Ulrich Kortenkamp · Birgit Brandt
Christiane Benz · Götz Krummheuer
Silke Ladel · Rose Vogel *Editors*

Early Mathematics Learning

Selected Papers of the POEM 2012 Conference

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Contents

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Part I
Introduction

Chapter 1
Introduction

Christiane Benz, Birgit Brandt, Ulrich Kortenkamp, Götz Krummheuer, Silke Ladel and Rose Vogel

This book is the result of a conference that took place from February 27 to 29 in Frankfurt am Main, Germany. Following up the Congress of the European Society for Research in Mathematics Education (CERME) conference 2011 in Rzészow, Poland, we, a group of German researchers from Frankfurt and Karlsruhe in early mathematics education, were faced with the question: In which way—and how much—should children be "educated" in mathematics before entering primary school. The European conference in Poland demonstrated that there are many opinions and research results, and the topic itself deserves further attention. We decided to organize an invitation-only workshop-conference to further investigate this question.

We wanted to address this question from a mathematics education perspective on early mathematics learning in the strain between instruction and construction. The topics of the conference included research on the design of learning opportunities, the development of mathematical thinking, the impact of the social setting and the professionalization of nursery teachers.

At the conference, we created a focused working atmosphere in the spirit of the CERME conferences, with only few paper presentations and allowing for more inter-

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action and exchange between the researchers. This book collects revised and extended versions of the conference papers, grouped in four parts that reflect major strands that emerged. These parts follow an introductory chapter by Norma Presmeg, "A dance of instruction with construction in mathematics education." Presmeg highlights in a very personal exposition the main theme of the book: The dual nature of instruction and construction, with each being necessary for the other, or as she phrases it, "Instruction and construction can mutually constitute each other in a fine-tuning awareness that I have called a dance." To us, there could not be a better description for the fundamental question of perspective on early mathematics (POEM) than this poetic one.

In the first part, the relation between instruction and construction is illuminated by case studies in different social settings. Case studies from three European countries with different curricular concepts for early (mathematics) education and with different institutional embedding of the interaction processes between children and adults are gathered in this part. Although the studies use different methodological approaches and theoretical backgrounds, they all use videotape as database and focus on situational aspects and different roles of the participants within the interactions.

Three of these contributions are concerned with interaction processes in institutional settings and particularly deal with the role of the teacher. Sayers and Barber examine one experienced teacher's practice in relation to the centrally imposed English mathematics curriculum. Within the framework of pedagogical content knowledge (Shulman 1986)[1], they focus on pedagogical issues related to the use of manipulatives and language emphasized when teaching place value to young children in a whole class interaction. Similarly, the contribution of Lange, Meaney, Riesbeck and Wernberg is concerned with one teacher's practice in relation to the national curriculum for early education. Using Anghileri's (2006) model of scaffolding, this case study focuses on how one teacher in a Swedish preschool recognizes and builds on mathematical teaching moments that arise from children's play with glass jars in a guided play set up by the kindergarten teacher. Brandt also discusses the acting of kindergarten teachers in guided play situations. Within the framework of folk pedagogy (Bruner 1996), she examines the pedagogical ideas of three kindergarten teachers arranging learning opportunities in the mathematical domain patterning and describes three basically different instruction models. The two last contributions of this part leave the institutional embedding of the interaction processes in preschool or kindergarten. Krummheuer examines the interface between cultural expectation and local realization in the social context of encounters that "serve" as mathematical learning opportunities for children. For this reason, he analyses peer interactions guided by an adult and a play situation in a family. Tracing back Super and Harkness (1986), he employs the concept of the "developmental niche in the development of mathematical thinking". This concept is adopted by Acar-Baraktar for her investigation of the interaction processes between children and adults and the reconstruction of mathematics learning in the familial context. In her contribution, she carries out the support system of a German–Turkish family in the domain of spatial thinking while playing a rule-based game.

[1] See the respective chapters for full references.

Thus, the cultural impact of early mathematical teaching and learning and the "dance of instruction and construction" become apparent through these case studies.

In Part 2, the focus will be on children's constructions. By investigating children's constructions, the learner's perspective will be focused on. The different insights in children's constructions can help to provide a basis for instruction in terms of realizing and using learning opportunities and creating learning environments within mathematical early childhood education. So the "dance between construction and instruction" also underlies this part, although the aspect of construction will be more exposed.

