Contemporary Mathematicians

Claude Brezinski
Ahmed Sameh
Editors

Walter Gautschi

Selected Works with Commentaries

Volume 1

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Contemporary Mathematicians

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Walter Gautschi, Volume 1

Selected Works with Commentaries

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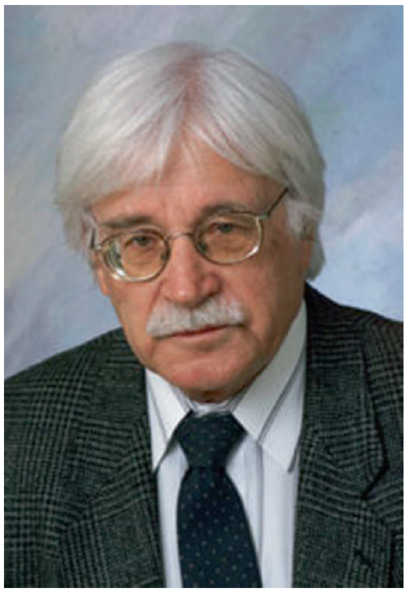

Walter Gautschi, 2001

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Preface

Claude Brezinski and Ahmed Sameh

Walter Gautschi is a world-renowned numerical analyst whose research contributions cover a wide range of topics including numerical conditioning, special functions, interpolation and approximation, orthogonal polynomials, quadrature, linear recurrence relations, ordinary differential equations, and history of mathematics. His contributions have had a significant impact on the field, and his papers are widely cited. Walter has published 3 books, 34 book chapters, 160 refereed journal papers, 7 refereed papers in conference proceedings, translated 3 books, and edited 5 conference proceedings. His papers are characterized by their clarity of exposition and will remain excellent resources for researchers in the field. Walter has 4820 citations in Google Scholar and 174,000 citations in Google. His two books: *Numerical analysis — an introduction*, published by Birkhäuser, and *Orthogonal polynomials — computation and approximation*, published by Oxford University Press, have set a high standard for graduate textbooks in their respective subjects.

Walter's 65th birthday was celebrated by a conference held in his honor in December 1993 at Purdue University, attended by leaders in the field, such as Richard Askey, Carl de Boor, John Butcher, Ward Cheney, Paul Erdős, Gene Golub, Bill Gragg, Arieh Iserles, Charles Micchelli, Frank Olver, John Rice, Ted Rivlin, Ed Saff, Frank Stenger, Richard Varga, Jet Wimp, among others. The proceedings of this conference were published by Birkhäuser in 1994. Since then, Walter has added significantly to his contributions to warrant this publication (also by Birkhäuser) of his selected works together with commentaries by foremost experts in the respective areas of Walter's contributions. Volume 1 collects papers on numerical conditioning, special functions, interpolation and approximation; Volume 2 those on orthogonal polynomials — on the real line and on the semicircle —, and quadrature — of Chebyshev, Gauss, and Kronrod type —; and Volume 3 papers on linear recurrence relations, ordinary differential eqations, computer algorithms and software packages, history and biography, and miscellaneous topics.

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The papers included are chosen by Walter, and the editors wish to thank the publishers of Walter's papers for permission to reprint them here. The editors also express their gratitude to the commentators for their excellent reviews and prompt response.

Finally, we wish to thank Birkhäuser for their wonderful cooperation to produce these volumes, thereby preserving and making easily accessible Walter's contribution to Computational Mathematics. We also thank Professor Michela Redivo-Zaglia of the University of Padua for lending a hand to one of the editors with Birkhäuser's latex style in the early phase of the work.

We present these volumes, honoring Walter and the memory of his late brother Werner, as a tribute to Walter — an inspiring and valued colleague. We are proud to call him a great friend.

Claude Brezinski
Ahmed Sameh

December 17, 2012

Part I

Walter Gautschi

In the article of Section 3, numbers in brackets refer to the numbered list of papers in Section 4, those in boldface type to papers included in these selected works.

