.62). In contrast to the smooth case, since in our setting the solution w of (1.61) does not exhibit any extra regularity on the Sobolev scale $H^s(\Omega)$, $s \geq 0$, other than membership to $L^2(\Omega; d^n x)$, a suitable interpretation of the boundary conditions in (1.61) should be adopted. (Here we shall rely on the recent progress from [89] where this issue has been resolved by introducing certain novel boundary Sobolev spaces, well adapted to the class of Lipschitz domains.) We nonetheless find this trade-off, between the 2nd-order boundary problem (1.60) which has nonlocal boundary conditions, and the boundary problem (1.61) which has local boundary conditions, but is of fourth-order, very useful. The reason is that (1.61) can be rephrased, in view of (1.38) and related regularity results developed in [89], in the form of

$$(-\Delta + V)^2 u = \lambda\, (-\Delta + V)u \ \text{ in } \ \Omega, \quad u \in H^2_0(\Omega). \tag{1.63}$$

In principle, this opens the door to bringing onto the stage the theory of generalized eigenvalue problems, that is, operator pencil problems of the form

$$Tu = \lambda\, Su, \tag{1.64}$$

where T and S are certain linear operators in a Hilbert space. Abstract results of this nature can be found for instance, in [133], [156], [175] (see also [129], [130], where this is applied to the asymptotic distribution of eigenvalues). We, however, find it more convenient to appeal to a version of (1.64) which emphasizes the role of the symmetric forms

$$a(u,v) := \int_\Omega d^n x\, \overline{(-\Delta + V)u}\, (-\Delta + V)v, \quad u,v \in H_0^2(\Omega), \tag{1.65}$$

$$b(u,v) := \int_\Omega d^n x\, \overline{\nabla u} \cdot \nabla v + \int_\Omega d^n x\, \overline{V^{1/2}u}\, V^{1/2}v, \quad u,v \in H_0^2(\Omega), \tag{1.66}$$

and reformulate (1.63) as the problem of finding $u \in H_0^2(\Omega)$ which satisfies

$$a(u,v) = \lambda\, b(u,v) \quad v \in H_0^2(\Omega). \tag{1.67}$$

This type of eigenvalue problem, in the language of bilinear forms associated with differential operators, has been studied by Kozlov in a series of papers [118], [119], [120]. In particular, in [120], Kozlov has obtained Weyl asymptotic formulas in the case where the underlying domain Ω in (1.65) is merely Lipschitz, and the lower-order coefficients of the quadratic forms (1.65)–(1.66) are only measurable and bounded (see Theorem 9.1 for a precise formulation). Our demand that the potential V is in $L^\infty(\Omega; d^n x)$ is therefore inherited from Kozlov's theorem. Based on this result and the fact that the problems (1.65)–(1.67) and (1.60) are spectral-equivalent, we can then conclude that (1.59) holds. Formulas (1.55)–(1.57) are also a byproduct of the connection between (1.60) and (1.61) and known spectral estimates for the buckling plate problem from [27], [28], [31], [60], [110], [150], [152], [153]. Similarly, (1.58) for convex domains is based on the connection between (1.60) and (1.61) and the eigenvalue inequality relating the first eigenvalue of a fixed membrane and that of the buckling problem for the clamped plate as proven in [151] (see also [152], [153]).

In closing, we wish to point out that in the C^∞-smooth setting, Grubb's remainder in (1.40), with the improvement to any $\theta < 1$ in [102], [143], [144], is sharper than that in (1.59). However, the main novel feature of our Theorem 1.1 is the low regularity assumptions on the underlying domain Ω, and the fact that we allow a nonsmooth potential V. As was the case with the Weyl asymptotic formula for the classical Dirichlet and Neumann Laplacians (briefly reviewed at the beginning of this section), the issue of regularity (or lack thereof) has always been of considerable importance in this line of work (as early as 1970, Birman and Solomyak noted in [40] that "*there has been recently some interest in obtaining the classical asymptotic spectral formulas under the weakest possible hypotheses.*"). The interested reader may consult the paper [44] by Birman and Solomyak (see also [42], [43]), as well as the article [63] by Davies for some very readable, highly informative surveys underscoring this point (collectively, these papers also contain more than 500 references concerning this circle of ideas).

We note that the results in Sections 4–6 originally appeared in [89], while those in Sections 7–11 originally appeared in [29].

Finally, a notational comment: For obvious reasons in connection with quantum mechanical applications, we will, with a slight abuse of notation, dub $-\Delta$ (rather than Δ) as the "Laplacian" in this survey.

