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# Parametric Array Loudspeakers

## From Theory to Application

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# Preface

“Sound Beaming”, a unique skill in martial arts novels, refers to the loudspeaker being able to condense the sound into a line and directly deliver it to the listener’s ears, achieving directional sound propagation, while others cannot hear it. In real-life applications, when a highly directional sound beam is required, the conventional approach is to employ loudspeaker arrays, but they suffer from poor low-frequency directivity, the presence of sidelobes, and large size, which limit their practical applications. The parametric array loudspeaker, as a new concept sound source, has the advantages of narrow beam, strong directivity, and very small sidelobes. Different from traditional loudspeakers, the sound production principle of the parametric array loudspeaker is based on the parametric array effect, modulating the audio signal with an ultrasonic carrier, and then amplifying it to obtain a modulated sound wave signal with a finite amplitude. After being emitted by the ultrasonic transducer, the modulated audio signal can self-demodulate due to the nonlinearity in the air during the propagation process, thereby obtaining a highly directional audio frequency signal.

Since the concept of the parametric array was proposed by Westervelt in 1963, it has received widespread attention and has continued to develop, from the initial underwater parametric array to the parametric array loudspeaker in the air. At present, the parametric array loudspeaker has become a research hotspot in the field of audio engineering and has received extensive research in both theory and application. Many domestic and international companies have also launched their own parametric array loudspeaker products, which have been increasingly used in exhibition halls, supermarkets, transportation, offices, and public places, bringing certain economic benefits. At the same time, the rapid development and application of parametric array loudspeakers can provide ideas for solving social problems caused by noise pollution and have certain social benefits.

The parametric array loudspeaker, as a typical application of nonlinear acoustics, involves a very wide range of knowledge, including modeling simulation, computational science, transducer technology, signal processing technology, and psychoacoustic technology, among many other disciplines. For newcomers, it is not easy to grasp so much relevant knowledge, and the existing related books are basically comprehensive books on nonlinear acoustics. There is currently no monograph on parametric array loudspeakers both domestic and international. This book focuses on parametric array loudspeakers, involves research in all major aspects, and details a large amount of related research content in basic theory, key technology development, and system practicality. This book mainly includes theoretical calculations, measurements, signal processing, beamsteering methods, implementation, and applications, and can provide comprehensive and valuable insights for researchers in the field of acoustics. This book can assist acoustic researchers in swiftly delving into the research areas related to parametric array loudspeakers, and is an indispensable reference book in the research of parametric array loudspeakers and even nonlinear acoustics.

This book is a summary and refinement of our research group's many years of work on parametric array loudspeakers. We would like to thank our research collaborators, Profs. Meng Hwa Er, Woon-Seng Gan, Tomoo Kamakura, and our research group members, Profs. Jing Tian, Ming Wu, and Associate Professor Kan Sha for their guidance and assistance in the related research process of this book. We also appreciate the extensive work done by students Wei Liu, Chao Ye, Zheng Kuang, Shuaibing Wu, Yongsheng Mu, Chenxi Huang, Wei Zhang, Wei Ji, Dengyong Ma, Yew Hin Liew, Furi Andi Karnapi, Khim-Sia Tan, and Kelvin Chee-Mun Lee in data collection, modeling calculations, and implementation. We are grateful to Dr. Chuang Shi for his assistance in the publication process of this book. We thank the responsible editor of the Science Press for the extensive work done for this book. The writing and publication of this book have received the care and support of the leaders and researchers of the Key Laboratory of Noise and Vibration of the Chinese Academy of Sciences, and have also received the support and funding of the National Natural Science Foundation of China (Nos. 11004217, 11674348). Here, we express our heartfelt thanks.

Due to the limitations of the authors' expertise, the book may contain shortcomings. We sincerely invite our colleagues to offer criticism and corrections.

Beijing, China
2021

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# Introduction

The parametric array loudspeaker, which can achieve directional sound radiation, has received widespread attention from researchers due to its advantages of narrow beam, highly directivity, and very small sidelobes. It has developed rapidly in theory and practical applications, and has formed several commercial products. At present, the parametric array loudspeaker has become a research hotspot in the field of audio engineering. This book mainly introduces the current research status of parametric array loudspeakers, including modeling calculations, measurements, signal processing, beamsteering, implementation, and applications. The introduction of the above aspects helps to deeply understand the parametric array loudspeaker and the scientific problems involved, which is of great guiding significance for further research on parametric array loudspeakers and provides a reference for other acoustic research.