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Biography of Walter Gautschi

Claude Brezinski and Ahmed Sameh

Walter Gautschi was born on December 11, 1927 in Basel, Switzerland, together with his twin brother Werner. He attended primary and secondary schools in Basel, graduating in 1947 from the Mathematisch-Naturwissenschaftlichen Gymnasium. He then enrolled at the University of Basel to study mathematics as the primary subject, with physics, physical chemistry, and actuarial mathematics as secondary subjects. In the early 1950s he became an assistant of Professor Alexander M. Ostrowski, obtaining a Ph. D. in 1953 under his supervision with a thesis on graphical integration of ordinary differential equations. He then received a two-year fellowship for study abroad from the Janggen-Poehn foundation in St. Gallen, of which he spent the first year at the *Istituto Nazionale per le Applicazioni del Calcolo* in Rome, founded and directed by Mauro Picone, and a second year at the Harvard Computation Laboratory. It was at the Harvard Computation Laboratory where he got his first hands-on experience with electronic computers, programming (in machine code) on Professor Aiken's MARK III computer. In 1956, under a contract with the American University, he joined the staff of the Computation Laboratory at the National Bureau of Standards in Washington, D. C. (now the National Institute of Standards and Technology). There, his major project was the preparation of two chapters of the *Handbook of Mathematical Functions* edited by Milton Abramowitz and Irene A. Stegun. Abramowitz introduced Walter to the work of J. C. P. Miller on backward recurrence, which became one of the early areas of emphasis in Walter's research. Because of employment difficulties related to Walter's Swiss citizenship, he had to leave the Bureau in 1959 and he joined Alston Householder's Mathematics Panel at the Oak Ridge National Laboratory. Through contacts with chemists at the laboratory, he became interested in the numerical aspects of Gaussian quadrature and orthogonal polynomials, which was to become one of the principal areas of Walter's research contributions. During the four years at the Oak Ridge laboratory he was twice invited to lecture at the Michigan University Engineering Summer Conferences then organized by Robert C. F. Bartels.

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In 1960, after the untimely death of Walter's twin brother Werner in 1959, he married his widow, Erika Wüst, and adopted their son Thomas, born only after Werner's death. The marriage brought forth three more children, Theresa, Doris, and Caroline, born respectively in 1961, 1965, and 1969.

In 1963, Walter started his academic career, accepting a professorship jointly at the then (1962) newly established Department of Computer Sciences and the Department of Mathematics at Purdue University. It was to become a life-long association, interrupted only by sabbatical years, 1970–1971 as a Fulbright scholar at the Technical University of Munich, and 1976–1977 at the University of Wisconsin. Walter regularly taught the beginning graduate course on Numerical Analysis, an advanced course on the numerical solution of ordinary differential equations, and occasionally courses on numerical linear algebra and optimization. Notes prepared over the years on the first two of these courses, and also notes prepared for summer courses taught repeatedly in Perugia, Italy, in the 1970s, led in 1997 to the publication of his book on Numerical Analysis by Birkhäuser Boston. A second edition of this book appeared in 2012. Another book, that grew out of seminars held on the constructive aspects and applications of orthogonal polynomials, was published by Oxford University Press in 2004.

Throughout his academic career, Walter participated and lectured at numerous national and international meetings and was a frequent visitor at other academic institutions, notably the Polytechnics of Milan and Turin, the University of Padua, the ETH in Zurich, and his alma mater, the University of Basel. For many years he was also a consultant at Argonne National Laboratory.

In 2001, Walter was elected a Foreign and Corresponding Member of two European Academies, respectively the Bavarian Academy of Sciences in Munich and the Turin Academy of Sciences (once the Royal Society). He was also named a SIAM Fellow in 2012.