2. The abstract Krein–von Neumann extension

To get started, we briefly elaborate on the notational conventions used throughout this survey and especially throughout this section which collects abstract material on the Krein–von Neumann extension. Let $\mathcal{H}$ be a separable complex Hilbert space, $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second factor), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$. Next, let T be a linear operator mapping (a subspace of) a Banach space into another, with $\operatorname{dom}(T)$ and $\operatorname{ran}(T)$ denoting the domain and range of T. The closure of a closable operator S is denoted by $\overline{S}$. The kernel (null space) of T is denoted by $\ker(T)$. The spectrum, essential spectrum, and resolvent set of a closed linear operator in $\mathcal{H}$ will be denoted by $\sigma(\cdot)$, $\sigma_{\rm ess}(\cdot)$, and $\rho(\cdot)$, respectively. The Banach spaces of bounded and compact linear operators on $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively. Similarly, the Schatten–von Neumann (trace) ideals will subsequently be denoted by $\mathcal{B}_p(\mathcal{H})$, $p\in(0,\infty)$. The analogous notation $\mathcal{B}(\mathcal{X}_1,\mathcal{X}_2)$, $\mathcal{B}_\infty(\mathcal{X}_1,\mathcal{X}_2)$, etc., will be used for bounded, compact, etc., operators between two Banach spaces $\mathcal{X}_1$ and $\mathcal{X}_2$. Moreover, $\mathcal{X}_1\hookrightarrow\mathcal{X}_2$ denotes the continuous embedding of the Banach space $\mathcal{X}_1$ into the Banach space $\mathcal{X}_2$. In addition, $U_1 \dotplus U_2$ denotes the direct sum of the subspaces U_1 and U_2 of a Banach space $\mathcal{X}$; and $V_1\oplus V_2$ represents the orthogonal direct sum of the subspaces V_j, $j=1,2$, of a Hilbert space $\mathcal{H}$.

Throughout this manuscript, if X denotes a Banach space, X^* denotes the *adjoint space* of continuous conjugate linear functionals on X, that is, the *conjugate dual space* of X (rather than the usual dual space of continuous linear functionals on X). This avoids the well-known awkward distinction between adjoint operators in Banach and Hilbert spaces (cf., e.g., the pertinent discussion in [71, p. 3, 4]).

Given a reflexive Banach space $\mathcal{V}$ and $T\in\mathcal{B}(\mathcal{V},\mathcal{V}^*)$, the fact that T is self-adjoint is defined by the requirement that

$$ {}_{\mathcal{V}}\langle u,Tv\rangle_{\mathcal{V}^*} = {}_{\mathcal{V}^*}\langle Tu,v\rangle_{\mathcal{V}} = \overline{{}_{\mathcal{V}}\langle v,Tu\rangle_{\mathcal{V}^*}}, \quad u,v\in\mathcal{V}, \tag{2.1}$$

where in this context bar denotes complex conjugation, $\mathcal{V}^*$ is the conjugate dual of $\mathcal{V}$, and ${}_{\mathcal{V}}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{V}^*}$ stands for the $\mathcal{V},\mathcal{V}^*$ pairing.

A linear operator $S:\operatorname{dom}(S)\subseteq\mathcal{H}\to\mathcal{H}$, is called *symmetric*, if

$$ (u,Sv)_{\mathcal{H}} = (Su,v)_{\mathcal{H}}, \quad u,v\in\operatorname{dom}(S). \tag{2.2}$$

If $\operatorname{dom}(S)=\mathcal{H}$, the classical Hellinger–Toeplitz theorem guarantees that $S\in\mathcal{B}(\mathcal{H})$, in which situation S is readily seen to be self-adjoint. In general, however, symmetry is a considerably weaker property than self-adjointness and a classical problem in functional analysis is that of determining all self-adjoint extensions in $\mathcal{H}$ of a given unbounded symmetric operator of equal and nonzero deficiency indices. (Here self-adjointness of an operator $\widetilde{S}$ in $\mathcal{H}$, is of course defined as usual by $\big(\widetilde{S}\big)^*=\widetilde{S}$.) In this manuscript we will be particularly interested in this question within the class of densely defined (i.e., $\overline{\operatorname{dom}(S)}=\mathcal{H}$), nonnegative operators (in fact, in most instances S will even turn out to be strictly positive) and we focus almost exclusively on self-adjoint extensions that are nonnegative operators. In the latter scenario, there are two distinguished constructions which we will briefly review next.

