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Andreas Johann  
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# Recent Trends in Dynamical Systems

Proceedings of a Conference in Honor  
of Jürgen Scheurle

 Springer

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Andreas Johann • Hans-Peter Kruse  
Florian Rupp • Stephan Schmitz  
Editors

# Recent Trends in Dynamical Systems

Proceedings of a Conference in Honor  
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*Editors*

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*To Jürgen Scheurle*



# Preface

In January 2012 the International Conference *Recent Trends in Dynamical Systems* was held in Munich on the occasion of Jürgen Scheurle's 60th birthday. As parts of this conference, a scientific colloquium took place at the Carl Friedrich von Siemens Stiftung in Munich from 11th to 13th of January and also a Festkolloquium at the Technische Universität München in the afternoon of January 13th. Besides numerous posters on recent advances in the field of dynamical systems, 25 highly recognized scholars gave plenary talks that were grouped according to the following themes:

- Stability and bifurcation
- Geometric mechanics and control theory
- Invariant manifolds, attractors, and chaos
- Fluid mechanics and elasticity
- Perturbations and multiscale problems
- Hamiltonian dynamics and KAM theory

These themes reflect the broad scientific interests of Jürgen Scheurle and his fascination of applying mathematics to real world situations, in particular from physics and mechanics. The volume at hand is an outgrowth of this conference, containing research articles about exciting new developments in the multifaceted subject of dynamical systems as well as survey articles. We are very happy that the authors accepted the invitation to contribute to this volume in honour of Jürgen Scheurle and we are sure that their exciting articles will be of interest not only to experts in the field of dynamical systems but also to graduate students and scientists from many other fields, including engineering. This is in the spirit of Jürgen Scheurle, who, besides his research activities, always puts a lot of emphasis on conveying the beauty of the Theory of Dynamical Systems and its applicability to real world problems in extremely well-prepared, beautiful lectures.

Munich, Germany  
January 2013

Andreas Johann  
Hans-Peter Kruse  
Florian Rupp  
Stephan Schmitz



## *Short Curriculum Vitae of Jürgen Scheurle*

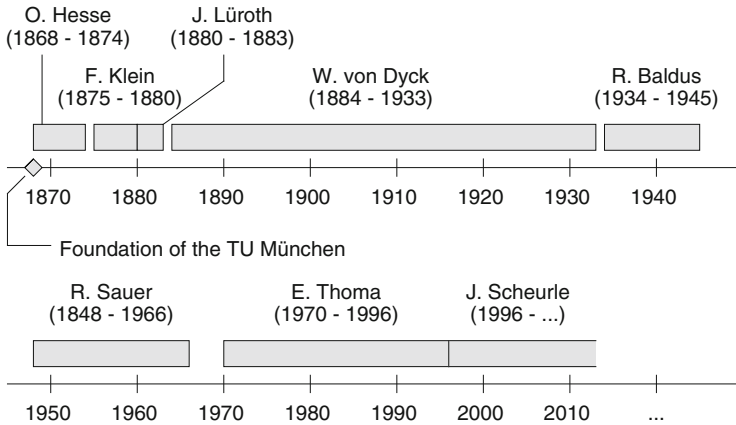
Jürgen Scheurle was born on September 26, 1951, in Schwäbisch Gmünd, Baden-Württemberg. He received his professional education at the University of Stuttgart, where he studied mathematics, physics, and computer science from 1970 until 1974, and finished his diploma degree in mathematics with a thesis entitled “Ein Antikonvergenzprinzip”. Some months later, in 1975, he completed his doctorate under the guidance of Klaus Kirchgässner. The title of his Ph.D. thesis is “Ein selektives Iterationsverfahren und Verzweigungsprobleme”. In 1981 he presented his Habilitation thesis on “Verzweigung quasiperiodischer Lösungen bei reversiblen dynamischen Systemen”.

From 1974 to 1985 Jürgen Scheurle held positions as a postdoctoral researcher, senior researcher, and assistant professor, at the University of Stuttgart. In 1982 he was visiting professor at the Department of Mathematics, University of California, Berkeley (USA), and in 1983 at the Division of Applied Mathematics, Brown University, Providence (USA). In 1985 Jürgen Scheurle moved to Fort Collins (USA), where he became an associate and later full professor at Colorado State University. In 1987 he accepted a full professorship and the Chair of Theory and Applications of Partial Differential Equations at the University of Hamburg. In 1996 Jürgen Scheurle was appointed full professor at the Technische Universität München (TUM) and since then holds the Chair of Advanced Mathematics and Analytical Mechanics. Notable predecessors at this chair were Felix Klein, Walter von Dyck, and Robert Sauer, see Fig. 1, which illustrates the special responsibility of Jürgen Scheurle for the mathematical education of engineering students.