In the introductory chapter, Bert van Oers highlights the aspect of construction, using the term of "productive mathematising" in contrast to mathematical activities with a re-productive aspect. Based on the cultural–historical activity theory, van Oers defends the position that "the activity of mathematising basically is a form of playful mathematics, embedded in young children's play". Regarding different aspects of play as a form of productive mathematising, both the aspect of children's construction and the aspect of instruction will be presented.

The productive and creative aspect in children's constructions is also analyzed by Melanie Münz. In her chapter, she focuses especially on the aspect of mathematical creativity. Mathematical creative ideas, which emerge in the interaction between children and the accompanying person, are illustrated. By including the analyses of the interaction of the accompanying person, the aspect of mathematical instruction is also mentioned.

A special expression of children's constructions are gestures. In the chapter by Melanie Huth, the interplay between gestures and speech used by second graders is illustrated while they are occupied with a geometrical problem in pairs. Constructions regarding the geometrical domain are also the subject of the chapter of Andrea Maier and Christiane Benz. Here, children of two educational settings, England and Germany, were interviewed. In addition to insights in children's geometrical competencies concerning shapes, hypotheses are formulated how the introduction of shapes might additionally influence the concept formation of the children. Another discernment of individual's constructions is given in the chapter of Christiane Benz. Here, different processes by recognizing or perceiving collections of objects and by identifying quantities of collections are investigated. On the basis of insights about children's constructions, different conclusions for instruction in early mathematics education will be drawn.

The third part concentrates on tools and interactions. There are three chapters in this part that all have a view on learning as a socio-cultural process. Therein, instructions are given as orders from the teachers to the students. The role of the teacher is to orchestrate mathematical learning opportunities for the children. The child has to construct the mathematical meaning. The artefacts determine this construction. They mediate between the subject, the child, and the object, the mathematical content. In the first and third chapter, the artefact is a digital tool. They focus on the use of technology to support mathematical teaching and learning. In the second chapter, mathematical conversation situations, impulses of the guiding adult, and of the materials are used as the starting point of the interaction.

Martin Carlsen, Ingvald Erfjord and Per Sigurd Hundeland analyse the children's engagement with mathematics in kindergarten mediated by the use of interactive whiteboards. In the research project, they survey in what ways digital tools may nurture children's appropriation processes relative to mathematics. In particular, they focus on the use of a digital pair of scales in kindergarten for comparison of weights. In the long-term study 'early Steps in Mathematics Learning' (erStMaL), Rose Vogel explores mathematical situations of play and exploration as an empirical research instrument. An especially developed description grid in the form of "design patterns of mathematical situations" achieves a comparability of the situations.

Silke Ladel and Ulrich Kortenkamp use information and communication technology (ICT), in particular multi-touch technology, to survey and to enhance the development of children's concepts of numbers. A special focus lies on the processes of internalization and externalization that constitute the construction of meaning. Also the instructions given by the (nursery) teacher as well as the partners have an influence on the child's internalization and externalization. As a basis for the design and analysis this research project refers to Artefact-Centric Activity Theory (ACAT).

In Part 4, "interventions" are presented that integrate both the principle of instruction and the principle of construction into processes of early mathematical education. Jie-Qi Chen and Jennifer McCray describe a yearlong training program for preschool teachers. The program "Early Mathematics Education" was launched in 2007. Starting with mathematical "Big Ideas", preschool teachers shall be enabled to understand children in their mathematical thinking and support children to build up mathematical knowledge. A variety of teaching strategies were developed in the program to encourage the preschool teachers.

Hedwig Gasteiger presents a professionalization program of early childhood educators as part of the project "TransKiGs Berlin". Early childhood educators are enabled to support the individual mathematical learning of children, here particularly in everyday learning situations. The further education program includes three modules in the domains number/counting/quantity, space and shape, and measurement and data. The fourth module is concerned with methodological components like observation, documentation and intervention measures.

Pessia Tsamir, Dina Tirosh, Esther Levenson, Michal Tabach and Ruthi Barkai examine 36 practising preschool teachers with regard to their mathematical knowledge and their self-efficacy. Based on the results of their study, they develop professional courses for preschool teachers. One important aspect of the program is to discuss with the teachers the different aims of mathematical tasks. In addition, the authors assume that the teacher's own learning experiences are important in supporting the children's learning.

All training programs presented focus on the development of mathematical and special methodological knowledge to support the learning of mathematics in early education.

