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QUANTITATIVE PORTFOLIO OPTIMIZATION

Advanced Techniques and Applications

MIQUEL NOGUER ALONSO, JULIÁN ANTOLÍN CAMARENA,
ALBERTO BUENO GUERRERO

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Preface

Quantitative portfolio optimization is a cornerstone of modern financial management, providing a rigorous framework for balancing risk and return in investment portfolios. This book, *Quantitative Portfolio Optimization: Advanced Techniques and Applications*, aims to serve as both a comprehensive introduction for those new to the field and a deep dive into the latest advancements for experienced practitioners and researchers.

The genesis of portfolio optimization can be traced back to Harry Markowitz's Modern Portfolio Theory (MPT) in the 1950s, which introduced the now-fundamental concepts of diversification and mean-variance optimization. Since then, the field has evolved significantly, integrating a wide array of quantitative methods, including Bayesian statistics, machine learning algorithms, and advanced optimization techniques. These methods have transformed portfolio management from a discipline grounded in basic statistical principles to one that leverages innovative computational techniques to solve increasingly complex problems.

This book is structured to reflect this evolution, beginning with foundational theories before progressing to advanced applications. We explore not only traditional models such as the Capital Asset Pricing Model (CAPM) and the Black-Litterman model but also the latest developments in areas such as reinforcement learning, deep learning, and graph-based portfolio construction. Additionally, we cover emerging topics like quantum computing's role in portfolio optimization and the integration of partial differential equations (PDEs) for modeling portfolio dynamics.

Each chapter is meticulously designed to bridge theory with practice, offering detailed explanations of the mathematical underpinnings of each technique, followed by practical applications using real-world data. The mathematical rigor is complemented by code implementations and case studies that demonstrate the practical utility of the methods discussed.

Whether you are a professional, researcher, or student in the field of finance, we hope this book enhances your understanding of quantitative portfolio optimization and equips you with the knowledge to apply these techniques effectively. As the financial markets continue to evolve, so must the methods we use to manage them. We believe that the approaches detailed in this book will be instrumental in addressing the challenges and opportunities of tomorrow's financial world.

Finally, we extend our deepest gratitude to our colleagues, students, and family members, whose support and encouragement have been invaluable throughout the creation of this book. It is our sincere hope that this work contributes meaningfully to the ongoing development of the field of quantitative portfolio optimization.

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Julián Antolín Camarena holds a bachelor's, master's and a PhD in physics. For his master's, he worked on the foundations of quantum mechanics examining alternative quantization schemes and their application to exotic atoms to discover new physics. His PhD dissertation work was on computational and theoretical optics, electromagnetic scattering from random surfaces, and nonlinear optimization. He then went on to a postdoctoral stint with the US Army Research Laboratory working on inverse reinforcement learning for human-autonomy teaming. He then worked as an Applied AI Researcher at Point72 Asset Management where his research was to develop algorithms for time series analysis using machine learning algorithms and developing deep learning architectures. Currently, works in applying AI to quantitative bioimaging and biophysics at Arizona State University.

Alberto Bueno Guerrero has two bachelor's degrees in physics and economics, and a PhD in banking and finance. Since he received his doctorate, he has dedicated himself to research in mathematical finance. His work has been presented at various international conferences and published in journals such as *Quantitative Finance; Journal of Derivatives, Mathematics; and Chaos, Solitons and Fractals*.

Among the topics addressed in his research are the term structure of interest rates, the valuation and hedging of derivatives, the immunization of bond portfolios, and a quantum-mechanical model for interest rate derivatives. His article "Bond Market Completeness Under Stochastic Strings with Distribution-Valued Strategies" has been considered a feature article in *Quantitative Finance*.

In 2020, he was interviewed for the article "Interviews with Researchers Who Started Their Career in Physics but Moved to Finance," for the special issue of the *Journal of Derivatives* titled "Physics and Financial Derivatives."

He has also functioned as anonymous reviewer for various journals, and regularly reviews articles for Mathematical Reviews of the American Mathematical Society.

CHAPTER 1

Introduction

1.1 EVOLUTION OF PORTFOLIO OPTIMIZATION

Portfolio optimization has undergone significant transformation since its inception. Initially, the focus was on maximizing returns without much regard for risk. This changed with the introduction of Modern Portfolio Theory (MPT) by Harry Markowitz in the 1950s, which introduced the concept of balancing risk and return. Markowitz's mean-variance optimization laid the groundwork for the systematic assessment of portfolio risk and diversification.