To set the stage, we recall that a linear operator $S : \mathrm{dom}(S) \subseteq \mathcal{H} \to \mathcal{H}$ is called *nonnegative* provided

$$(u, Su)_{\mathcal{H}} \geq 0, \quad u \in \mathrm{dom}(S). \tag{2.3}$$

(In particular, S is symmetric in this case.) S is called *strictly positive*, if for some $\varepsilon > 0$, $(u, Su)_{\mathcal{H}} \geq \varepsilon \|u\|_{\mathcal{H}}^2$, $u \in \mathrm{dom}(S)$. Next, we recall that $A \leq B$ for two self-adjoint operators in $\mathcal{H}$ if

$$\begin{aligned} &\mathrm{dom}\big(|A|^{1/2}\big) \supseteq \mathrm{dom}\big(|B|^{1/2}\big) \text{ and} \\ &\big(|A|^{1/2}u, U_A|A|^{1/2}u\big)_{\mathcal{H}} \leq \big(|B|^{1/2}u, U_B|B|^{1/2}u\big)_{\mathcal{H}}, \quad u \in \mathrm{dom}\big(|B|^{1/2}\big), \end{aligned} \tag{2.4}$$

where U_C denotes the partial isometry in $\mathcal{H}$ in the polar decomposition of a densely defined closed operator C in $\mathcal{H}$, $C = U_C|C|$, $|C| = (C^*C)^{1/2}$. (If in addition, C is self-adjoint, then U_C and $|C|$ commute.) We also recall ([75, Section I.6], [114, Theorem VI.2.21]) that if A and B are both self-adjoint and nonnegative in $\mathcal{H}$, then

$$0 \leq A \leq B \text{ if and only if } (B + aI_{\mathcal{H}})^{-1} \leq (A + aI_{\mathcal{H}})^{-1} \text{ for all } a > 0, \tag{2.5}$$

(which implies $0 \leq A^{1/2} \leq B^{1/2}$) and

$$\ker(A) = \ker\big(A^{1/2}\big) \tag{2.6}$$

(with $C^{1/2}$ the unique nonnegative square root of a nonnegative self-adjoint operator C in $\mathcal{H}$).

For simplicity we will always adhere to the conventions that S is a linear, unbounded, densely defined, nonnegative (i.e., $S \geq 0$) operator in $\mathcal{H}$, and that S has nonzero deficiency indices. In particular,

$$\mathrm{def}(S) = \dim(\ker(S^* - zI_{\mathcal{H}})) \in \mathbb{N} \cup \{\infty\}, \quad z \in \mathbb{C}\backslash[0, \infty), \tag{2.7}$$

is well known to be independent of z. Moreover, since S and its closure $\overline{S}$ have the same self-adjoint extensions in $\mathcal{H}$, we will without loss of generality assume that S is closed in the remainder of this section.

The following is a fundamental result to be found in M. Krein's celebrated 1947 paper [124] (cf. also Theorems 2 and 5–7 in the English summary on page 492):

Theorem 2.1. *Assume that S is a densely defined, closed, nonnegative operator in $\mathcal{H}$. Then, among all nonnegative self-adjoint extensions of S, there exist two distinguished ones, S_K and S_F, which are, respectively, the smallest and largest* (*in the sense of order between self-adjoint operators, cf.* (2.4)) *such extension. Furthermore, a nonnegative self-adjoint operator $\widetilde{S}$ is a self-adjoint extension of S if and only if $\widetilde{S}$ satisfies*

$$S_K \leq \widetilde{S} \leq S_F. \tag{2.8}$$

In particular, (2.8) *determines S_K and S_F uniquely.*

In addition, if $S \geq \varepsilon I_{\mathcal{H}}$ *for some* $\varepsilon > 0$*, one has* $S_F \geq \varepsilon I_{\mathcal{H}}$*, and*

$$\operatorname{dom}(S_F) = \operatorname{dom}(S) \dotplus (S_F)^{-1}\ker(S^*), \tag{2.9}$$

$$\operatorname{dom}(S_K) = \operatorname{dom}(S) \dotplus \ker(S^*), \tag{2.10}$$

$$\begin{aligned}\operatorname{dom}(S^*) &= \operatorname{dom}(S) \dotplus (S_F)^{-1}\ker(S^*) \dotplus \ker(S^*) \\ &= \operatorname{dom}(S_F) \dotplus \ker(S^*),\end{aligned} \tag{2.11}$$

in particular,

$$\ker(S_K) = \ker\big((S_K)^{1/2}\big) = \ker(S^*) = \operatorname{ran}(S)^{\perp}. \tag{2.12}$$

Here the operator inequalities in (2.8) are understood in the sense of (2.4) and hence they can equivalently be written as

$$(S_F + aI_{\mathcal{H}})^{-1} \leqslant \big(\widetilde{S} + aI_{\mathcal{H}}\big)^{-1} \leqslant (S_K + aI_{\mathcal{H}})^{-1} \text{ for some (and hence for all) } a > 0. \tag{2.13}$$