Andrea Peter Koop and Meike Grüßing focus on the children themselves. Their study examines 5-year-old preschoolers. By different methods of testing, they identify 73 children out of 947 that are "potentially at risk learning school mathematics". The 73 children are split in two groups, which are promoted with different programs in prior to school entry. The results of the study are examined in detail, with a special interest in children with a migrant background.

We hope that this book will be able to carry over not only the results of the conference, but also its spirit and atmosphere to a broader audience. May we ask you for the next dance?

Chapter 2
A Dance of Instruction with Construction in Mathematics Education

Norma Presmeg

Setting the Scene

Our field, our baby field that is brand new in comparison with the millennia for which mathematics has existed as a discipline, has seen some dramatic changes in its half-century of being a field in its own right, with its own journals and conferences. We have come a long way, even since the early 1980s, when "illuminative evaluation" (McCormick 1982) was slowly replacing or, initially at least, supplementing the psychometric experiments that used "subjects" (*people*) who were being taught mathematics. Before that period, in the old paradigm, no research that did not aim for objectivity by means of carefully controlled experiments and statistical analysis was considered scientific in our field. In connection with the research methods of this period, Krutetskii (1976) gave a pungent critique:

> It is hard to understand how theory or practice can be enriched by, for instance, the research of Kennedy, who computed, for 130 mathematically gifted adolescents, their scores on different kinds of test and studied the correlations between them, finding that in some cases it was significant and in others not. The process of solution did not interest the investigator. But what rich material could be provided by the process of mathematical thinking in 130 mathematically able adolescents! (p. 14)

Krutetskii's interview methods, in Soviet Russia, were in many ways a precursor to the qualitative methodologies that followed this early period. Slowly, the qualitative research paradigm gained credence. After all, we are dealing with human beings in their teaching and learning of mathematics, with all the complexities and uncertainties that that fact implies! Even Krutetskii (1976), aware as he was of individual differences, wrote of "perfect teaching methods" (p. 6), terminology that we might use more circumspectly today. With regard to useful and believable research (rather than reliable and valid experiments), initial crude attempts at quality control became strengthened. Thus, *triangulation* of various types (Stake 1995) was needed to ensure that research results and insights reported more than merely the researcher's opinions. We learned to go back and ask the mathematics teachers and their stu-

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dents whether they agreed with the results of our observations and interviews, in "member checks" that were a means of respondent validation. By the 1990s, such qualitative research was the prominent methodology, and it was in this climate that radical constructivism became the dominant theoretical framework for research in our field. Radical constructivism was salutary in its critique of the behaviorism that had preceded it. And this theoretical precedence leads me to the topic of this talk.

Construction and Instruction

I remember, in the early 1990s sitting on a stone seat in the garden of The Florida State University with Ernst von Glasersfeld and asking him about the status of *conventional* knowledge in mathematics education according to radical constructivism. It seemed obvious that attempts by teachers to give their students space to construct their ideas of mathematics in more personal ways (e.g., by discussion in groups) would lead to a kind of knowledge that could be more *meaningful* to learners in terms of their mathematical identities and ownership. It is *not* that some kinds of instruction lead to construction and others do not. What other ways of "appropriation of knowledge" (van Oers 2002) do we have than by construction? We are constructing even in the choice of what we make of a straightforward lecture as we sit and listen. We may listen, but what do we *hear*? It was concerns such as these, in part, that caused debates on whether or not radical constructivism was epistemological, and whether or not it made claims about the ontology of mathematical knowledge. Nell Noddings, in the 1990s, called it "post-epistemological" (Janvier 1996).

But to return to my conversation with Ernst von Glasersfeld in the garden, Ernst acknowledged that there are different kinds of knowledge, and that knowledge of conventions had a different status, belonging as it does to accidents of cultural historicity rather than to the logic of rational thinking. Even the ability to use conventional knowledge would entail construction by an individual; but telling by somebody who knows the convention (aurally or in written form) is required, simply because there is no logical necessity for this kind of knowledge, except perhaps in a historical sense. Why, for instance, do we have 360° in a complete revolution? 100 degrees would be much more convenient. Reporting on some of his work with Les Steffe, in one of his many publications during this period, von Glasersfeld (1994) gave a short synopsis of the radical constructivist position concerning early mathematics concepts such as number; and early mathematical learning is of particular relevance in this conference, although it is clear that mathematics learning between the poles of instruction and construction is an important topic at all levels.

