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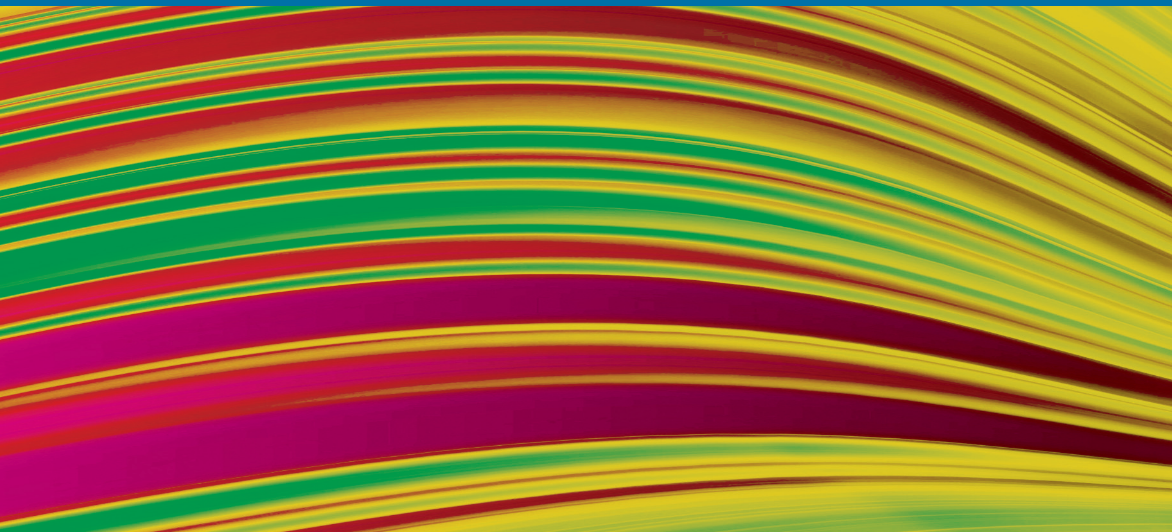

Quaternion Fourier Transforms for Signal and Image Processing

Todd A. Ell, Nicolas Le Bihan
and Stephen J. Sangwine

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Contents

Nomenclature

Symbol	Meaning	Section	Equation
$\mathbb{R}$	Set of real numbers	section 1.1	
$\mathbb{C}$	Set of complex numbers	section 1.1	
$\mathbb{H}$	Set of quaternions	section 1.1	
$\mathbf{V}(\mathbb{H})$	Set of pure quaternions	section 1.1	
$\mathbb{C}_{\boldsymbol{\mu}}$	Subfield of $\mathbb{H}$	section 1.6	
I	Complex root of -1	(See note below.)	
$\Re(.)$	Real part	section 1.1	
$\Im(.)$	Imaginary part	section 1.1	
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	Quaternion basis elements	section 1.1	[1.2]
$\boldsymbol{\mu}$	Pure quaternion	section 1.1	[1.2]
$\Im_{\boldsymbol{i}}$	$\boldsymbol{i}$-imaginary part	section 1.1	[1.4]
$\Im_{\boldsymbol{j}}$	$\boldsymbol{j}$-imaginary part	section 1.1	[1.4]
$\Im_{\boldsymbol{k}}$	$\boldsymbol{k}$-imaginary part	section 1.1	[1.4]
$\mathcal{M}_{\mathbb{R}}(.)$	Real 4×4 matrix representation	section 1.4	[1.69]
$\mathcal{M}_{\mathbb{C}}(.)$	Complex 2×2 matrix representation	section 1.4	[1.70]
$[\![.]\!]$	Alternate real 4×4 matrix representation	section 2.3	[2.20]
$[.]$	Real 4×1 vector representation	section 2.3	[2.20]
$\mathrm{S}(.)$	Scalar part	section 1.1	[1.3]
$\mathbf{V}(.)$	Vector part	section 1.1	[1.3]
(q_1, q_2)	$\mathbb{C}_{\boldsymbol{j}}$-pair notation of q	section 1.4	
$\widehat{\mathcal{A}\mathcal{B}}$	Arc of great circle between $\mathcal{A}$ and $\mathcal{B}$	section 2.2	
$\overline{q}$	Quaternion conjugate	section 1.2	[1.18]
$z^{\star}$	Complex conjugate	section 1.4	[1.54]
$\overline{q}^{\boldsymbol{\mu}}$	Involution	section 1.2	[1.23]
q^{-1}	Inverse	section 1.2	[1.26]
$\langle p, q \rangle$	Inner product	section 1.2	[1.10]
$\boldsymbol{\mu} \times \boldsymbol{\eta}$	Cross product	section 1.2	[1.9]
$\boldsymbol{\mu} \perp \boldsymbol{\eta}$	Orthogonality	section 1.2	