From 1966 to 1999, Walter was a member of the Editorial Committee of *Mathematics of Computation* and its Managing Editor from 1984 to 1995. His meticulous attention to details was legendary. Other journals for which he served as an Associate Editor are *Numerische Mathematik*, 1971 to the present (Honorary Editor since 1991), the *SIAM Journal on Mathematical Analysis*, 1970–1973, and *Calcolo*, 1975–1987. In addition, in 1981–1983, Walter served as a Special Editor of *Linear Algebra and its Applications.* On the 50th anniversary of *Mathematics of Computation*, Walter edited an AMS proceedings volume entitled *A half-century of computational mathematics*, and he was co-editor of a number of other proceedings volumes. He was also active as a translator, translating (jointly with R. Bartels and C. Witzgall) the text *Numerische Mathematik* by J. Stoer, preparing an annotated translation of H. Rutishauser's *Vorlesungen über numerische Mathematik*, and (jointly with his wife Erika) an English translation of E. A. Fellmann's *Leonhard Euler.*

Walter officially retired from Purdue University in 2000 with the title of Professor Emeritus, but both his research and lecturing activities continued unabatedly ever since.

For more details on Walter's life, and especially his early research activities, see also Walter Gautschi's "Reflections and recollections" in *Approximation and Computation — a festschrift in honor of Walter Gautschi* (R. V. M. Zahar, ed.), pp. xvii–xxxv, Birkhäuser, Boston, 1994.

3

A brief summary of my scientific work and highlights of my career

Walter Gautschi

I have worked in a number of different areas of (mostly computational) mathematics. They are organized here in thirteen sections. For the sake of brevity, when referring to joint papers, coauthors are not identified explicitly.

1. *Numerical conditioning.* The general theme here is to analyze the sensitivity of a problem to small perturbations in the data. This has been an area of continued interest to me, given my predilection to fundamental issues.

An example of this is the extensive work on the condition of Vandermonde and Vandermonde-like matrices. The former [**16**, **19**, **34**, **51**, **52**, **62**, **110**] are shown to be always ill-conditioned, exponentially so or worse, if the nodes are real. They are usually well-conditioned if the nodes are complex. A noteworthy example [**120**] is the $n \times n$ Vandermonde matrix whose nodes are the first n members of an infinite sequence of complex numbers on the unit circle, for example the Van der Corput sequence. The (spectral) condition number is then shown to be bounded by $\sqrt{2n}$. In the case of (real) Vandermonde-like matrices whose entries are not powers of the nodes, but orthogonal polynomials evaluated at the nodes, the matter depends on the Christoffel numbers, or Christoffel function (evaluated at the nodes) of the underlying measure, more precisely, on the ratio of their arithmetic and harmonic means [**83**]. Another interesting problem treated very recently pertains to optimally scaled and optimally conditioned Vandermonde and Vandermonde-like matrices [**200**]. For a survey, see also [**118**].

Other instances of work in this area are the condition of polynomial bases [**43**, **66**], the condition of algebraic equations [**45**], and most notably, the condition of moment maps in the theory of orthogonal polynomials and related quadratures [**40**, **81**, **98**].

2. *Special functions.* My contributions to this subject are four-fold: numerical evaluation, inequalities, asymptotics, and expository work.

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In the first of these categories, the influential work [**29**] should be mentioned on computational aspects of three-term recurrence relations. This centers around the concept of minimal solution of three-term recurrence relations and related algorithms involving continued fractions. The latter have been successfully applied to the computation of many special functions, such as Bessel functions [23], Legendre functions [24], Coulomb wave functions [28, 33, **35**], incomplete beta and gamma functions [22, 158], repeated integrals of the error function [**13**, 59, 60], and Stieltjes transforms of orthogonal polynomials [**75**]. Special mention deserves an efficient algorithm developed for computing the complex error function [36, **39**] (a Stieltjes transform of the Hermite weight function), which relies on similar ideas and which has found widespread use in the physics and nuclear engineering communities. In the paper [63], attention is drawn to a continued fraction of Perron as a useful alternative to the more customary Gauss-type continued fraction for evaluating ratios of modified Bessel functions of a real argument. A variety of techniques, including Taylor series and continued fraction expansions, are employed in the calculation of incomplete gamma functions [**68**, 69, 70]. Applying Gaussian quadrature led to useful procedures for computing hypergeometric and confluent hypergeometric functions [**168**], Bessel and Airy functions [**169**], modified Bessel functions of complex orders [**178**], and Kontorovich–Lebedev integral transforms [**181**]. High-precision nonstandard Gaussian quadrature rules are also employed to compute certain integrals involving the Lambert W-function [**199**].