The founders of theoretical edifices, such as von Glasersfeld, are thus aware of the contingencies and intricacies inherent in building theories. But Peirce (1992) had insight into what happens to such theories over time. He cast light on what he meant by continuity in his *law of mind*:

> Logical analysis applied to mental phenomena shows that there is but one law of mind, namely, that ideas tend to spread continuously and to affect certain others which stand to them in a peculiar relation of affectability. In this spreading they lose intensity, and especially the power of affecting others, but gain generality and become welded with other ideas. (Peirce 1992, p. 313)

Some followers of radical constructivism took the theory to be a prescription for instruction. The mantra became, "Teachers mustn't tell!" (I have an anecdote about a professor and her primary school mathematics education prospective teachers, who just smiled and moved on when her students decided in groups that doubling the length of a particular similar figure must, automatically, double the area.) It is to the credit of deep scholars in our field, such as Paul Cobb and Erna Yackel (e.g., Yackel and Cobb 1996) that they recognized even in the heyday of radical constructivism, that instruction has an indispensable role, and that there is a delicate blending of instruction and construction that is a fine-tuning of the teacher's craft. It is this blending that I am calling the *dance* of instruction with construction.

In an email conversation with Götz Krummheuer, it emerged that when we considered the metaphor of the dance in this regard, we were viewing different aspects of dance that had relevance. He was interested in the swirling motion as the dancers moved—and certainly there is movement if we are considering teachers and their pupils in interaction in a dynamic way that leads to deep contemplation of mathematical ideas and changes in cognition, ideally also with a positive affective component. I had been thinking more of dance involving canonical moves by people in interaction—although both aspects are relevant to instruction and construction in mathematics education. Within the set moves of a particular dance there is freedom, creativity, and vigor. Certainly, a dancer can decide to construct a different set of movements, and they may be harmonious and beautiful, but if they are too far from the set moves, that dancer cannot be considered to be doing that particular dance. As is the case with all metaphors, there are elements in which the source domain (in this case dance) resonates with the target domain (mathematics education), and this common structure constitutes the *ground* of the metaphor. But every metaphor also involves ways in which the source and target domains are different, and these constitute the *tension* of the metaphor (Presmeg 1997). The dance metaphor does not take into account that there is a knowledge differential between teachers and their students who are learning mathematics. Teachers know the conventions of reasoning and representation that are involved in the patterns of mathematical thinking: Students initially may not have this awareness. There is also thus a power differential involved. However, effective instruction can facilitate students' making of constructions that lie within the canons of mathematically accepted knowledge, and yet there is room for creativity and enjoyment. I present two examples of such instruction in the next sections.

An Example of the Dance

As an example of an effective dance, I would like to highlight the doctoral research of Andrejs Dunkels (1996) in Luleå in the north of Sweden, in the mid-1990s. But for the untimely and tragic death of Andrejs, it is likely that he would have been the very first mathematics *education* professor in Sweden, who was appointed at the University of Luleå in 2001. After establishing his credibility as a mathematician with publications in pure mathematics (which was a necessity in that academic climate), Andrejs set out to teach his section of an engineering calculus course in a way that was very different from the traditional lecture format. Of the 5 or 6 sections of the course, with students arranged in the sections according to their previous accomplishments, Andrejs chose a section for his research that was just one up from the bottom in the hierarchy (i.e., many of these students had experienced difficulty in mathematics courses previously). He collected baseline data, so that he could compare these data with the achievements of his class at the end of the course, using exploratory data analysis (EDA) as well as observations and interviews. Thus, the research design used mixed methods (quantitative and qualitative), prefiguring a balanced swing of the pendulum to methodologies that became more common in the 2000s.

How did Andrejs teach his class? Firstly, he arranged them in groups of four for ease of communication. Secondly, he told them in advance what would be the mathematical topic of a particular class session, and he expected them to read and try to make sense of the relevant material in the textbook of the course. Thirdly, they were expected to come to the session prepared to talk about their current constructions. Finally, in the session, he circulated among the groups, listened to their conversations, and answered their questions although not always directly; he sometimes answered a question by posing another question. He sometimes pointed the group in directions they had not considered—with suggestions, not as the all-knowing teacher, and without taking away their ownership and agency. He had instinctively mastered the difficult and delicate dance of instruction with construction.