$q \odot p$	Bicomplex product	section 1.4	[1.65]
$\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$	Canonical basis in $\mathbb{H}$	section 1.1	
$\|q\|$	Norm	section 1.2	[1.14]
$\lvert q \rvert$	Modulus	section 1.2	[1.16]
$\tilde{q}$	Unit quaternion $\lvert\tilde{q}\rvert = 1$	section 1.2	[1.30]
$f * g$	Convolution	section 3	[3.1]
$f \star g$	Correlation	section 3	[3.1]
$\mathcal{F}\{.\}$	Fourier transform	section 3	[3.1]
$\mathcal{P}_{\boldsymbol{\eta}}[.]$	Reflection operator	section 2.1	[2.1]
$\mathcal{R}_q[.]$	Rotation operator	section 2.1	[2.3]
$\mathcal{S}_{\alpha,\beta,\boldsymbol{\mu}}[.]$	Shear operator	section 2.1	[2.9]
$\mathcal{D}_{\boldsymbol{\mu},\alpha}[.]$	Dilation operator	section 2.1	[2.10]
$L^1(\mathbb{G}; \mathbb{K})$	Space of absolutely integrable $\mathbb{K}$-valued functions taking arguments in $\mathbb{G}$	section 3.1	
$L^2(\mathbb{G}; \mathbb{K})$	Space of square integrable $\mathbb{K}$-valued functions taking arguments in $\mathbb{G}$	section 4.3	
$\mathrm{sgn}(.)$	Signum function	section 4.3	[4.31]
$p.v.(.)$	Principal value of an integral	section 4.3	[4.32]

NOTE.– The complex root of -1 which is usually denoted i, or, in engineering texts j, is denoted throughout this book by a capital letter I, in order to avoid any confusion with the first of the three quaternion roots of -1, all three of which are denoted throughout in bold font like this: $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$.

Preface

This book aims to present the state of the art, together with the most recent research results in the use of quaternion Fourier transforms (QFTs) for the processing of color images and complex-valued signals. It is based on the work of the authors in this area since the 1990s and presents the mathematical concepts, computational issues and some applications to signals and images. The book, together with the MATLAB® toolbox developed by the authors, [SAN 13b] allows the readers to make use of the presented concepts and experiment with them in practice through the examples provided.

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Introduction

This book covers a topic that combines two branches of mathematical theory to provide practical tools for the analysis and processing of signals (or images) with three- or four-dimensional samples (or pixels). The two branches of mathematics are not recent developments, but their combination has occurred only within the last 25–30 years, and mostly since just before the millennium.

I.1. Fourier analysis

Fourier analysis was, in 1822, with Joseph Fourier's development of techniques, the first to analyze mathematical functions into sinusoidal components. In signal and image processing, Fourier's ideas underpin the two fundamental representations of a signal: one in the *time (or image) domain* where the signal (or image) is represented by samples (or pixels) with amplitudes and the other in the *frequency domain* where the signal (or image) is represented by sinusoidal frequency components, each with an amplitude and a phase. Mathematically, these concepts are not limited to time and frequency: one can use Fourier analysis on a function of any variable, resulting in a representation in terms of sinusoidal functions of that variable. However, this book is concerned with signal and image processing, and we will therefore use the terms *time* and *frequency* rather than more general concepts. It should be understood throughout that when we talk of images, the concept of time is replaced by the two spatial coordinates that define pixel position within an image.

Today, Fourier analysis is classically taught to mathematicians, scientists and engineers in several related ways, each applicable to a specific subset of mathematical functions or signals:

1) Fourier *series* analysis [SNE 61] in which continuous *periodic* functions of time, with infinite duration, are represented as sums of cosine and sine functions, each with infinite duration;

– Fourier *integrals* or transforms [BRA 00, ROB 68] in which continuous (but aperiodic) functions of time are represented as continuous functions of frequency (or *vice versa*);

2) Discrete Fourier transforms in which signals defined at discrete intervals in time are represented in the frequency domain by cosine and sine functions. This topic is broken down into:

– discrete-time Fourier transforms, in which discrete-time signals of limited duration are represented as continuous frequency-domain distributions;

– discrete Fourier transforms, in which discrete-time, discretized (that is *digital*) signals of finite duration are represented by a finite-length array of digital frequency coefficients. (These are usually computed numerically using the *fast Fourier transform* (FFT)).

The key to all of the above ideas is the representation of a signal using complex exponentials, often known as *harmonic analysis*, although this term has a somewhat wider meaning in mathematics than its usage in signal and image processing. The complex exponential with angular frequency ω and phase ϕ: $f(t) = A\exp(\omega t + \phi) = A\left(\cos(\omega t + \phi) + I\sin(\omega t + \phi)\right)$ has cosine and sine components in its real and imaginary parts, respectively. Since, in this book, we are concerned with signals that have three- or four-dimensional samples, it is helpful to consider classical Fourier analysis in terms of complex exponentials rather than in terms of separate cosines and sines.

Figure I.1 shows a real-valued signal (on the left-hand side of the plot, with time increasing away from the viewer). The signal is a sawtooth waveform reconstructed from its first five non-zero harmonics, which are plotted in the center of the figure as helices. (The horizontal spacing between the helices is introduced simply to make them clearer: there is no mathematical significance to it). The five helices on the left are the positive frequency complex exponentials and the five helices on the right are the negative frequencies. Note that the positive and negative frequency exponentials have opposite directions of rotation. The real parts of the harmonics are projected onto the right-hand side of the figure (these sum to give the reconstructed waveform on the left) and the imaginary parts of the harmonics are projected onto the base of the figure (these cancel out because the exponentials occur in complex conjugate pairs at positive and negative frequencies, a symmetry due to the original signal being real-valued).

In general, with a complex signal analyzed into complex exponentials in the same way, there would be no symmetry between the positive and negative frequency exponentials. This case is a useful model for what follows in this book, where we consider signals and images with three- and four-dimensional samples. Figure I.2 shows a complex signal constructed by bandlimiting a random complex signal.