Over the years, portfolio optimization has evolved to incorporate various advanced techniques and models. These include the Capital Asset Pricing Model (CAPM), Arbitrage Pricing Theory (APT), and more sophisticated approaches like the Black-Litterman model, risk parity, and hierarchical risk parity. Recently, machine learning methods have also been integrated into portfolio optimization, providing new ways to manage complex data and uncover hidden patterns in financial markets. Moreover, the integration of reinforcement learning, and graph-based methods has opened new avenues for dynamic and complex portfolio strategies. Sensitivity-based portfolios, which focus on the sensitivity of portfolio returns to changes in underlying factors, have also become an important aspect of modern portfolio management.

1.2 ROLE OF QUANTITATIVE TECHNIQUES

Quantitative techniques play a crucial role in modern portfolio optimization. These techniques allow for the systematic analysis and management of risk, the development of models to predict asset returns, and the optimization of portfolios to achieve desired outcomes. Key quantitative methods used in portfolio optimization include:

- **Mean-Variance Optimization**: This foundational technique balances expected return against risk, measured as the variance of returns. It involves calculating the expected returns and covariances of all assets, then solving for the weights that minimize portfolio variance subject to a desired return. The efficient frontier is derived from this process, representing the set of optimal portfolios.
- **Factor Models**: These models, such as the CAPM and multifactor models, explain asset returns based on various macroeconomic factors or firm-specific factors. The CAPM, for example, relates an asset's return to the return of the market portfolio, adjusted for the asset's sensitivity to market movements.
- **Bayesian Methods**: Bayesian techniques incorporate prior beliefs and observed data to update the estimation of expected returns and risks. The Black-Litterman model is a popular application in portfolio optimization, combining market equilibrium with investor views to produce more stable and diversified portfolios. Bayesian methods are particularly useful for handling parameter uncertainty and incorporating subjective views.
- **Machine Learning**: Machine learning algorithms are used to identify patterns in large datasets, making them valuable for predictive modeling in portfolio optimization. Techniques like neural networks, decision trees, and generative models can uncover complex relationships between asset returns and various predictors. These methods can enhance the forecasting of returns and risks as well as optimize trading strategies.
  - **Neural Networks:** These are used to model nonlinear relationships between inputs and outputs. In portfolio optimization, they can predict asset returns based on historical data and other variables.
  - **Decision Trees:** These algorithms split the data into subsets based on feature values, creating a tree-like model of decisions. They are useful for capturing the nonlinear relationships in financial data and can be used to identify important variables influencing asset returns.
  - **Generative Models:** These models, such as Generative Adversarial Networks (GANs) and Variational Autoencoders (VAEs), are used to generate new data samples that are like the training data. In portfolio optimization, generative models can be used to simulate realistic market scenarios and generate synthetic data for stress testing and risk management.

- **Reinforcement Learning (RL):** RL involves training algorithms to make sequences of decisions by rewarding desirable actions and penalizing undesirable ones. In portfolio optimization, RL can dynamically adjust the asset allocation based on market conditions and investment goals. An RL agent learns a policy that maximizes cumulative rewards, which can correspond to returns in a portfolio context. Techniques like Q-learning and policy gradients are commonly used in RL for portfolio management.
  - **Q-learning:** This algorithm learns the value of actions in different states and aims to maximize the expected reward over time. It updates its estimates using the Bellman equation.
  - **Policy Gradients:** These methods optimize the policy directly by computing gradients of the expected reward with respect to the policy parameters.
- **Graph-based Methods**: These methods use graph theory to represent and analyze the relationships between assets. Graphs can model the dependencies and correlations among assets, aiding in the construction of diversified and robust portfolios.
  - **Graph Theory:** This involves studying graphs, which are mathematical structures used to model pairwise relations between objects. In portfolio optimization, nodes can represent assets, and edges can represent the correlations or co-movements between them.
  - **Hierarchical Risk Parity (HRP):** This approach uses clustering and tree structures to construct portfolios. It aims to distribute risk more evenly across different clusters of assets, improving diversification.
- **Sensitivity-based Portfolios**: These portfolios focus on the sensitivity of portfolio returns to changes in underlying factors, such as economic variables or market indices. By analyzing how small changes in these factors impact on the portfolio, managers can better understand and manage risk.
  - **Sensitivity Analysis:** This involves examining how the variation in the output of a model can be attributed to different variations in the inputs. In portfolio optimization, sensitivity analysis helps in understanding the impact of changes in asset returns and other factors on the portfolio performance.
  - **Partial Differential Equations (PDEs):** PDEs can be used to model the dynamics of portfolio values over time, considering factors like interest rates and asset prices. Solving these equations provides insights into the optimal portfolio allocation under different market conditions.
- **Risk Measures and Management**: Techniques like Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) are used to quantify the risk