He was the founding director of the Center for Mathematics at TUM and later dean of the Faculty of Mathematics. As dean, he continued the reform-oriented politics of his predecessors. During his term in office, the faculty voluntarily conducted a peer assessment and was awarded the title “Reformfakultät” by the “Stifterverband der Deutschen Wissenschaft”. Such assessments are common nowadays but were completely novel 10 years ago. Moreover, far ahead before such procedures were put into law, the Bavarian Ministry of Research and Teaching allowed the faculty to introduce an “Experimentierklausel” to assess prospective for the admission of students.

Jürgen Scheurle was responsible for the introduction of the “Master of Science in Industrial & Financial Mathematics” at the off-shore campus of TUM in Singapore. He was a member of the planning team for the new mathematics building at the research campus Garching and in charge of the relocation from downtown Munich to Garching in 2002. Finally, Jürgen Scheurle was and is member of numerous expert committees appointed by the president of the TUM and the faculty of mathematics. Inter alia he is representative of the “Bayerische Eliteakademie”, member of the “Hurwitz-Gesellschaft zur Förderung der Mathematik an der TU München” and its president since 2011.

Jürgen Scheurle authored and co-authored several pioneering publications, and among them the following are highly influential articles:



**Fig. 1** Genealogy of the chair “Analytische Mechanik und Angewandte Mathematik” at the Technische Universität München

- *On the bounded solutions of a semilinear elliptic equation in a strip* (together with K. Kirchgässner). *J. Diff. Equat.* 32 (1) (1979), 119–148.
- *Smoothness of bounded solutions of non-linear evolution equations* (together with J. Hale). *J. Diff. Equat.* 56 (1) (1985), 142–163.
- *Chaotic solutions of systems with almost periodic forcing*. *ZAMP* 37 (1986), 12–26.
- *The construction and smoothness of invariant manifolds by the deformation method* (together with J. Marsden). *SIAM J. Math. Anal.* 18 (5) (1987), 1261–1274.
- *Exponentially small splittings of separatrices in KAM theory and degenerate bifurcations* (together with P. Holmes and J. Marsden). *Cont. Math.* 81 (1988), 213–243.
- *Existence of perturbed solitary wave solutions to a model equation for water waves* (together with J. Hunter). *Physica D* 32 (1988), 253–268.
- *Lagrangian reduction and bifurcations of relative equilibria of the double spherical pendulum* (together with J. Marsden). *ZAMP* 44 (1993), 17 - 43.
- *The reduced Euler-Lagrange equations* (together with J. Marsden). *Fields Inst. Comm.* 1 (1993), 139–164.
- *Pattern evocation and geometric phases in mechanical systems with symmetry* (together with J. Marsden), *Dyn. and Stab. of Systems* 10 (1995), 315–338.
- *Discretization of homoclinic orbits and “invisible” chaos* (together with B. Fiedler). *Memoirs of the AMS* vol. 119, nb. 570 (3), Providence 1996.
- *Reduction Theory and the Lagrange-Routh equations* (together with J. Marsden and T. Ratiu). *J. Math. Phys.* 41(6) (2000), 3379–3429.
- *The orbit space method* (together with M. Rumberger). In *Ergodic Theory, Analysis and Efficient Simulation of Dynamical Systems*, B. Fiedler ed., Springer-Verlag 2001, 649–689.

- *On the generation of conjugate flanks for arbitrary gear geometries* (together with A. Johann). GAMM-Mitt. 32, No. 1, 2009, 61–79.

His teaching covers a wide spectrum of subjects, ranging from mathematics for engineering students, functional analysis, ordinary differential equations and partial differential equations to dynamical systems, bifurcation theory, hamiltonian dynamics, geometric mechanics, mathematical methods in continuum mechanics, and mathematical modeling in biology and ecology. He supervised more than 20 dissertations and habilitations in these areas.

Jürgen Scheurle was a member of the advisory board of the book series *Dynamics Reported* and an executive editor of the *International Journal of Nonlinear Mechanics*. He is currently a member of the editorial board of the *Journal of Nonlinear Science*, *Nonlinear Science Today*, *Journal of Applied Mathematics and Mechanics* (ZAMM), and *Journal of Geometric Mechanics*.