With regard to inequalities, the two-sided inequalities for gamma function ratios [**9**], published in 1959, have been most widely noted (and now bear my name), although they were obtained in the context of more general two-sided inequalities for the incomplete gamma function. Of a quite different nature are the harmonic mean inequalities for the gamma function [**47**, **48**], obtained in the 1970s. In [**72**], classical inequalities of Laguerre for the largest zero of Jacobi, Laguerre, and Hermite polynomials are sharpened. Beginning in 2007, in a series of papers [**182**, **190**, **191**, **192**, **203**], a number of far-reaching conjectures are set forth regarding inequalities for zeros of Jacobi polynomials, all based on extensive numerical computation. Bernstein's inequality for Jacobi polynomials is analyzed in [**193**] with regard to sharpness and extended to larger domains of the Jacobi parameters. The computational work therein also suggests a numerical value for the best constant in the Erdélyi–Magnus–Nevai conjecture on orthonormal Jacobi polynomials.

There is one short paper on asymptotics [**10**] generalizing an asymptotic formula of G. Blanch for exponential integrals.

Most important among my expository work on special functions are the two chapters [20, 21] on the exponential integrals and the error function in the famous handbook of Abramowitz and Stegun.

3. *Interpolation and approximation.* An early paper [11] deals with bivariate linear interpolation of an analytic function in the complex plane and the respective error committed.

According to a classical result of Erdős and Turán, Lagrange interpolation of any continuous function on $[-1, 1]$ at the n zeros of an orthogonal polynomial of degree n converges in the mean as $n \to \infty$. Does the same conclusion hold if one inserts $n+1$ additional points in a well-specified manner (similar to Kronrod's method in the theory of quadrature)? This is explored in [**132**] with mixed success: the answer is conjectured to be "yes" for Jacobi polynomials $P_n^{(\alpha,\beta)}$ with parameters α, β suitably restricted, but is proved to be "no" for Chebyshev polynomials of the first, third, and fourth kind (the answer being trivially "yes" for Chebyshev polynomials of the second kind). For quadrature convergence in the sense of Erdős and Turán, however, the answer is "yes" for all four Chebyshev polynomials, as is proved in [**147**]. Under an additional interlacing condition on the interpolation points, we also established necessary and sufficient conditions for quadrature convergence to hold and conjectured them to be satisfied for Jacobi polynomials with parameters $|\alpha| \leq \frac{1}{2}$, $|\beta| \leq \frac{1}{2}$.

Inspired by work in physics, I became interested in approximating a function in such a way that as many of its moments as possible are preserved. I began by considering functions f on $\mathbb{R}_+$ and approximation by piecewise constant functions with both the location and height of their jumps being freely variable [**89**]. The problem was generalized in [**100**] to approximation on $\mathbb{R}_+$ by spline functions of fixed degree and variable (positive) knots. Interestingly, under appropriate conditions the problem has a unique solution expressible in terms of the nodes and weights of a Gaussian quadrature formula relative to a weight function which depends on f. Unique existence is always assured if f is completely monotonic on $\mathbb{R}_+$. Analogous problems on a finite interval can also be solved [**102**] and involve generalized Gauss–Radau and Gauss–Lobatto formulae. For a summary of this work and related work by others, see [131].

Other approximation-theoretic problems considered pertain to continued fractions [61, **87**, 127], Padé approximation [**86**], Fourier analysis [**41**], and the summation of slowly convergent series [**93**, **124**, **125**, **175**].