At the end of the course, the statistical EDA revealed that his students had improved their accomplishments so significantly that their section was now almost at the top of the hierarchy, second only to one other section. But even more convincingly, the analysis of data from interviews with students showed that the *quality* of the mathematical knowledge the students had constructed had improved immeasurably. There was no longer memorization of *rules without reasons*; they knew *why* the rules worked, and above all, they experienced greater enjoyment of the mathematical content, and more self-confidence than previously. This doctoral research study thus provided convincing evidence, both quantitative and qualitative, of the efficacy of balancing instruction with construction in mathematics education.

The Purported "epistemological paradox"

An issue that is relevant at this point is the oft-quoted paradox of instruction and construction (e.g., Simon 1995) that students can actively work only with what they have *already* constructed: How then is new knowledge possible? I shall argue shortly that there really is no paradox; the seeming paradox hinges on a false dichotomy. However, let me first give an example of a related phenomenon from my own research on ethnomathematics. I asked students in a masters-level course in mathematics education to take a personally meaningful cultural activity, and to construct mathematics from it. I gave examples from ethnomathematics literature and my own experiences to show them how to use several steps of semiotic chaining (Presmeg 2006a) to build connections between a cultural activity and mathematical ideas suitable for teaching at some level in a mathematics classroom. The process is akin in many ways to the horizontal mathematization, followed by vertical mathematization, used by the Freudenthal group (e.g., Treffers 1993; Gravemeijer 1994) in *Realistic Mathematics Education* (RME). The students in my course took ownership of the project, and the activities they chose were diverse and personally meaningful to them. However, it was evident that the mathematical ideas that students recognized in their chosen cultural activities depended heavily on what mathematics they already knew. For example, Vivienne, a primary school teacher, did not recognize the hyperbola that resulted when she analyzed the gear ratios and distances traveled by her mountain bicycle: Vivienne called the graph "a nice curve." In contrast, David constructed a "dihedral group of order 4" when he analyzed the symmetries of a tennis court: He was a teacher of college-level number theory. And in the data there were many more examples of this phenomenon. How then might teachers use the connections of horizontal mathematization to facilitate students' construction of *new* mathematical ideas? This question might be particularly vexing for a teacher who feels under pressure to 'cover' the topics listed in a mathematics syllabus.

I can do no more here (the topic has been addressed in several papers or book chapters, e.g., Presmeg 1998, 2007) than to report that the ethnomathematics course had the effect of broadening participants' beliefs about the *nature of mathematics*, which was no longer seen as a "bunch of rules to be memorized" (initial student characterization of what mathematics is), with or without understanding. Many students expressed in reflective journals that after the course they saw mathematics as inherent in patterns and regularities that they could identify also in their daily lives and activities. This change of beliefs prefigures what Tony Brown (2011) is accomplishing in his "weekly session centred on broadening the students' perceptions of mathematics and of how mathematics might be taught" (p. 18). Brown does not use the conceptual framework of semiotics, but the contemporary theoretical lenses of Zizek and Badiou, in his work, but the aim of his teaching resonates with a dance of instruction with construction.

To return to the so-called *learning paradox,* as I hinted, there really is no paradox at all if mathematics education is reconceptualized as a dance of construction with instruction. The crux of the matter is the relationship between the constructions made by an individual, and the broader societal context, the culture in which established mathematical ideas reside: These might be characterized as Karl Popper's (1974, 1983) worlds 2 and 3, respectively. Radford (2012) has trenchantly pointed out that the seeming dilemma results from what he calls the "antinomies" in epistemological views that we have accepted: "Unfortunately, we have become used to thinking that either students construct their own knowledge or knowledge is imposed upon them" (p. 4). As he points out, this conception is a misleading oversimplification. Radford poses the paradox in terms of *emancipation* in mathematics education rather than in terms of construction, but the ideas are relevant to both. He points out that the antinomies reside in two epistemological ideas: "First, knowledge is something that subjects *make*. Second, the making of knowledge must be carried out free from authority" (p. 102, italics in original). What is problematic is the relationship between freedom and truth. Radford points out convincingly that the paradox results from "a subjectivist view of the world espoused by modernity (a world thought of as made and known through the individual's deeds) and the cultural regimes of reason and truth that precede the individual's own activity" (p. 104). Although Radford does not cast it in these terms, it is the mistaken notion that Popper's worlds 2 and 3 are colliding. But all individual constructions (world 2) are made in the *context* of a cultural milieu (world 3). This relationship is inescapable. Seen in this light, the paradox disappears, and this relationship has its practical manifestation in a delicate blending of freedom and truth, a dance of instruction with construction. It is not necessary for the teacher's role to conform to an irreducible and contradictory dichotomy of "the sage on the stage" versus "the guide on the side," because elements of both these metaphors are evident in the dance, as the following example illustrates.