We will call the operator S_K the *Krein–von Neumann extension* of S. See [124] and also the discussion in [14], [23], [24]. It should be noted that the Krein–von Neumann extension was first considered by von Neumann [183] in 1929 in the case where S is strictly positive, that is, if $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$. (His construction appears in the proof of Theorem 42 on pages 102–103.) However, von Neumann did not isolate the extremal property of this extension as described in (2.8) and (2.13). M. Krein [124], [125] was the first to systematically treat the general case $S \geq 0$ and to study all nonnegative self-adjoint extensions of S, illustrating the special role of the *Friedrichs extension* (i.e., the "hard" extension) S_F of S and the Krein–von Neumann (i.e., the "soft") extension S_K of S as extremal cases when considering all nonnegative extensions of S. For a recent exhaustive treatment of self-adjoint extensions of semibounded operators we refer to [22]–[25].

For classical references on the subject of self-adjoint extensions of semibounded operators (not necessarily restricted to the Krein–von Neumann extension) we refer to Birman [37], [38], Friedrichs [79], Freudenthal [78], Grubb [94], [95], Krein [125], Štraus [173], and Višik [182] (see also the monographs by Akhiezer and Glazman [10, Sect. 109], Faris [75, Part III], and the recent book by Grubb [100, Sect. 13.2]).

An intrinsic description of the Friedrichs extension S_F of $S \geq 0$ due to Freudenthal [78] in 1936 describes S_F as the operator $S_F : \operatorname{dom}(S_F) \subset \mathcal{H} \to \mathcal{H}$ given by

$$\begin{aligned}&S_F u := S^* u, \\ &u \in \operatorname{dom}(S_F) := \big\{v \in \operatorname{dom}(S^*) \,\big|\, \text{there exists } \{v_j\}_{j\in\mathbb{N}} \subset \operatorname{dom}(S), \\ &\quad \text{with } \lim_{j\to\infty} \|v_j - v\|_{\mathcal{H}} = 0 \text{ and } ((v_j - v_k), S(v_j - v_k))_{\mathcal{H}} \to 0 \text{ as } j,k \to \infty\big\}.\end{aligned} \tag{2.14}$$

Then, as is well known,

$$S_F \geq 0, \tag{2.15}$$

$$\begin{aligned} &\operatorname{dom}\big((S_F)^{1/2}\big) = \big\{v \in \mathcal{H} \,\big|\, \text{there exists} \{v_j\}_{j\in\mathbb{N}} \subset \operatorname{dom}(S), \\ &\quad \text{with} \lim_{j\to\infty} \|v_j - v\|_{\mathcal{H}} = 0 \text{ and } ((v_j - v_k), S(v_j - v_k))_{\mathcal{H}} \to 0 \text{ as } j,k \to \infty\big\}, \end{aligned} \tag{2.16}$$

and

$$S_F = S^*|_{\operatorname{dom}(S^*)\cap\operatorname{dom}((S_F)^{1/2})}. \tag{2.17}$$

Equations (2.16) and (2.17) are intimately related to the definition of S_F via (the closure of) the sesquilinear form generated by S as follows: One introduces the sesquilinear form

$$q_S(f,g) = (f, Sg)_{\mathcal{H}}, \quad f,g \in \operatorname{dom}(q_S) = \operatorname{dom}(S). \tag{2.18}$$

Since $S \geq 0$, the form q_S is closable and we denote by Q_S the closure of q_S. Then $Q_S \geq 0$ is densely defined and closed. By the first and second representation theorem for forms (cf., e.g., [114, Sect. 6.2]), Q_S is uniquely associated with a nonnegative, self-adjoint operator in $\mathcal{H}$. This operator is precisely the Friedrichs extension, $S_F \geq 0$, of S, and hence,

$$\begin{aligned} &Q_S(f,g) = (f, S_F g)_{\mathcal{H}}, \quad f \in \operatorname{dom}(Q_S),\, g \in \operatorname{dom}(S_F), \\ &\operatorname{dom}(Q_S) = \operatorname{dom}\big((S_F)^{1/2}\big). \end{aligned} \tag{2.19}$$