Conference photo in the garden of the Carl Friedrich von Siemens Stiftung at the Schloß Nymphenburg, Munich

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# Contents

## Part I Stability, Bifurcation and Perturbations

<b>1 The Birth of Chaos</b> .....	3
John Guckenheimer	
1.1 Introduction .....	3
1.2 The Forced Van der Pol Equation .....	5
1.3 Background on Slow–Fast Dynamical Systems .....	7
1.4 Folds and Folded Saddles.....	12
1.5 Return Maps.....	16
1.6 Structural Stability, Hyperbolic Invariant Sets, and Axiom A .....	18
1.7 Structural Stability of the Forced Van der Pol Equation .....	19
1.8 Afterword .....	21
References.....	22
<b>2 Periodic Orbits Close to Grazing for an Impact Oscillator</b> .....	25
D.R.J. Chillingworth and A.B. Nordmark	
2.1 The Impact Oscillator .....	25
2.1.1 Nordmark’s Criteria .....	26
2.1.2 The Impact Surface Approach.....	29
2.1.3 Single Impact Period $T$ Orbits.....	32
2.1.4 Single Impact $2T$ -Periodic Orbits .....	35
2.1.5 Conclusion.....	36
References.....	37
<b>3 Branches of Periodic Orbits in Reversible Systems</b> .....	39
André Vanderbauwhede	
3.1 Introduction .....	39
3.2 Reversible Systems .....	40
3.3 Reversible Hopf Bifurcation.....	42

3.4	Generic Subharmonic Branching.....	42
3.4.1	Period Doubling.....	43
3.4.2	Subharmonic Branching.....	44
3.5	Degenerate Subharmonic Branching.....	46
3.6	Change of Stability Without Bifurcation.....	47
	References.....	48
<b>4</b>	<b>Canard Explosion and Position Curves</b> .....	<b>51</b>
	Freddy Dumortier	
4.1	Introduction.....	51
4.2	Setting of the Problem and Statement of Results.....	53
4.2.1	Generic Breaking Mechanisms and Nearby Transition Maps.....	53
4.2.2	Control Curves and Manifold of Closed Orbits.....	57
4.2.3	Position Curves and Statement of Results.....	58
4.3	Typical Shape of Generic Position Curves.....	61
4.3.1	Flying Canards.....	61
4.3.2	Simple Zeros of $I(Y, \mu_0)$ .....	65
4.4	Catastrophes of Canard Type Limit Cycles.....	68
4.5	Consequences of Theorem 4.1 and Remaining Problems.....	75
	References.....	77
<b>5</b>	<b>Bifurcation for Non-smooth Dynamical Systems via Reduction Methods</b> .....	<b>79</b>
	T. Küpper, H.A. Hosham, and D. Weiss	
5.1	Introduction.....	80
5.2	General Setting.....	83
5.3	Concept of Generalized Center Manifolds.....	87
5.3.1	Brake Model as PWS.....	91
5.3.2	Detecting Crossing and Sliding Regions.....	92
5.4	Piecewise Smooth Linear System.....	92
5.4.1	Concepts of Invariant Cones.....	92
5.5	PWLS with Sliding.....	97
5.6	Nonlinear Piecewise Smooth Systems (PWNS).....	102
	References.....	104
<b>6</b>	<b>Homoclinic Flip Bifurcations in Conservative Reversible Systems</b> .....	<b>107</b>
	Björn Sandstede	
6.1	Introduction.....	107
6.2	Main Results.....	109
6.3	Proof of Theorem 6.1.....	112
6.4	Application to a Fifth-Order Model for Water Waves.....	115
6.5	Open Problems.....	118
	References.....	123

**7 Local Lyapunov Functions for Periodic and Finite-Time ODEs** ..... 125  
 Peter Giesl and Sigurdur Hafstein

7.1 Introduction ..... 125

7.2 Autonomous System ..... 128

7.3 Periodic Time ..... 130

7.3.1 Linear Systems ..... 130

7.3.2 Nonlinear Systems ..... 133

7.4 Finite Time ..... 134

7.4.1 Dini Derivative ..... 137

7.4.2 Linear Systems ..... 140

7.4.3 Nonlinear Systems ..... 144

7.4.4 Norm  $\|x\|^2 = x^T Nx$  ..... 145

7.5 Relations Between Autonomous, Periodic and Finite-Time Systems ..... 147

7.5.1 Periodic Systems as Finite-Time Systems ..... 147

7.5.2 Autonomous Systems as Periodic and Finite-Time Systems ..... 148

7.6 Conclusions and Outlook ..... 150

References ..... 151

**8 Quasi-Steady State: Searching for and Utilizing Small Parameters** ..... 153  
 Alexandra Goeke and Sebastian Walcher