Blending Popper's Worlds in the Teaching of Trigonometry

I would like to present here an instance of teaching high school trigonometry that uses the dance of instruction and construction to the fullest, thereby—at least in some measure—resolving the apparent paradox suggested in the previous section.

Sue Brown (2005) carried out a powerful dissertation study in which she analyzed high school students' understanding of connections among trigonometric definitions (particularly of sine and cosine) that move from right triangles to the coordinate plane and unit circle, and then to definitions that establish sine and cosine as functions. Following this research (which involved quantitative as well as qualitative methods), she and I set out to examine further, pedagogy that might facilitate the students' constructions of such connections in trigonometry. In this postdoctoral phase, I served as the researcher in Sue's trigonometry class in the spring of 2006, in Chicago, USA. The research question was as follows: *How may teaching facilitate students' construction of connections among registers in learning the basic concepts of trigonometry?* The main goal in Sue's trigonometry class was to foster skill in converting among signs as students build up comprehensive knowledge of trigonometry concepts.

The methodology of this teaching experiment included cycles of joint reflection based on interviews with students, followed by further teaching. Early in our collaboration, Sue listed ways in which she tried to facilitate connected knowledge in her class—actions that were confirmed in my observations of her lessons, and in documents such as tests and quizzes. In the analysis of data, her list was compared with the connections constructed—or the lack of connections—by four students in a series of six interviews conducted with each student at intervals during the semester. The four students were purposively chosen by the teacher in collaboration with the researcher to ensure a range of learning styles and proficiency.

Some of Sue's facilitative principles that have the intent of helping students to move freely and flexibly among trigonometric registers are summarized as follows:

- Connecting old knowledge with new, starting with the "big ideas," providing contexts that demand the use of trigonometry, allowing ample time, and moving into complexity slowly
- Connecting visual and nonvisual registers, e.g., numerical, algebraic, and graphical signs, and requiring or encouraging students to make these connections in their classwork, homework, tests, and quizzes
- Supplementing problems with templates that make it easy for students to draw and use a sketch, or asking students to interpret diagrams that are given
- Providing contextual ("real world") signs that have an iconic relationship with trigonometric principles, e.g., a model of a boom crane that rotates through an angle θ, $0^o < \theta < 180°$, on a half plane
- Providing memorable summaries in diagram form, which have the potential of becoming for the students prototypical images of trigonometric objects, because these inscriptions are sign vehicles for these objects
- Providing or requiring students to construct static or dynamic computer simulations of trigonometric principles and their connections, in many cases giving a sense of physical motion; and
- Using metaphors that are sometimes based on the students' contextual experiences, e.g., a bow tie and the boom crane, for trigonometric ratios in the unit circle.

An analysis of the complete corpus of data in terms of Sue's full list (abridged here) assessed the effectiveness of these principles in accomplishing their goal, at least for the four students who were interviewed (Presmeg 2006b). On the surface, Sue's list appears to relate to the *instruction* pole of the dance; however, it was her long experience of students' *constructions*—informed also by her intensive doctoral research—that formed the foundation for her principles of instruction in this list. And many instances were present of ways that Sue incorporated idiosyncratic constructions of students in her teaching. An example of this inclusion is the bow tie metaphor, which was introduced in class by Sue, but originated in interviews with students in a task in which they were finding the sine of angles in the second and third quadrants. Sue's pedagogy provides an illustration of principles that alternate flexibly and sensitively between instruction and construction in learning trigonometry.

Some Conclusions

In this introduction to the topic of *a mathematics education Perspective On Early Mathematics learning between the poles of instruction and construction (POEM)*, I have introduced a brief overview of the way our field has moved from a behaviorist emphasis on instruction, to an opposite concern with pupils' constructions, and further to the realization that instruction and construction can mutually constitute each other in a fine-tuning awareness that I have called a dance. Other writers have used different terminology, although the ideas resonate with the notions of construction and instruction: Hewitt (2012) makes the distinction between arbitrary and necessary knowledge, which he characterizes as knowledge that has the function of assisting memory and knowledge that is necessary in educating awareness of the accepted canons of a discipline, respectively. In any case, learning mathematics involves not only becoming aware of conventions and standards of the mathematics that has been accepted as such through the ages but also making sense of the logic of these canons in a personally and individually meaningful way.