An intrinsic description of the Krein–von Neumann extension S_K of $S \geq 0$ has been given by Ando and Nishio [16] in 1970, where S_K has been characterized as the operator $S_K : \operatorname{dom}(S_K) \subset \mathcal{H} \to \mathcal{H}$ given by

$$\begin{aligned} &S_K u := S^* u, \\ &u \in \operatorname{dom}(S_K) := \big\{v \in \operatorname{dom}(S^*) \,\big|\, \text{there exists} \{v_j\}_{j\in\mathbb{N}} \subset \operatorname{dom}(S), \\ &\quad \text{with} \lim_{j\to\infty} \|Sv_j - S^*v\|_{\mathcal{H}} = 0 \text{ and } ((v_j - v_k), S(v_j - v_k))_{\mathcal{H}} \to 0 \text{ as } j,k \to \infty\big\}. \end{aligned} \tag{2.20}$$

By (2.14) one observes that shifting S by a constant commutes with the operation of taking the Friedrichs extension of S, that is, for any $c \in \mathbb{R}$,

$$(S + cI_{\mathcal{H}})_F = S_F + cI_{\mathcal{H}}, \tag{2.21}$$

but by (2.20), the analog of (2.21) for the Krein–von Neumann extension S_K fails.

At this point we recall a result due to Makarov and Tsekanovskii [134], concerning symmetries (e.g., the rotational symmetry exploited in Section 11), and more generally, a scale invariance, shared by S, S^*, S_F, and S_K (see also [105]). Actually, we will prove a slight extension of the principal result in [134]:

Proposition 2.2. *Let $\mu > 0$, suppose that $V, V^{-1} \in \mathcal{B}(\mathcal{H})$, and assume S to be a densely defined, closed, nonnegative operator in $\mathcal{H}$ satisfying*

$$VSV^{-1} = \mu S, \tag{2.22}$$

and

$$VSV^{-1} = (V^*)^{-1}SV^* \ \ (\textit{or equivalently, } (V^*V)^{-1}S(V^*V) = S\,). \tag{2.23}$$

Then also S^*, S_F, and S_K satisfy

$$(V^*V)^{-1}S^*(V^*V) = S^*, \quad VS^*V^{-1} = \mu S^*, \tag{2.24}$$

$$(V^*V)^{-1}S_F(V^*V) = S_F, \quad VS_FV^{-1} = \mu S_F, \tag{2.25}$$

$$(V^*V)^{-1}S_K(V^*V) = S_K, \quad VS_KV^{-1} = \mu S_K. \tag{2.26}$$

Proof. Applying [185, p. 73, 74], (2.22) yields $VSV^{-1} = (V^*)^{-1}SV^*$. The latter relation is equivalent to $(V^*V)^{-1}S(V^*V) = S$ and hence also equivalent to $(V^*V)S(V^*V)^{-1} = S$. Taking adjoints (and applying [185, p. 73, 74] again) then yields $(V^*)^{-1}S^*V^* = VS^*V^{-1}$; the latter is equivalent to $(V^*V)^{-1}S^*(V^*V) = S^*$ and hence also equivalent to $(V^*V)S^*(V^*V)^{-1} = S$. Replacing S and S^* by $(V^*V)^{-1}S(V^*V)$ and $(V^*V)^{-1}S^*(V^*V)$, respectively, in (2.14), and subsequently, in (2.20), then yields that

$$(V^*V)^{-1}S_F(V^*V) = S_F \text{ and } (V^*V)^{-1}S_K(V^*V) = S_K. \tag{2.27}$$

The latter are of course equivalent to

$$(V^*V)S_F(V^*V)^{-1} = S_F \text{ and } (V^*V)S_K(V^*V)^{-1} = S_K. \tag{2.28}$$

Finally, replacing S by VSV^{-1} and S^* by VS^*V^{-1} in (2.14) then proves $VS_FV^{-1} = \mu S_F$. Performing the same replacement in (2.20) yields $VS_KV^{-1} = \mu S_K$. □

If in addition, V is unitary (implying $V^*V = I_{\mathcal{H}}$), Proposition 2.2 immediately reduces to [134, Theorem 2.2]. In this special case one can also provide a quick alternative proof by directly invoking the inequalities (2.13) and the fact that they are preserved under unitary equivalence.

Similarly to Proposition 2.2, the following results also immediately follow from the characterizations (2.14) and (2.20) of S_F and S_K, respectively:

Proposition 2.3. *Let $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ be unitary from $\mathcal{H}_1$ onto $\mathcal{H}_2$ and assume S to be a densely defined, closed, nonnegative operator in $\mathcal{H}_1$ with adjoint S^*, Friedrichs extension S_F, and Krein–von Neumann extension S_K in $\mathcal{H}_1$, respectively. Then the adjoint, Friedrichs extension, and Krein–von Neumann extension of the nonnegative, closed, densely defined, symmetric operator USU^{-1} in $\mathcal{H}_2$ are given by*

$$[USU^{-1}]^* = US^*U^{-1}, \quad [USU^{-1}]_F = US_FU^{-1}, \quad [USU^{-1}]_K = US_KU^{-1} \tag{2.29}$$

in $\mathcal{H}_2$, respectively.