This book can be used as a reference book for graduate students in acoustics and acoustic researchers. We hope this book can help readers in their scientific research work.

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# Chapter 1
# Overview

## 1.1 History and Current Status of Nonlinear Acoustics in Parametric Arrays

### *1.1.1 History of Nonlinear Acoustics*

Acoustics mainly studies the emission, propagation, reception of sound, the properties of sound, and the interaction between sound and other substances. Acoustics is one of the oldest disciplines in natural science, with descriptions of acoustics dating back to ancient times. With the invention and application of radio technology, acoustics has moved from classical acoustics to the development period of modern acoustics. Modern acoustics has a strong permeability and intersects with many other disciplines, engineering technologies, and artistic fields. It plays an important and unique role in these fields, and further develops the corresponding theories and technologies, and gradually forms independent branches of acoustics, such as physical acoustics, nonlinear acoustics, quantum acoustics, molecular acoustics, ultrasonics, optoacoustics, electroacoustics, architectural acoustics, environmental acoustics, language acoustics, bioacoustics, underwater acoustics, atmospheric acoustics, geosonics, physiological acoustics, psychoacoustics, musical acoustics, and sonochemistry, etc.

In early acoustic research, the motion equation of acoustics could satisfy people's basic needs by solving it on a linear basis, thus ignoring the nonlinearity of motion and medium. Therefore, the assumption of linear acoustics is mainly based on small amplitude sound waves. In linear acoustics, two sound waves propagating at the same time will not interact with each other, and the vibration of the medium caused by them is equal to the linear superposition of the vibration of the medium caused by them when they exist separately, satisfying the superposition principle. However, if the acoustic equation retains nonlinear terms, the superposition principle is no longer



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applicable, and it no longer follows the rules of linear acoustics, thus giving rise to a new branch of discipline—nonlinear acoustics.

Nonlinear acoustics is a science that studies intensive sound. In nonlinear acoustics, intensive sound is usually referred to as finite amplitude sound waves or large amplitude sound waves, which is a phenomenon between small amplitude sound waves and weak shock waves. The main research object of nonlinear acoustics is phenomena related to the propagation of finite amplitude sound waves, such as shock formation, harmonic distortion, non-constant propagation speed, etc. These phenomena cannot be explained by general linear acoustics. They are caused by the nonlinear effects of the medium on finite amplitude sound waves [1].

The field of nonlinear acoustics is a discipline with a long history, especially in the past sixty years, it has made great progress. Reference [2] gives a detailed introduction to the development of nonlinear theory before the 1930s. In 1755, Euler first gave the nonlinear wave equation describing finite amplitude acoustics, but did not give a strict solution to this equation. Soon after, Lagrange derived a general solution, but he believed that the changing sound wave speed would destroy the theory of sound propagation. In 1808, Poisson obtained the exact solution of the propagation of finite amplitude plane waves. Stokes [3] explained the meaning of the non-constant propagation speed that troubled Lagrange and Poisson, and he published the first explanation about waveform distortion in 1848. Stokes was also the first to give an analysis of shock waves, and proposed that viscosity and heat conduction are the inevitable loss causes of shock front. Two important papers published around 1860 [2, 4] can be regarded as the beginning of nonlinear acoustic research, that is, the simple wave theory of Riemann-Earnshaw, which is the strict solution form of the nonlinear one-dimensional wave equation independently published by Riemann and Earnshaw. Beyer compiled some early papers into the famous collection of nonlinear acoustics [6].

In addition to the analysis of shock waves by Rankine [7], Hugoniot [1], Rayleigh [8] and Taylor [9], the publication of three papers in the 1930s marked a significant progress in the understanding of finite amplitude sound. One paper by Fubini [10] is applicable to waves in lossless fluids before shock formation, and another by Fay [11] provides an asymptotic solution for finite amplitude waves with viscous losses. These two papers first provided explicit models for harmonic generation in sound waves, and the third paper by Thuras et al. [12] first provided experimental research on this phenomenon.