8.1 Introduction ..... 153

8.2 Background and Statement of Problem ..... 154

8.2.1 Chemical Reactions and ODEs ..... 154

8.2.2 Quasi-Steady State ..... 156

8.2.3 The Ad Hoc Reduction from QSS ..... 157

8.3 Reduction in the Presence of Small Parameters ..... 158

8.3.1 Singular Perturbations ..... 159

8.3.2 Computing a Reduction ..... 161

8.3.3 Slow and Fast Reactions ..... 168

8.3.4 Why Does the Ad Hoc Method Persist? ..... 171

8.4 Finding Small Parameters ..... 172

8.4.1 Underlying Assumptions: QSS vs. Slow–Fast ..... 172

8.4.2 The Role of Scaling ..... 173

8.4.3 Near-Invariance Heuristics ..... 174

References ..... 177

**9 On a Global Uniform Pullback Attractor of a Class of PDEs with Degenerate Diffusion and Chemotaxis in One Dimension** ..... 179  
 Messoud Efendiev and Anna Zhigun

9.1 Introduction ..... 180

9.2 Dissipative Estimates (Proof of *Theorem 9.2*) ..... 184



9.3	Global Uniform Pullback Attractor (Proof of <i>Theorem 9.3</i> ) .....	196
	References .....	203
<b>10</b>	<b>A Guided Sequential Monte Carlo Method for the Assimilation of Data into Stochastic Dynamical Systems</b> .....	<b>205</b>
	Sebastian Reich	
10.1	Introduction .....	206
10.2	Bayes' Theorem, Filtering, and Coupling of Random Variables .....	209
10.3	A GSMC Method .....	214
10.4	Brownian Dynamics Under a Double Well Potential .....	216
10.5	Conclusions .....	219
	References .....	219
<b>11</b>	<b>Deterministic and Stochastic Dynamics of Chronic Myelogenous Leukaemia Stem Cells Subject to Hill-Function-Like Signaling</b> .....	<b>221</b>
	Tor Flå, Florian Rupp, and Clemens Woywod	
11.1	Introduction .....	222
11.2	Definition of the Governing Probabilistic Four-Dimensional Model (Model C).....	224
11.2.1	Biological Aspects of the Model .....	224
11.2.2	Formulation of the Building Blocks of Model C .....	227
11.2.3	The Approximate Fokker-Planck Equation for Model C .....	229
11.2.4	The Stochastic Version of Model C in Terms of Itô/Langevin Equations .....	231
11.3	Equilibria and Their Stability in the Deterministic Small Noise Limit .....	234
11.3.1	Model A: The Dynamics of Two Competing Clones ....	235
11.3.2	Model B: The Formation of Cancer—Competition Between Normal and Wild-Type Leukaemic Stem Cells .....	246
11.3.3	Model C: The Full Four-Dimensional Problem, Including Cycling and Noncycling Normal Stem Cells Plus Two Cycling Leukaemic Stem Cell Clones .....	252
11.4	Summary and Outlook .....	257
	References .....	261

**Part II Hamiltonian Dynamics, Geometric Mechanics and Control Theory**

**12 Singular Solutions of Euler–Poincaré Equations on Manifolds with Symmetry** ..... 267  
 D.D. Holm, J. Munn, and S.N. Stechmann

12.1 Introduction ..... 268

    12.1.1 Motivation and Problem Statement ..... 268

    12.1.2 The Camassa–Holm Equation on a Riemannian Manifold ..... 269

    12.1.3 Main Results of the Paper ..... 272

    12.1.4 Plan of the Paper ..... 273

12.2 EPDiff Equations on Einstein Spaces ..... 273

12.3 The EPDiff Equation on the Sphere ..... 275

    12.3.1 Rotationally Invariant Solutions ..... 275

    12.3.2 The Basic Irrotational Puckon ..... 280

    12.3.3 Rotating Puckons ..... 282

    12.3.4 The Basic Rotating Puckon ..... 286

    12.3.5 Puckons and Geodesics ..... 288

    12.3.6 Further Hamiltonian Aspects of Radial Solutions of EPDiff on the Riemann Sphere ..... 289

12.4 Numerical Solutions for EPDiff on the Sphere ..... 291

    12.4.1 Overview ..... 291

    12.4.2 Numerical Specifications ..... 292

12.5 Generalizing to Other Surfaces ..... 297

    12.5.1 Rotationally Symmetric Surfaces ..... 298

    12.5.2 Rotationally Invariant Diffeons on Hyperbolic Space ..... 302

    12.5.3 Horotationally Invariant Diffeons on Hyperbolic Space ..... 305

    12.5.4 Translation Invariant Diffeons on Hyperbolic Space ..... 306

12.6 EPDiff on Warped Product Spaces ..... 308

    12.6.1 Warped Products ..... 309

    12.6.2 Singular Fibers ..... 313

12.7 Conclusions ..... 314

References ..... 315

**13 On the Destruction of Resonant Lagrangean Tori in Hamiltonian Systems** ..... 317  
 Henk W. Broer, Heinz Hanßmann, and Jiangong You