Proposition 2.4. *Let $J \subseteq \mathbb{N}$ be some countable index set and consider $\mathcal{H} = \bigoplus_{j\in J}\mathcal{H}_j$ and $S = \bigoplus_{j\in J} S_j$, where each S_j is a densely defined, closed, nonnegative operator in $\mathcal{H}_j$, $j \in J$. Denoting by $(S_j)_F$ and $(S_j)_K$ the Friedrichs and Krein–von Neumann extension of S_j in $\mathcal{H}_j$, $j \in J$, one infers*

$$S^* = \bigoplus_{j\in J}(S_j)^*, \quad S_F = \bigoplus_{j\in J}(S_j)_F, \quad S_K = \bigoplus_{j\in J}(S_j)_K. \tag{2.30}$$

The following is a consequence of a slightly more general result formulated in [16, Theorem 1]:

Proposition 2.5. *Let S be a densely defined, closed, nonnegative operator in $\mathcal{H}$. Then S_K, the Krein–von Neumann extension of S, has the property that*

$$\operatorname{dom}\big((S_K)^{1/2}\big) = \left\{u \in \mathcal{H} \,\middle|\, \sup_{v \in \operatorname{dom}(S)} \frac{|(u, Sv)_{\mathcal{H}}|^2}{(v, Sv)_{\mathcal{H}}} < +\infty\right\}, \tag{2.31}$$

and

$$\big\|(S_K)^{1/2}u\big\|_{\mathcal{H}}^2 = \sup_{v \in \operatorname{dom}(S)} \frac{|(u, Sv)_{\mathcal{H}}|^2}{(v, Sv)_{\mathcal{H}}}, \quad u \in \operatorname{dom}\big((S_K)^{1/2}\big). \tag{2.32}$$

A word of explanation is in order here: Given $S \geq 0$ as in the statement of Proposition 2.5, the Cauchy–Schwarz-type inequality

$$|(u, Sv)_{\mathcal{H}}|^2 \leq (u, Su)_{\mathcal{H}}(v, Sv)_{\mathcal{H}}, \quad u, v \in \operatorname{dom}(S), \tag{2.33}$$

shows (due to the fact that $\operatorname{dom}(S) \hookrightarrow \mathcal{H}$ densely) that

$$u \in \operatorname{dom}(S) \text{ and } (u, Su)_{\mathcal{H}} = 0 \text{ imply } Su = 0. \tag{2.34}$$

Thus, whenever the denominator of the fractions appearing in (2.31), (2.32) vanishes, so does the numerator, and one interprets 0/0 as being zero in (2.31), (2.32).

We continue by recording an abstract result regarding the parametrization of all nonnegative self-adjoint extensions of a given strictly positive, densely defined, symmetric operator. The following results were developed from Krein [124], Višik [182], and Birman [37], by Grubb [94], [95]. Subsequent expositions are due to Faris [75, Sect. 15], Alonso and Simon [14] (in the present form, the next theorem appears in [89]), and Derkach and Malamud [65], [135]. We start by collecting our basic assumptions:

Hypothesis 2.6. *Suppose that S is a densely defined, symmetric, closed operator with nonzero deficiency indices in $\mathcal{H}$ that satisfies*

$$S \geq \varepsilon I_{\mathcal{H}} \text{ for some } \varepsilon > 0. \tag{2.35}$$

Theorem 2.7. *Suppose Hypothesis 2.6. Then there exists a one-to-one correspondence between nonnegative self-adjoint operators $0 \leq B : \operatorname{dom}(B) \subseteq \mathcal{W} \to \mathcal{W}$, $\overline{\operatorname{dom}(B)} = \mathcal{W}$, where $\mathcal{W}$ is a closed subspace of $\mathcal{N}_0 := \ker(S^*)$, and nonnegative self-adjoint extensions $S_{B,\mathcal{W}} \geq 0$ of S. More specifically, S_F is invertible, $S_F \geq \varepsilon I_{\mathcal{H}}$, and one has*

$$\begin{aligned} \operatorname{dom}(S_{B,\mathcal{W}}) = \big\{ & f + (S_F)^{-1}(Bw + \eta) + w \,\big|\, \\ & f \in \operatorname{dom}(S),\, w \in \operatorname{dom}(B),\, \eta \in \mathcal{N}_0 \cap \mathcal{W}^{\perp} \big\}, \\ S_{B,\mathcal{W}} = {} & S^*|_{\operatorname{dom}(S_{B,\mathcal{W}})}, \end{aligned} \tag{2.36}$$