4. *Orthogonal polynomials on the real line.* The constructive theory of orthogonal polynomials is an area of work for which I am probably best known. (I have been called Mr. Orthogonal Polynomials by some of my colleagues!) I was the first to take up the problem of computationally generating orthogonal polynomials relative to essentially arbitrary weight functions or measures. While the solution via moments, in principle, is classically known, it is problematic computationally because of severe ill-conditioning. The major effort, indeed, was to carefully analyze the degree of ill-conditioning and to find methods that successfully surmount this ill-conditioning. The approach I have taken was to either replace moments by so-called modified moments (an idea that had been floating around at the time) and study the condition number of the relevant moment map; or else, to discretize the underlying inner product and take the corresponding discrete orthogonal polynomials to approximate the desired ones. The former approach led to two algo-

rithms, one based on Cholesky decomposition [**40**], and another, more efficient one [**81**, §2.4], given the name modified Chebyshev algorithm, because I could trace its origin to an 1859 memoir of Chebyshev dealing with ordinary moments of a discrete measure. The second appoach, often more effective, is entirely original with me [**31**]. It led to what I called a discretized Stieltjes procedure [**81**, §2.2], since Stieltjes in 1884 briefly alluded to an algorithm of this kind (without discretization). A Fortran program implementing the method has been published in [32]. Both algorithms are extended in [**145**] to Sobolev orthogonal polynomials, which are orthogonal with respect to an inner product also containing derivatives and accompanying measures. They are applied in [**153**] to illustrate theoretical results about the asymptotic distribution of zeros of Sobolev orthogonal polynomials and their derivatives. Very special Sobolev orthogonal polynomials involving a derivative of fixed order with an associated one-point atomic measure are discussed in [**151**] along with their zeros.

The algorithms thus developed, sometimes in conjunction with analytic or symbolic variable-precision tools, have been used to generate (recursion coefficients of) orthogonal polynomials with special, sometimes unusual, weight functions, for example the reciprocal gamma function [**80**], weight functions of interest in theoretical chemistry that are supported on two separate intervals [**90**], Einstein and Fermi functions [**93**], Freud and half-range Hermite weight functions [**195**], refinable [**161**] and densely oscillating, or rapidly exponentially decaying, weight functions [**176**], and sub-range Jacobi weight functions [**205**].

Other important algorithms studied pertain to modifications of the weight function, for example multiplying it by a positive rational function [**77**], [**179**, §2.6]. A notable special case is multiplication by the square of the respective orthogonal polynomial, which gives rise to what in [134] are called induced orthogonal polynomials. They are relevant, e.g., in the problem of extended interpolation mentioned in §3. Repeated modifications by linear divisors are studied in [**206**] and applied to generate special Gaussian quadrature rules for dealing with nearby poles. Some of these algorithms can also be used to “neutralize” singularities other than poles [**207**].

5. *Orthogonal polynomials on the semicircle.* An entirely new kind of (complex) orthogonal polynomials was introduced in 1985: polynomials orthogonal on the semicircle [**95**]. The novelty here is the non-Hermitian nature of the underlying inner product. Yet, many properties of these new polynomials, and also of the respective zeros, resemble properties known for classical orthogonal polynomials with positive definite or Hermitian weight functions. This was further developed in a number of papers, [**97**, **104**, **113**] and summarized in [116].

6. *Chebyshev quadrature.* The majority of my papers is dedicated to problems of quadrature. My early work in this area, suggested by a visitor (Hiroki Yanagiwara) from Japan, deals extensively with weighted Chebyshev and Chebyshev-type

formulae. The former are weighted quadrature rules with equal (real) coefficients, distinct (real) nodes, and polynomial degree of exactness equal to the number of nodes. From a celebrated result of Bernstein (relative to constant weight functions) one can expect such quadrature rules to exist only for a finite, typically small, number of nodes. A severe case in point is exhibited in [**50**]. In all remaining instances one can try to find substitute formulae by relaxing the exactness condition in one way or another. This is the kind of problem studied by me and co-workers in the mid-1970s [**46**, **50**, **53**, **57**, **58**]. A historical summary is provided in [**55**].