13.1 Introduction ..... 318

13.2 Kolmogorov Hamiltonians ..... 322

13.3 An Umbilic Example ..... 329

13.4 Rüssmann Hamiltonians ..... 330

13.5 Conclusions ..... 331

References ..... 332

<b>14</b>	<b>Deformation of Geometry and Bifurcations of Vortex Rings</b> .....	335
	James Montaldi and Tadashi Tokieda	
14.1	Smooth Family of Geometries .....	337
	14.1.1 Lie Algebras .....	337
	14.1.2 Surfaces .....	339
	14.1.3 Hamiltonians for Point Vortices .....	342
14.2	Nondegenerate Analysis of Vortex Rings .....	344
	14.2.1 Regular Ring .....	344
	14.2.2 Hessians .....	345
	14.2.3 Symplectic Slice.....	347
14.3	Bifurcations Across the Degeneracy .....	351
	14.3.1 Dihedral Group Action.....	351
	14.3.2 Dihedral Bifurcations .....	352
	14.3.3 Bifurcations of Vortex Rings .....	356
	14.3.4 Geometry of Bifurcating Rings .....	358
	14.3.5 Degenerate Critical Points .....	359
	14.3.6 Bifurcations from the Equator .....	367
14.4	What Happens with Other Hamiltonians .....	368
	14.4.1 Green's Function $G = \log  z - w ^2$ .....	368
	14.4.2 Green's Function $G = \log \frac{ z-w ^2}{ 1+\lambda z\bar{w} ^2}$ .....	369
	References.....	370
<b>15</b>	<b>Gradient Flows in the Normal and Kähler Metrics and Triple Bracket Generated Metriplectic Systems</b> .....	371
	Anthony M. Bloch, Philip J. Morrison, and Tudor S. Ratiu	
15.1	Introduction .....	372
15.2	Metrics on Adjoint Orbits of Compact Lie Groups and Associated Dynamical Systems .....	373
	15.2.1 Double Bracket Systems .....	373
	15.2.2 The Finite Toda System .....	374
	15.2.3 Lie Algebra Integrability of the Toda System.....	376
	15.2.4 The Toda System as a Double Bracket Equation .....	377
	15.2.5 Riemannian Metrics on $\mathcal{O}$ .....	377
15.3	Gradient Flows on the Loop Group of the Circle .....	379
	15.3.1 The Loop Group of $S^1$ .....	379
	15.3.2 The Based Loop Group of $S^1$ .....	380
	15.3.3 $L(S^1)$ as a Weak Kähler Manifold .....	381
	15.3.4 Weak Riemannian Metrics on $L(S^1)$ .....	384
	15.3.5 Vector Fields on $L(S^1)$ and $L(\mathbb{R})$ .....	385
	15.3.6 The Gradient Vector Fields in the Three Metrics of $L(S^1)$ .....	387
	15.3.7 Symplectic Structure on Periodic Functions .....	392
15.4	Metriplectic Systems.....	394
	15.4.1 Definition and Consequences .....	395
	15.4.2 Metriplectic Systems Based on Lie Algebra Triple Brackets .....	397