where $\mathcal{W}^{\perp}$ denotes the orthogonal complement of $\mathcal{W}$ in $\mathcal{N}_0$. In addition,

$$\operatorname{dom}\big((S_{B,\mathcal{W}})^{1/2}\big) = \operatorname{dom}\big((S_F)^{1/2}\big) \dotplus \operatorname{dom}\big(B^{1/2}\big), \tag{2.37}$$

$$\big\|(S_{B,\mathcal{W}})^{1/2}(u+g)\big\|_{\mathcal{H}}^2 = \big\|(S_F)^{1/2}u\big\|_{\mathcal{H}}^2 + \big\|B^{1/2}g\big\|_{\mathcal{H}}^2, \tag{2.38}$$
$$u \in \operatorname{dom}\big((S_F)^{1/2}\big),\ g \in \operatorname{dom}\big(B^{1/2}\big),$$

implying,

$$\ker(S_{B,\mathcal{W}}) = \ker(B). \tag{2.39}$$

Moreover,

$$B \le \widetilde{B} \ \textit{ implies } \ S_{B,\mathcal{W}} \le S_{\widetilde{B},\widetilde{\mathcal{W}}}, \tag{2.40}$$

where

$$\begin{aligned} &B\colon \operatorname{dom}(B) \subseteq \mathcal{W} \to \mathcal{W}, \quad \widetilde{B}\colon \operatorname{dom}\big(\widetilde{B}\big) \subseteq \widetilde{\mathcal{W}} \to \widetilde{\mathcal{W}}, \\ &\overline{\operatorname{dom}\big(\widetilde{B}\big)} = \widetilde{\mathcal{W}} \subseteq \mathcal{W} = \overline{\operatorname{dom}(B)}. \end{aligned} \tag{2.41}$$

In the above scheme, the Krein–von Neumann extension S_K of S corresponds to the choice $\mathcal{W} = \mathcal{N}_0$ and $B = 0$ (with $\operatorname{dom}(B) = \operatorname{dom}\big(B^{1/2}\big) = \mathcal{N}_0 = \ker(S^*)$*). In particular, one thus recovers* (2.10)*, and* (2.12)*, and also obtains*

$$\operatorname{dom}\big((S_K)^{1/2}\big) = \operatorname{dom}\big((S_F)^{1/2}\big) \dotplus \ker(S^*), \tag{2.42}$$

$$\big\|(S_K)^{1/2}(u+g)\big\|_{\mathcal{H}}^2 = \big\|(S_F)^{1/2}u\big\|_{\mathcal{H}}^2, \quad u \in \operatorname{dom}\big((S_F)^{1/2}\big),\ g \in \ker(S^*). \tag{2.43}$$

Finally, the Friedrichs extension S_F corresponds to the choice $\operatorname{dom}(B) = \{0\}$ (*i.e., formally, $B \equiv \infty$*)*, in which case one recovers* (2.9)*.*

The relation $B \le \widetilde{B}$ in the case where $\widetilde{\mathcal{W}} \subsetneqq \mathcal{W}$ requires an explanation: In analogy to (2.4) we mean

$$\big(|B|^{1/2}u, U_B|B|^{1/2}u\big)_{\mathcal{W}} \le \big(|\widetilde{B}|^{1/2}u, U_{\widetilde{B}}|\widetilde{B}|^{1/2}u\big)_{\mathcal{W}}, \quad u \in \operatorname{dom}\big(|\widetilde{B}|^{1/2}\big) \tag{2.44}$$

and (following [14]) we put

$$(|\widetilde{B}|^{1/2}u, U_{\widetilde{B}}|\widetilde{B}|^{1/2}u)_{\mathcal{W}} = \infty \ \text{ for } \ u \in \mathcal{W}\backslash \operatorname{dom}\big(|\widetilde{B}|^{1/2}\big). \tag{2.45}$$

For subsequent purposes we also note that under the assumptions on S in Hypothesis 2.6, one has

$$\dim(\ker(S^* - zI_{\mathcal{H}})) = \dim(\ker(S^*)) = \dim(\mathcal{N}_0) = \operatorname{def}(S), \quad z \in \mathbb{C}\backslash[\varepsilon,\infty). \tag{2.46}$$

The following result is a simple consequence of (2.10), (2.9), and (2.20), but since it seems not to have been explicitly stated in [124], we provide the short proof for completeness (see also [135, Remark 3]). First we recall that two self-adjoint extensions S_1 and S_2 of S are called *relatively prime* if $\operatorname{dom}(S_1) \cap \operatorname{dom}(S_2) = \operatorname{dom}(S)$.