7. *Kronrod and other quadratures.* Gauss–Kronrod formulae give rise to intriguing problems of existence, that is, of determining if and when all nodes are real and distinct. This has been studied, using algebraic tools, for Jacobi weight functions in [**109**]. Other instances of such formulae, [**111**, **114**], involve weight functions of Bernstein–Szegő type, i.e., Chebyshev weights of any of the four kinds divided by a quadratic polynomial which remains positive on $[-1,1]$, or weights whose orthogonal polynomials have a three-term recurrence relation with ultimately constant coefficients [148]. Computing Gauss–Kronrod formulae is a topic discussed in [**99**, **108**]. For a review up to about 1987, see [**107**].

A convergence result for interpolatory quadrature rules with Chebyshev nodes (already studied by Fejér), when applied to improper integrals having monotonic singularities at the endpoints ± 1, is proved in [**30**]. The result is of interest in the generation of orthogonal polynomials by the discretized Stieltjes procedure (cf. §4).

Evaluating the Hilbert transform (a Cauchy principal value integral) of the classical Jacobi, Laguerre, and Hermite measures and of the respective orthogonal polynomials is discussed in [**103**, **166**]. The latter satisfy the same three-term recurrence relation as the one for the orthogonal polynomials themselves, but exhibit a phenomenon of pseudostability (cf. §9). A case of computing singular integrals is studied in [105].

In [**160**], a new look is taken at adaptive quadrature employing, among other devices, a 4-point Gauss–Lobatto formula and two successive Kronrod extensions thereof. The procedure has been incorporated into one of the Matlab quadrature routines, `quadl`.

A challenging integral involving an integrand that is densely oscillating near one of the endpoints of the interval of integration, with amplitudes tending to infinity, is evaluated in [188] by elementary means.

8. *Gauss-type quadrature.* The larger part of my work on numerical quadrature, however, concerns Gauss-type quadrature rules and, apart from [**31**, **40**, **65**], began to appear around 1981 after my long historical essay [**74**] on Gauss-Christoffel quadrature rules, written on the occasion of Christoffel's 150th anniversary of birth. The work can be divided into five parts: (i) geometric properties, (ii) explicit formulae and computation, (iii) validation, (iv) error estimation for analytic functions, and (v) polynomial/rational formulae.

(i) In 1961, P. J. Davis and Philip Rabinowitz proved that the classical Gauss–Jacobi formula has weights which, when suitably normalized and plotted over the corresponding nodes, come to lie on the upper half of the unit circle, asymptotically for large orders. In 2006, I have shown [**180**] that this pretty "circle theorem" is true for a much larger class of weight functions, essentially the Szegő class, not only for Gauss formulae, but also for Gauss–Radau, Gauss–Lobatto, and, under more restictive conditions, even for Gauss–Kronrod formulae.

(ii) There is a large number of papers dealing with the numerical calculation not only of Gaussian formulae (which essentially amounts to the numerical generation of the respective orthogonal polynomials – see §4 – followed by an eigenvalue/vector computation involving the Jacobi matrix of the orthogonal polynomials), but also of ordinary [**163**, **164**] and generalized [**173**], [**194**] Gauss–Radau and Gauss–Lobatto formlae, especially of very high order, as well as Gauss–Turán formulae [**154,** 211] (which involve derivative values of the function to be integrated up to some even order). For Gauss–Radau and Gauss–Lobatto formulae with double endpoints and Chebyshev weight functions of all four kinds, explicit formulae for the boundary weights are derived in [**126**].

(iii) The problem of validation, considered in [**84**], consists in assessing *a posteriori* the accuracy of the nodes and weights of a Gaussian quadrature formula, once computed in one way or another. In view of the severe ill-conditioning mentioned in §4, this is a nontrivial problem.

(iv) The remainder term of weighted Gaussian quadrature formulae over a finite interval applied to analytic functions can be estimated by contour integration techniques. This is the subject of the frequently cited work [**85**] and of [**119**]. Additional work on this topic is done in [**121**, **123**], where the same techniques are applied to Gauss–Radau and Gauss–Lobatto quadratures.