15.4.3	The Toda System Revisited .....	403
15.4.4	Metriplectic Systems for PDEs: Metriplectic Brackets and Examples .....	404
15.4.5	Hybrid Dissipative Structures .....	410
	References .....	412
<b>16</b>	<b>Boundary Tracking and Obstacle Avoidance Using Gyroscopic Control</b> .....	<b>417</b>
	Fumin Zhang, Eric W. Justh, and P.S. Krishnaprasad	
16.1	Introduction .....	418
16.2	Planar Boundary Tracking .....	419
	16.2.1 Models .....	419
	16.2.2 Boundary-Curve Frame Convention .....	421
16.3	Planar Bertrand Mate Strategy .....	423
	16.3.1 Lyapunov Function and Steering Law .....	423
	16.3.2 Shape Variables .....	425
	16.3.3 Convergence Result .....	426
16.4	Curve Tracking with Obstacle Avoidance in Three Dimensions .....	428
	16.4.1 Curves and Moving Frames .....	429
	16.4.2 Spherical Curves and Natural Frames .....	429
	16.4.3 Free-Particle Interaction with the Spherical Curve .....	430
	16.4.4 Lyapunov Function and Control Law Derivation .....	431
	16.4.5 Control Law Interpretation .....	434
	16.4.6 Strategy and Invariant Submanifold .....	435
	16.4.7 Shape Variables .....	436
	16.4.8 Convergence Result .....	438
	16.4.9 Simulation Example .....	441
16.5	Conclusions .....	442
	References .....	445
<b>17</b>	<b>Random Hill's Equations, Random Walks, and Products of Random Matrices</b> .....	<b>447</b>
	Fred C. Adams, Anthony M. Bloch, and Jeffrey C. Lagarias	
17.1	Introduction .....	448
17.2	Matrices from Hill's Equation in the Unstable Regime .....	451
	17.2.1 Growth Rates for Positive Matrix Elements .....	452
	17.2.2 Matrix Elements with Varying Signs .....	454
17.3	Products of Randomly Rotated Matrices .....	456
	17.3.1 Deterministic Formulas for Product Matrices .....	458
	17.3.2 Uniformly Distributed Rotations Case .....	460
	17.3.3 Uniformly Distributed Case with Constant $x_k$ .....	462
17.4	Comparison of the Unstable Regime Hill Equation Model and Random Rotation Model with all $x_k = 1$ .....	464
17.5	Concluding Remarks .....	466
	References .....	468

### Part III Continuum Mechanics: Solids, Fluids and Other Materials

<b>18</b>	<b>The Three-Dimensional Globally Modified Navier–Stokes Equations: Recent Developments</b> .....	473
	T. Caraballo and P.E. Kloeden	
18.1	Introduction .....	473
18.1.1	Notation .....	475
18.2	Existence and Regularity of Solutions .....	476
18.2.1	Weak Solutions .....	476
18.2.2	Strong Solutions .....	477
18.3	Global Attractor in $V$ : Existence and Dimension Estimate .....	479
18.3.1	Autonomous Case .....	479
18.3.2	Nonautonomous Case .....	481
18.4	Globally Modified NSE with Delays .....	482
18.5	Statistical Solutions of GMNSE .....	486
18.5.1	Time-Averages Solutions in the Autonomous Case .....	486
18.5.2	Stationary Statistical Solutions of the Autonomous GMNSE .....	487
18.6	Numerical Solution of the Globally Modified NSE .....	488
18.7	Weak Solutions of the Three-Dimensional Navier–Stokes Equations .....	489
18.7.1	Weak Kneser Property of the Attainability Set of Weak Solutions .....	489
18.7.2	Convergence to Weak Solutions of the Three-Dimensional NSE .....	489
18.7.3	Existence of Bounded Entire Weak Solutions of Three-Dimensional NSE .....	490
	References .....	491
<b>19</b>	<b>Simulation of Hard Contacts with Friction: An Iterative Projection Method</b> .....	493
	Christoph Glocker	
19.1	Introduction .....	493
19.2	The Normal Cone and Proximal Points .....	495
19.3	Exact Regularization of the Set-Valued Sign Function .....	496
19.4	Equations of Motion in Lagrangian Mechanics .....	498
19.5	The Contact Model .....	499
19.6	Formulation of the Contact Laws by Normal Cone Inclusions ...	502
19.7	Embedding Impact Dynamics and the Impact Laws .....	504
19.8	Time Discretization .....	507
19.9	Numerical Solution of the Inclusion Problem .....	509
19.10	Applications .....	512
	References .....	513