I tried to initiate conversations on the topic with reference to two examples: one in a university-level calculus class and the other in a high school trigonometry class. I look forward to examples our colleagues will present in early childhood teaching and learning of mathematics. But I hope the cases presented here exemplify my belief that the topic is important at all levels of learning mathematics and that attention to this topic is required at both theoretical and empirical levels, the former, for example, with regard to the so-called paradoxes of our field and the latter in the day-to-day lives of teachers and students.

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Part II
Case Studies

Chapter 3
It is quite confusing isn't it?

Judy Sayers and Patti Barber

Implications of a national policy on mathematics teaching on the dance of instruction and construction of knowledge in the teaching of place value through manipulatives.

Introduction

Firstly, it is important to distinguish between two different constructs of place value; namely, the 'quantity' value aspect and 'column' value aspect. Thompson (2009) informs us that using manipulatives, such as base ten apparatus, reinforces the 'column' aspect of place value, while emphases on partitioning reinforce the quantity aspect. However, there are important questions about whether we should be teaching column value to young children at all, for it is not a necessary prerequisite for early calculation, whereas an understanding of quantity value is (Thompson 2009).

In this chapter, we discuss how Jane, an experienced teacher with good mathematics subject knowledge and considered locally to be effective, presented the topic of place value to 5–6-year-old children. We wanted to examine the various resources, including manipulatives, she used and the specific language she privileged in her support of her young children's mathematical thinking. In so doing, we wished to understand how her practice was constructed by the curriculum discourse imposed centrally on teachers in England. The depth of case study colleagues' subject knowledge was an important defining characteristic, not least because it is an essential prerequisite for good mathematics teaching (Rowland and Ruthven 2011). In particular, deep pedagogical subject knowledge (Shulman 1986) is needed to understand the developmental stages of number sense (Howell and Kemp 2005) foundational to place value and its relationship to quantity. Thus, the dance between the instruction and construction of knowledge could be identified through the case.

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The Context

In England, since the introduction of a national curriculum in 1987, there has been substantial change in the teaching of mathematics, culminating in the launch of the National Numeracy Strategy and the adoption by primary schools of the ‘daily mathematics lesson’ in 1999. There was much debate over many years prior to its introduction about the teaching and learning of mathematics, but this was the first shift towards a public education emphasis which decreed equal treatment for all students and all teachers (Brown 2010). It was also the first time a pedagogical prescription was introduced, which emphasised whole class, or direct, teaching, and specified modes of instruction. However, the research base for the innovation was weak (Brown et al. 1998) and, in the years that have followed, further government-sponsored pedagogical intervention has been introduced to the extent that teachers no longer appear to trust their own judgement, believing themselves compelled to follow their interpretations of the national strategy.

Theoretical Background

A long, and still current, debate in education questions whether a constructivist approach is sufficient in teaching and learning, for example, Kirschner et al. (2006) question the efficacy of all constructivist approaches to learning, whether discovery, experiential, problem-based and inquiry-based teaching. Tobias and Duffy (2009) challenge this view. Drawing on the research foundations of Vygotsky (1978), Piaget (1952) and Bruner (1966), and more recent perspectives such as situated learning (Resnick 1987) and its derivative, communities of practice (Lave and Wenger 1991), they suggest that there are many, not unrelated, characteristics of constructivist approaches to learning.

Recent research into early years practice in England, for example, the Effective Practice of Preschool Education (EPPE) project (Siraj-Baltchford 2002), refers to the continued work of Weikart’s (2000) model, as a typology of commonly applied ‘early childhood education curriculum models’ where high teacher initiative is described in terms of the highly structured pedagogy and high child initiative in terms of the learner’s control of the curriculum. The model is based on two continua reflecting the interactions of both teacher and child perspectives in the ownership of the learning trajectory, implying there should be a balance between the two, where an instructional orientation can interplay, or dance, with a constructivist approach (Siraj-Baltchford 2002).

What the early years literature appears to suggest is that effective early childhood pedagogy must still be ‘instructive’, but should be interpreted as incorporating all of those processes that occur within the classroom that aim to initiate or maintain learning processes, and to be effective means to achieve educational goals (Creemers 1994). This does not mean the rejection of a constructivist approach, on the contrary. According to the English system education guidance for teachers

(DfES, 1999), effective teaching encompasses direct teaching that makes effective use of unexpected and unforeseen opportunities for children's learning. This will include both instructive and constructive approaches to learning. When considering the guidance, the message to teachers in England is clear, there are structured opportunities to be identified and planned which should provide a balance of approach between instruction and construction in order to be an effective practitioner.