(v) Quadrature formulae with polynomial degrees of exactness are of limited use when the function to be integrated has poles, especially poles near the interval of integration. In such cases, it is more meaningful to include among the functions that are integrated exactly also rational functions having the same, or at least the more important, poles. It turns out that such polynomial/rational n-point quadrature formulae (that exactly integrate m rational functions, $0 < m \leq 2n$, with prescribed poles of given multiplicities and polynomials of degree $2n - m - 1$) can be constructed in terms of classical (polynomial) Gauss formulae with modified weight functions and hence can be computed by methods described earlier. This is discussed, and illustrated by a number of examples, in [**137**], and implemented in a computer algorithm in [159]. Integrals over half-infinite intervals and exact for special rational functions are considered in [**128**]. Polynomial/rational versions of other quadrature rules, specifically Gauss–Kronrod, Gauss–Turán rules, and quadrature procedures for Cauchy principal value integrals, are developed in [162] and (favorably) compared with the polynomial counterparts. For an updated summary that includes also estimates of the remainder term and an additional example, see [**167**]. Applications to Fermi–Dirac and Bose–Einstein integrals and comparisons with results in the physics literature are made in [**136**].

The theory and algorithms described in §§4–5 and §§7–8 and various applications thereof, are the subject of a monograph [B3] published in 2004 by Oxford University Press. For additional surveys, see also [65, 92, **94**, 112, 115, 117, **122**, 130, 140, 157].

9. *Linear difference equations.* As already mentioned in §2, linear homogeneous difference equations of order two (i.e., three-term recurrence relations) are an important tool for computing special functions. So are inhomogeneous difference equations of order one (for example, see [12, **26**, 27, **37**, 38, 44]). In [42, **150**], the numerical stability of initial and boundary value problems for such difference equations is discussed systematically using the concept of amplification factors. Special attention is given to a phenomenon of "pseudostability" (stability in theory, but instability in practice). Its adverse effects on computing are illustrated in [**135**] in the case of discrete orthogonal polynomials when computed by their three-term recurrence relation (cf. also [**150**, §3.4.2]).

10. *Ordinary differential equations.* I acquired an interest in this topic early on, already during my doctoral thesis work. I obtained error bounds [5] for special Runge–Kutta methods developed by Zurmühl for single differential equations of arbitrary order, following work of Bieberbach on the classical Runge–Kutta method for first-order differential equations. In 1961 I developed numerical methods based on trigonometric rather than algebraic polynomials [**14**], anticipating methods later called "exponentially fitted". Only recently, they have attracted renewed interest in the context of oscillatory second-order differential equations and also in time integration schemes for Maxwell equations in three dimensions. An early expository account of the theory of one-step and multistep methods is given in [15], which for the first time includes Dahlquist's theory of stability and convergence of linear multistep methods. Later in 1975, in the paper [**54**] dedicated to Mauro Picone, it is proposed to estimate the global error (not the local error, as is usually done) of one-step methods by integrating numerically the variational differential equation along with the main differential equation. For multistep methods, this is done in my book [B2, §6.3.5 of the 2d edition]. Asymptotic estimates are derived in [56] for certain coefficients of interest in Adams, Störmer, and Cowell multistep methods. The last paper on differential equations [**73**] appeared in 1980. Within a class of stable multistep methods it determined the method which has minimum coefficient in the asymptotic formula for the global error.

11. *Software.* Much of my work on computing special functions is supported by pieces of software, initially written in Algol and Fortran and published in separate algorithms, and later written in Matlab and placed on my homepage. I also wrote major software packages in support of my work on orthogonal polynomials and quadrature: the Fortran package `ORTHPOL` [**141**] and the

Matlab package `OPQ` [174, **179**, `http://www.cs.purdue.edu/ archives/2002/wxg /codes/OPQ.html`]. Some of the routines in the latter package have been rewritten in symbolic Matlab and collected in the package `SOPQ` [`http://www.cs.purdue.edu /archives/2002/wxg/codes/SOPQ.html`], and can therefore be run in variable-precision arithmetic. This, incidentally, provides another approach to overcome the ill-conditioning mentioned in §4: simply do the computation with as many digits as are required to compensate for the loss of accuracy caused by ill-conditioning.