<b>20</b>	<b>Dynamics of Second Grade Fluids: The Lagrangian Approach</b> .....	517
	M. Paicu and G. Raugel	
20.1	Introduction .....	518
20.2	Existence Results for the Second Grade Fluid Equations .....	527
20.2.1	The Transport Equation .....	527
20.2.2	An Auxiliary Problem.....	534
20.2.3	Local Existence and Uniqueness of Solutions in $V^{3,p}$ , $p > 1$ .....	536
20.2.4	Global Existence of Solutions in $V^{3,p}$ , $p > 1$ .....	541
20.3	Dynamics of the Second Grade Fluids in the 2D Torus .....	544
20.3.1	Existence of a Compact Global Attractor .....	544
20.3.2	Regularity of the Compact Global Attractor .....	546
20.3.3	Finite-Dimensional Properties .....	550
	References.....	551
<b>21</b>	<b>Dissipative Quantum Mechanics Using GENERIC</b> .....	555
	Alexander Mielke	
21.1	Introduction .....	556
21.2	The GENERIC Framework.....	559
21.2.1	The Structure of GENERIC .....	560
21.2.2	Properties of GENERIC Systems .....	561
21.2.3	Isothermal Systems .....	561
21.3	Coupling of Quantum and Dissipative Mechanics .....	562
21.3.1	Quantum Mechanics .....	562
21.3.2	Dissipative Evolution .....	564
21.3.3	Coupling of the Models .....	565
21.4	Canonical Correlation .....	566
21.4.1	The Kubo–Mori Metric .....	566
21.4.2	GENERIC Systems with Canonical Correlation.....	569
21.4.3	Steady States .....	571
21.4.4	Comparison to the Lindblad Equation .....	572
21.5	A Simple Coupled System .....	573
21.5.1	The Case of One Heat Bath .....	573
21.5.2	Elimination of the Temperature .....	574
21.5.3	The Case $\dim \mathbf{H} = 2$ .....	575
21.6	Existence and Convergence into Equilibrium .....	577
21.6.1	Existence via a Modified Explicit Euler Scheme .....	577
21.6.2	Convergence into the Thermodynamic Equilibrium.....	580
21.7	Comparison to Stochastic Gradient Structures .....	582
	References.....	584
<b>22</b>	<b>Modelling of Thin Martensitic Films with Nonpolynomial Stored Energies</b> .....	587
	Martin Kružík and Johannes Zimmer	
22.1	Introduction .....	587
22.1.1	Shape Memory Alloys .....	589
22.1.2	Variational Models for Shape Memory Alloys .....	589

- 22.2 Thin Films ..... 590
  - 22.2.1 Static Problems ..... 590
  - 22.2.2 Evolutionary Problems ..... 595
- 22.3 Problems Involving Concentration ..... 598
  - 22.3.1 DiPerna–Majda Measures ..... 599
  - 22.3.2 DiPerna–Majda Measures Depending on the Inverse ... 601
  - 22.3.3 Application to a Thin Film Model ..... 603
- 22.4 Open Problems ..... 606
- References ..... 606
  
- 23 Linear Stability of Steady Flows of Jeffreys Type Fluids ..... 609**  
Michael Renardy
  - 23.1 Introduction ..... 609
  - 23.2 Statement of Results ..... 611
  - 23.3 Proof of Theorem 23.1 ..... 613
  - 23.4 Some Comments on the Proof of Theorem 23.2 ..... 615
- References ..... 615

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**Part I**  
**Stability, Bifurcation and Perturbations**

# Chapter 1

## The Birth of Chaos

John Guckenheimer

### 1.1 Introduction

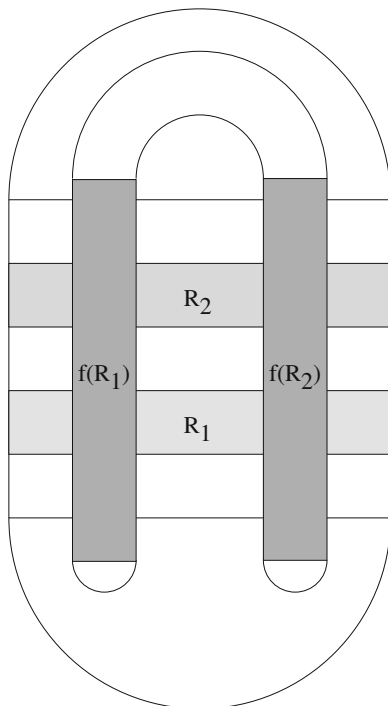
The word *chaos* has become firmly embedded in the literature on dynamical systems. Indeed, James Gleick's book, *Chaos Theory* [17], established that term as a description of the entire subject in the public mind. Nonetheless, there is no authoritative technical meaning of "chaos" in dynamical systems. Li and Yorke first used the word in the title of their paper "Period three implies chaos" [31], but it does not appear in the text. They refer to trajectories that are "nonperiodic and might be called 'chaotic'." Ruelle and Takens [45] used the longer phrase "sensitive dependence to initial conditions" and the two terms have largely been regarded as synonyms [15, 57]. The informal definition of sensitive dependence to initial conditions is that nearby initial conditions separate; the technical definition is that there are sets of trajectories with positive Lyapunov exponents [15] that measure the exponential rate of separation of nearby trajectories. What is not often specified in the definition is *how many* trajectories have positive Lyapunov exponents. For example, if a dynamical system has a saddle point, this point has a positive Lyapunov exponent, but the presence of a single saddle point (or even more complicated normally hyperbolic sets) does not make the system chaotic. There appears to be little consensus on the minimal requirements for sets of trajectories with positive Lyapunov exponents that make a system chaotic, but there is a sufficient criterion formulated by Smale [48] that is often used as a practical test: namely, that the system possesses a "transversal intersection of stable and unstable