of loss in a portfolio. These measures are essential for understanding potential downside risks and making informed decisions about risk mitigation strategies. Advanced risk measures also consider tail risks and the distribution of returns.

- **Optimization Algorithms**: Several optimization algorithms are employed to solve portfolio optimization problems. These include:
  - **Quadratic Programming:** Used in mean-variance optimization to find the optimal asset weights that minimize portfolio variance for a given return.
  - **Monte Carlo Simulation:** Used to model the probability of different outcomes in a process that cannot easily be predicted due to the intervention of random variables. In portfolio optimization, it is used to simulate the performance of different portfolio strategies under various market conditions.
  - **Genetic Algorithms:** These algorithms mimic natural selection processes to generate high-quality solutions for optimization problems. They are particularly useful in finding optimal portfolios in large, complex investment universes.
  - **Dynamic Programming:** Applied in multi-period portfolio optimization to make decisions that consider the evolution of the portfolio over time.

1.3 ORGANIZATION OF THE BOOK

This book is structured to provide a comprehensive understanding of quantitative portfolio optimization techniques, from foundational theories to advanced applications. The chapters are organized as the list describes:

- **Chapter 2: History of Portfolio Optimization:** A review of the key developments in portfolio optimization, from early theories to modern advancements.
- **Chapter 3: Modern Portfolio Theory:** A detailed study of mean-variance analysis, the CAPM, and APT, including their applications and limitations. We introduce a new framework Mean Variance with CVAR constraints.
- **Chapter 4: Bayesian Methods in Portfolio Optimization:** An exploration of Bayesian techniques and their application to portfolio optimization.
- **Chapter 5: Risk Models and Measures:** A discussion on various risk measures, including Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR), and their estimation methods.

- **Chapter 6: Factor Models and Factor Investing:** Examination of single and multifactor models, factor risk, and performance attribution.
- **Chapter 7: Market Impact, Transaction Costs, and Liquidity:** Insights into market impact, transaction costs, and liquidity considerations in portfolio optimization.
- **Chapter 8: Optimal Control:** Coverage of dynamic programming, optimal control, and their applications in portfolio optimization.
- **Chapter 9: Markov Decision Processes:** Discussion on fully observed and partially observed MDPs, infinite and finite horizon problems, and the Bellman equation.
- **Chapter 10: Reinforcement Learning:** Examination of reinforcement learning techniques and their applications in portfolio optimization.
- **Chapter 11: Deep Learning in Portfolio Management:** Introduction to deep learning methods and their integration into portfolio management.
- **Chapter 12: Graph-based Portfolios:** Exploration of graph theory-based portfolios and their applications.
- **Chapter 13: Sensitivity-based Portfolios:** Insights into modeling portfolio dynamics with partial differential equations and sensitivity analysis.
- **Chapter 14: Backtesting in Portfolio Management:** Discussion on backtesting methods, trading rules, and transaction costs.
- **Chapter 15: Scenario Generation:** Techniques for generating scenarios and their application in portfolio optimization.

This structure ensures a logical progression from basic concepts to advanced techniques, providing readers with the tools and knowledge necessary to optimize portfolios effectively in today's complex financial markets.

CHAPTER 2

History of Portfolio Optimization

This chapter is dedicated to a non-exhaustive review of the main contributions to portfolio optimization, from Markowitz's precursors to the modern machine learning methods, including Markowitz's mean-variance approach, the Black-Litterman model, Risk Parity, and Hierarchical Risk Parity. Only the Black-Litterman model and the Risk Parity approach will be addressed in depth. An alternative derivation of the Black-Litterman model, based on Bayesian methods, will be presented in Chapter 4. A more detailed treatment of the Markowitz model, the Hierarchical Risk Parity algorithm and some machine learning methods will be postponed to subsequent chapters.