In general, primary teachers in England do not study early years educational philosophy in their training; they receive a general training delivered through the curriculum subjects and a national strategy. However, recent independent reviews of primary practice in England have shown how the national strategy has failed to provide an appropriately meaningful pedagogy for early years teaching (Williams 2008), in its offering just a series of notes that implicitly adopt a broadly constructivist approach to teaching mathematics. The Cambridge Primary Review (2009) also reported on the need for a proper debate about primary education, in particular research-based approaches to how mathematics should be presented to children for deep learning experiences.

A significant issue of the national strategy model of planning is that 'teaching starts to be assumed to be the reality of learning. Children are not assessed on what they have learned, but on whether they have learned very specific objectives. Rather than the attained curriculum—in the sense of what children actually learn—being a guide to help shape further teaching it has become a tick list' (Askew 2011, p. 23). Furthermore, Aubrey et al. (2006) reported how the national strategy advantages some pupils more than others, with low attainers in particular, being least advantaged. Although primary mathematics teaching was presented as informed by constructivist, in reality it appeared to have become an impoverished list of things for the teacher to do.

Importantly, Askew et al. (2002) also found that if the emphasised mental mathematical images suggested in the guidance did not fit with those predetermined for the lesson then these were, at best, judged by teachers as not so relevant, rather than being a resource for the class to discuss and build upon, and were often ignored. Askew and Brown (2004) later confirmed that informed interpretation of the given objectives, and a move to more strategic ways of working, were challenging for teachers to understand and implement. Although professional development training was offered by the national strategy department, different interpretations were conceived by those training and by those being trained.

Knowledge for Teaching

The understanding of subject knowledge necessary for teaching mathematics is not disputed; effectiveness of mathematics teaching is not only due to the depth of a teacher's knowledge but also to how explicit connections are made within the subject (Askew et al. 1997). Importantly, teachers should have a deep knowledge of mathematics at the level they were teaching rather than having knowledge of advanced mathematics' (Ma 1999, p. 120).

Furthermore, Ball et al.'s (2001) view is that not only should mathematical conceptual knowledge be revisited but also pre-service teachers may need to unlearn what they know about teaching and learning of mathematics. Indeed, case studies, such as Goulding et al. (2002), report that an early years specialist with good subject knowledge does not guarantee successful teaching of mathematics. Their case of Frances revealed that a lack in confidence was also an issue, for although she knew the theory behind the teaching of subtraction, due to problems with her management, she resorted to time-filling activities.

In 1986, Lee Shulman et al. introduced different kinds of knowledge, in particular 'pedagogical content knowledge'. This term called attention to a special kind of teacher knowledge that links content and pedagogy. Ball and Bass (2000) describe how pedagogical content knowledge characterises the representations of particular topics and how children tend to interpret them. Children will often have difficulties with mathematical ideas or procedures and so teachers will need a unique subject-specific body of knowledge highlighting the need for close interweaving of subject matter and pedagogy in teaching (Ball and Bass 2000).

Teaching Place Value

Place value is taken to mean the value assigned to a digit according to its position in a number, e.g. 2 represents 2 units in the number 42, 2 tens in the number 125 and 2 hundreds in the number 274. Teaching place value has been a part of the mathematics curriculum since the introduction of a numeracy strategy (Department for Education and Employment (DfEE) 1999). With respect to early mathematics, it has been emphasised through extensive work on partitioning and recombining two or more digit numbers. Young children, who are only just developing an early understanding of quantity value while working with two-digit numbers, are expected to recognise a more formal perspective on these numbers, in order to prepare them for the formal written method.

English teachers are encouraged to support young children's mathematical thinking by taking a constructivist approach and provide a range of manipulatives that will model the concepts under scrutiny and support mathematical thinking, but little guidance on why this might be done. Documentation available encourages teachers to use a range of models and images to support the teaching of mathematics to young children. What they appear not to have achieved is the provision of appropriate pedagogical content knowledge and guidance as to how best to teach the contradictory elements of place value. In England, young children are expected to develop and refine counting skills, which are then abandoned in favour of a completely different approach based on place value through partitioning the tens from the units. Research contradicts this perspective (Sugarman 2007; Thompson 2000; Beishuizen 2004), in that young children need to bridge this understanding of quantity value and column value, but much later in their development, when they are fluent in their understanding of quantities (Beishuizen 2004).