Lemma 2.8. *Suppose Hypothesis* 2.6*. Then S_F and S_K are relatively prime, that is,*

$$\operatorname{dom}(S_F) \cap \operatorname{dom}(S_K) = \operatorname{dom}(S). \tag{2.47}$$

Proof. By (2.9) and (2.10) it suffices to prove that $\ker(S^*)\cap(S_F)^{-1}\ker(S^*) = \{0\}$. Let $f_0 \in \ker(S^*) \cap (S_F)^{-1}\ker(S^*)$. Then $S^* f_0 = 0$ and $f_0 = (S_F)^{-1} g_0$ for some $g_0 \in \ker(S^*)$. Thus one concludes that $f_0 \in \operatorname{dom}(S_F)$ and $S_F f_0 = g_0$. But $S_F = S^*|_{\operatorname{dom}(S_F)}$ and hence $g_0 = S_F f_0 = S^* f_0 = 0$. Since $g_0 = 0$ one finally obtains $f_0 = 0$. □

Next, we consider a self-adjoint operator

$$T : \operatorname{dom}(T) \subseteq \mathcal{H} \to \mathcal{H}, \quad T = T^*, \tag{2.48}$$

which is bounded from below, that is, there exists $\alpha \in \mathbb{R}$ such that

$$T \geq \alpha I_{\mathcal{H}}. \tag{2.49}$$

We denote by $\{E_T(\lambda)\}_{\lambda\in\mathbb{R}}$ the family of strongly right-continuous spectral projections of T, and introduce, as usual, $E_T((a,b)) = E_T(b_-) - E_T(a)$, $E_T(b_-) = \text{s-lim}_{\varepsilon\downarrow 0} E_T(b-\varepsilon)$, $-\infty \leq a < b$. In addition, we set

$$\mu_{T,j} := \inf\big\{\lambda \in \mathbb{R} \,|\, \dim(\operatorname{ran}(E_T((-\infty,\lambda)))) \geq j\big\}, \quad j \in \mathbb{N}. \tag{2.50}$$

Then, for fixed $k \in \mathbb{N}$, either:

(i) $\mu_{T,k}$ is the kth eigenvalue of T counting multiplicity below the bottom of the essential spectrum, $\sigma_{\rm ess}(T)$, of T,

or,

(ii) $\mu_{T,k}$ is the bottom of the essential spectrum of T,

$$\mu_{T,k} = \inf\{\lambda \in \mathbb{R} \,|\, \lambda \in \sigma_{\rm ess}(T)\}, \tag{2.51}$$

and in that case $\mu_{T,k+\ell} = \mu_{T,k}$, $\ell \in \mathbb{N}$, and there are at most $k-1$ eigenvalues (counting multiplicity) of T below $\mu_{T,k}$.

We now record a basic result of M. Krein [124] with an important extension due to Alonso and Simon [14] and some additional results recently derived in [30]. For this purpose we introduce the *reduced Krein–von Neumann operator* $\widehat{S}_K$ in the Hilbert space (cf. (2.12))

$$\widehat{\mathcal{H}} = [\ker(S^*)]^\perp = \big[I_{\mathcal{H}} - P_{\ker(S^*)}\big]\mathcal{H} = \big[I_{\mathcal{H}} - P_{\ker(S_K)}\big]\mathcal{H} = [\ker(S_K)]^\perp, \tag{2.52}$$

by

$$\widehat{S}_K := S_K|_{[\ker(S_K)]^\perp} \tag{2.53}$$

$$\begin{aligned} &= S_K[I_{\mathcal{H}} - P_{\ker(S_K)}] \ \text{ in } [I_{\mathcal{H}} - P_{\ker(S_K)}]\mathcal{H} \\ &= [I_{\mathcal{H}} - P_{\ker(S_K)}]S_K[I_{\mathcal{H}} - P_{\ker(S_K)}] \ \text{ in } [I_{\mathcal{H}} - P_{\ker(S_K)}]\mathcal{H}, \end{aligned} \tag{2.54}$$

where $P_{\ker(S_K)}$ denotes the orthogonal projection onto $\ker(S_K)$ and we are alluding to the orthogonal direct sum decomposition of $\mathcal{H}$ into

$$\mathcal{H} = P_{\ker(S_K)}\mathcal{H} \oplus [I_{\mathcal{H}} - P_{\ker(S_K)}]\mathcal{H}. \tag{2.55}$$