2.1 EARLY BEGINNINGS

Harry Markowitz is unanimously recognized as the father of Modern Portfolio Theory (MPT). His seminal works, Markowitz (1952, 1959), settled the foundations of the mean-variance analysis upon which other aspects of MPT were built.

The groundbreaking nature of Markowitz (1952) can be inferred from its scarce set of references: Uspensky (1937), Williams (1938) and Hicks (1939). The first one is a text on Mathematical Probability, to which the reader was referred. Of the two remaining references, only Williams (1938) played a role in the development of the 1952 paper, as we will see soon. Therefore, it is not surprising that Mark Rubinstein stated the following: "What has always impressed me most about Markowitz's 1952 paper is that it seemed to come out of nowhere" (Rubinstein, 2002). Nevertheless, the key ideas of mean-variance analysis (i.e., diversification, mean returns, and a risk measure as variables) were present in previous literature. The rest of the section will be dedicated to analyzing some of these early contributions.

According to Markowitz (1999), written references to the concept of diversification can be traced back to Shakespeare's *The Merchant of Venice*, in which the following passage can be found (Act I, Scene I):

My ventures are not in one bottom trusted,
Nor to one place; nor is my whole state
Upon the fortune of this present year
Therefore my merchandise makes me no sad.

In the academic field, Daniel Bernouilli, in his 1738 paper on the Saint Petersburg paradox (Bernouilli, 1954), offered this diversification advice for risk-averse investors: "... it is advisable to divide goods which are exposed to some small danger into several portions rather than to risk them all together." In the twentieth century, the first reference related to Portfolio Theory appears in *The Nature of Capital and Income* (Fisher, 1906), in which variance is suggested as a measure of economic risk. Hicks, in his treatise *Theory of Money* (Hicks, 1935), introduces a qualitative analysis regarding the probabilities associated with the risk of an investment, concluding that they can be represented by a mean value and an appropriate measure of risk (although he does not mention any specific measure).

Marschak (1938) takes a step forward specifying the nature of the parameter measuring risk: "It is sufficiently realistic, however, to confine ourselves, for each yield, to two parameters only: the mathematical expectation ('lucrativity') and the coefficient of variation ('risk')." Although Marschak was the advisor of Markowitz's thesis, we cannot state (as Markowitz himself acknowledges) that Marschak had an influence on Markowitz's work. In fact, Marschak did not inform to Markowitz of the existence of Marschak (1938).

The only previous work that had a clear impact on Markowitz is Williams (1938). According to Markowitz (1991): "The basic principles of portfolio theory came to me one day while I was reading John Burr Williams *The Theory of Investment Value*. Williams was remarkably prescient. He provided the first derivation of the 'Gordon growth formula,' the Modigliani-Miller Capital Structure Irrelevancy Theorem, and strongly advocated the dividend discount model. But Williams had very little to say about the effects of risk on valuation (pp. 67–70), because he believed that all risk could be diversified away." In the opinion of Mark Rubinstein: "Markowitz had the brilliant insight that, while diversification would reduce risk, it would not generally eliminate it" (Rubinstein, 2002).

We cannot conclude this section dedicated to the predecessors of MPT without considering Roy (1952), of which Markowitz wrote the following: "On the basis of Markowitz (1952), I am often called the father of modern portfolio theory (MPT), but Roy can claim an equal share of this honor" (Markowitz, 1999). In fact, Roy (1952) presents an analysis very similar to that of Markowitz (1952). Specifically, Roy includes correlations between asset prices in the analysis and, like Markowitz, realizes that "the principle of maximising expected return does not explain the well-known phenomenon of the diversification of resources among a wide range of assets." Moreover, Roy also considers the expected value of returns, m; and the standard deviation of returns, σ, as the only parameters on which investment decisions are based. However, instead of minimizing the standard deviation, as Markowitz did, Roy maximizes $(m - d)/\sigma$, where d is a level of returns that can be considered a disaster. This maximization procedure leads Roy to obtain the efficient frontier as a hyperbola in the (σ, m) space. According to Markowitz (1999), the main differences between Roy's paper and his 1952 paper are that Markowitz (1952) worked only with long positions and allowed the investors to select one portfolio from the efficient frontier, whereas Roy (1952) worked also with short positions and recommended one specific portfolio.