12. *History and biography.* Every so often, I took time out to review the history of a subject I had been working on myself. This led to a number of special-topic surveys, for example on computational methods in special functions [**49**], advances in Chebyshev quadrature [**55**], questions of numerical condition related to polynomials [**64**], Gauss-Christoffel quadrature formulae [**74**], Gauss-Kronrod quadrature [**107**], remainder estimates for analytic functions [**129**], applications and computation of orthogonal polynomials [**146**], the incomplete gamma function since Tricomi [**155**] (written on the occasion of Tricomi's 100th anniversary of birth), and the interplay between classical analysis and numerical linear algebra [**170**], a tribute to Gene H. Golub. There are also appreciations of the work, and sometimes the life, of individual personalities, for example Yudell L. Luke [**91**] (an obituary), Philip Rabinowitz [**143**], Luigi Gatteschi [**144**, **189**], Alexander M. Ostrowski [82], [**196**] (published on the occasion of the 100th anniversary of the Swiss Mathematical Society), Joseph-Louis Lagrange [210], and, above all, Leonhard Euler [**187**]. Two of my articles deal specifically with Euler's handling of slowly convergent series [**183**] and with Euler's curious attempt [**186**] (communicated in a 1734 letter to his friend Daniel Bernoulli) to interpolate the common logarithm from the known values $\log 10^k = k$, $k = 0, 1, 2, 3, \ldots$.

13. *Miscellanea.* Additional work not easily subsumed under any of the categories above concerns a proof, under weaker assumptions, of a necessary condition of Picone in the calculus of variation [7] and an extension thereof to double integrals [6], families of algebraic test equations [**71**], the error behavior in optimal relaxation methods [78], monotonicity and complete monotonicity properties related to the successive remainder terms of the exponential series [79], an algorithm for simultaneous orthogonal transformation of several positive definite matrices to nearly diagonal form [**96**], summation procedures [**175**] for evaluating the interesting Hardy–Littlewood function $H(x) = \sum_{k=1}^{\infty} \sin(x/k)/k$, and the analytic smoothing of the discrete spiral of Theodorus [**197**].

———ooOOoo———

Apart from my election in 2001 to two prominent European Academies – the prestigious Bavarian Academy of Sciences and Humanities, founded in 1759, and the

Turin Academy of Sciences, once the Royal Academy, founded in 1761 by Lagrange and others – and the designation of SIAM Fellow in 2012, there are two highlights in my career that stand out. The first is my collaboration in 1984 with Louis de Branges on the proof of the Bieberbach conjecture [**101**]. De Branges knew that the validity of the Bieberbach conjecture for the nth coefficient hinges on the validity of a system of n inequalities involving integrals of Jacobi polynomials (in fact ${}_3F_2$ hypergeometric functions). I was able to use my software package [**141**] for orthogonal polynomials to verify computationally that these inequalities are indeed valid for all n up to 30. More importantly, I made a now famous telephone call to Richard Askey, which led to the incredible discovery that these inequalities are true not only for $n \leq 30$ but for all n, being a special case of results proved several years earlier by Askey and Gasper. That finished off de Branges's proof of the Bieberbach conjecture. The second highlight was the Euler lecture I was invited to give as part of the Euler 300th anniversary year before an audience of some 3,000 attendees of the ICIAM 2007 Congress in Zürich. An expanded version of the lecture has been published in [**187**], and a preliminary rendition thereof given, and recorded, at Purdue University; the video is made available with permission at `springer.com` (type in the ISBN of Vol. 1, 978-1-4614-7033-5, and click on `EulerLect.avi`).

4

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Walter Gautschi

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