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manifolds of a periodic orbit.” This concept is explained below. Such *homoclinic* orbits were first discovered by Poincaré in 1890 in a prize winning essay [43] motivated by the question, is the solar system stable? The intriguing history of Poincaré’s discovery has been studied and recounted by Barrow-Green [3]. The work of Poincaré and later Birkhoff was directed at *conservative* dynamical systems arising in celestial mechanics. Within the setting of systems that preserve a symplectic structure, they investigated the presence of transversal intersections of the stable and unstable manifolds of periodic orbits. The first mathematical analysis of transversal homoclinic orbits in the context of dissipative systems that are not conservative was carried out by Cartwright and Littlewood, beginning during World War II [9–12] and culminating in Littlewood’s long two part paper of 1957 [32–34]. The personal aspects of the Cartwright–Littlewood collaboration are also fascinating and have been described by McMurrin and Tattersall [37, 38] as well as by Cartwright herself [8, 50].

The initial presentations of significant mathematical discoveries seldom appear in full clarity. The path to a new discovery is often tortuous, so reformulation is typically needed to distill the essence of new insights. This has been true in dynamical systems theory: the papers of Poincaré and Littlewood cited above are excellent examples. The work of Cartwright–Littlewood has a dual character, containing detailed analysis of the forced Van der Pol differential equation as well as a description of the dynamical consequences of transversal homoclinic orbits in dissipative systems. There was a long period of abstraction and simplification of the arguments of Cartwright–Littlewood that led to piecewise linear vector fields studied by Levinson [30] and later Levi [29], the geometric discrete time Smale horseshoe [47, 49] and the concept of *hyperbolic invariant sets* [46]. Figure 1.1 illustrates the horseshoe. These developments provided tremendous insight into chaotic dynamics, but they draw upon only a small portion of the Cartwright–Littlewood analysis of the forced Van der Pol differential equation. Thus, there is a disparity between mathematical awareness of these two aspects of the Cartwright–Littlewood discovery of chaos in dissipative systems. The horseshoe and its symbolic dynamics are a beautiful geometric example of chaotic dynamics, simple enough to be included routinely in undergraduate courses. Littlewood’s analysis of the forced Van der Pol equation remains obscure despite its central role in the book of Grasman [18]. This paper visualizes horseshoes in the forced Van der Pol equation from the perspective of *geometric singular perturbation theory* and describes recent extensions of the work of Cartwright–Littlewood by myself and collaborators [6, 19, 20] that culminated in the thesis of Radu Haiduc [22, 23]. Haiduc proved that there are parameter values for which the forced Van der Pol equation is structurally stable and possesses a chaotic invariant set. This paper gives an extended outline of this work, presenting the key geometric constructions used in the analysis of the forced Van der Pol equation.



**Fig. 1.1** The horseshoe is an invariant set  $A$  of the discrete map  $f$  depicted in this figure. The map  $f$  stretches the background oval vertically, compresses it horizontally, and maps it back into itself. The rectangles  $R_1$  and  $R_2$  shaded in *light gray* are mapped rectilinearly into their images shaded in *dark gray*. Inside the intersection  $(R_1 \cup R_2) \cap (f(R_1) \cup f(R_2))$ , there is an invariant Cantor set  $A$  consisting of points whose  $f$ -trajectories (both forward and backward) remain inside the intersection. The vertical distance between points that lie on different horizontal lines increases until one of the points lands in  $R_1$  at the same time that the other lands in  $R_2$ . This expresses the *sensitivity to initial conditions* of this map. There is a one-to-one correspondence between points of  $A$  and bi-infinite sequences of 1 and 2 that encode which rectangle  $R_j$  each iterate lies in

## 1.2 The Forced Van der Pol Equation

The main object of this paper is analysis of the system of differential equations

$$\begin{aligned}
 \varepsilon \dot{x} &= y + x - \frac{x^3}{3} \\
 \dot{y} &= -x + a \sin(2\pi\theta) \\
 \dot{\theta} &= \omega
 \end{aligned}
 \tag{1.1}$$

where the variable  $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ , so we identify  $\theta$  and  $\theta + 1$ . We are interested in the parameter regime where  $\varepsilon > 0$  is small. The limit  $\varepsilon = 0$  produces a system