Given that Roy arrived at the same results as Markowitz independently and using similar methods, it is reasonable to wonder why Roy did not also receive the Nobel Prize. Markowitz thought the reason was his greater visibility: Roy's paper was his last (and only) publication in finance, whereas Markowitz wrote two books and a collection of papers related to this subject (Markowitz, 1999).

2.2 HARRY MARKOWITZ'S MODERN PORTFOLIO THEORY (1952)

Before delving into the analysis of Markowitz's paper, we introduce some of the definitions and the notation that we will use. We consider portfolios composed of n risky assets, with returns $r_1, \ldots, r_n$; which are random variables with expectations $R_i = \mathbb{E}\left[r_i\right]$ and covariances $\sigma_{ij} = cov\left(r_i, r_j\right)$, $i, j = 1, \ldots, n$. We will usually work in matrix form, with the vector of expected returns $\mathbf{R} = \left(R_1, \ldots, R_n\right)^T$ and the covariance matrix $\boldsymbol{\Sigma}$ with $(\boldsymbol{\Sigma})_{ij} = \sigma_{ij}$. The variances of asset returns are given by the diagonal elements of $\boldsymbol{\Sigma}$, $\sigma_i^2 = \sigma_{ii}$. As the assets in the portfolio are risky assets, we have $\sigma_i^2 > 0$, $i = 1, \ldots, n$, and then, $\boldsymbol{\Sigma}$ is a positive definite matrix, that is, $\mathbf{x}^T\boldsymbol{\Sigma}x > 0$ for any nonzero n-vector $\mathbf{x}$. In addition, we will assume that $\boldsymbol{\Sigma}$ is nonsingular, $|\boldsymbol{\Sigma}| \neq 0$, meaning that none of the asset returns is perfectly correlated with the return of the portfolio composed of the remaining assets.

Each portfolio is determined by a vector $\mathbf{w} = \left(w_1, \ldots, w_n\right)^T$ in which w_i is the proportion of investor's wealth allocated to the i-th asset. By this definition we have $\sum_{i=1}^{n} w_i = 1$, or in matrix notation, $\mathbf{w}^T\mathbf{1} = 1$, where $\mathbf{1}$ is the n-vector given by $\mathbf{1} = (1, \ldots, 1)^T$. The expected return of the portfolio is given by $R_p = \mathbf{w}^T\mathbf{R}$, and the variance of its return is $\sigma_p^2 = \mathbf{w}^T\mathbf{\Sigma}\mathbf{w}$.

In his 1952 paper, Markowitz begins by rejecting the maximization of expected returns as a guiding principle for investment behavior. This decision is rooted in the often-understated principle of diversification, a key aspect of Markowitz's work. In his own words: "[...] a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim."

Diversification, as a rule leading to the reduction of the risk of a portfolio, measured by the variance of its return, finds support in theoretical arguments. As the following simple result shows, under certain assumptions, perfectly diversified portfolios become asymptotically risk-free. In other words, the variance of the portfolio return diminishes as the number of component assets increases.

Proposition 2.1 *Consider an equally weighted portfolio whose asset returns are independent random variables with the same variance σ^2. Then, the variance of the portfolio return, σ_p^2, satisfies*

$$\lim_{n\to\infty} \sigma_p^2 = 0,$$

Proof. As the portfolio is equally weighted, we have $\mathbf{w} = \frac{1}{n}\mathbf{1}$, and by the independence assumption, $\mathbf{\Sigma} = \sigma^2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Then, the portfolio variance is

$$\sigma_p^2 = \mathbf{w}^T\mathbf{\Sigma}\mathbf{w} = \frac{1}{n^2}\sigma^2\mathbf{1}^T\mathbf{I}\mathbf{1} = \frac{\sigma^2}{n},$$

from where we obtain the desired result. ■

While the previous result is intellectually appealing, its assumptions are hardly satisfied in actual markets. First, there is empirical evidence suggesting that returns follow stable Paretian distributions, characterized by infinite variances (Mandelbrot, 1963; Fama, 1965a).[1] Second, it is difficult, if not impossible, to find a substantial number of assets with totally independent returns and equal variance. Therefore, as Markowitz himself pointed out: "The returns from securities are too intercorrelated. Diversification cannot eliminate all variance."

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[1]According to Fama (1965b), in the context of stable Paretian distributions, there is a range of conditions under which diversification is a meaningful economic activity.