Eckart [13], Lighthill [14] and Mendousse [15] derived the wave equation around 1950, marking the beginning of the modern era of nonlinear acoustics. The equations of Eckart and Lighthill allow the study of non-planar finite amplitude waves, while Mendousse proved that the Burgers equation can be used to simulate plane waves in viscous fluids. Khokhlov and his collaborators demonstrated that the generalization of the Burgers equation can simulate cylindrical waves [16]. Blackstock [17] studied the detailed solution of the Burgers equation in detail, using the intermediate values of nonlinear wave solutions to smoothly connect the Fubini solution and the Fay solution, solving the problem of nonlinear propagation of single-frequency large amplitude waves. Fenlon [18] started from the lossless Burgers equation, using

Fourier expansion to represent the sound waves of each frequency component after interaction in the form of a spectrum. So far, in the most famous practical applications, the most noteworthy contribution is Westervelt's work on sound of scattering by sound [19, 20], which eventually formed the theory of acoustic parametric arrays in the 1960s [21, 22]. In the parametric array, the nonlinear interaction of two high-frequency sound beams produces a narrow low-frequency sound beam with almost no sidelobes. This process allows high directivity sound to be radiated from a relatively small transducer, with the additional benefit of being able to transmit a wide frequency range of sound. The proposal of the acoustic parametric array has brought broad prospects for the application of nonlinear acoustics. At a conference of the Acoustical Society of America in 1962, papers following Westervelt reported the experimental verification of the parametric array by Bellin and Beyer [23].

The parametric array was first conceived and tested in the United States, followed by many related works in the UK and Norway. Berktay [24–26] studied various possible application examples of using the parametric array, but later found that some of his theoretical predictions [7, 27, 28] were somewhat overly optimistic. When it was discovered from Berktay's work that the attractive features of the parametric array often exceeded the inherent low efficiency of the phenomenon, more related research work began to appear. By the early 1970s, the parametric array had been applied in civilian and military sonar systems, and the number of papers on the parametric array published at the Acoustical Society meetings and seminars each year had exceeded 100. The next turning point in the practical application of nonlinear effects in sound beams came from the theoretical work carried out inthe former Soviet Union around 1970 by Zabolotskaya and Khokhlov [29] and Kuznetsov [30]. Their research resulted in a parabolic nonlinear wave equation, known as the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation, which provides an excellent model for the combined effects of nonlinearity, diffraction, and dissipation in directional sound beams. Throughout the 1970s, numerous works based on the KZK equation appeared in the former Soviet Union, most of which were aimed at Gaussian beams. However, the most authoritative subsequent research on the nonlinear effects of sound beams was a series of important works published by Norwegian mathematicians Jacqueline Naze Tjøtta, Sigve Tjøtta and their collaborators [31–36], who explained the complex diffraction effects present in high-intensity sound fields radiated from circular sound sources, which characterize the geometric characteristics of most ultrasonic transducers. The book written by Beyer [37] emphasizes many achievements in the field of nonlinear acoustics, including experimental work. Rudenko and Soluyan [38] reported the latest progress in the former Soviet Union. Hamilton and Blackstock's [1] work on nonlinear acoustics is a good introductory textbook on nonlinear acoustic theory and applications. Qian systematically introduced the propagation of finite amplitude sound waves in unbounded and bounded spaces [39], he carried out a lot of work in many areas such as broadband parametric array research, interaction between sound waves research [40–52], greatly promoting the development of nonlinear acoustics both domestic and international. Based on Fenlon theory, Du et al. [53] mainly conducted theoretical research on the interaction between finite amplitude sound waves and small amplitude plane sound waves and sound energy suppression, and conducted related experimental verification in the sound wave tube. Based on the Burgers equation, Yang's research group mainly used the spectral decomposition method to derive the mechanism of the interaction process between low-frequency signal waves and high-frequency pump waves, and studied the energy transfer problem after the nonlinear action of variable parameter sound waves [54−57].

In the past few decades, the development of high-power sound sources and the improvement of digital signal processing technology have increasing supported practical applications using high-power sound sources. Parametric arrays with unique technical advantages such as "wideband, high directivity, small size" have been widely used in the field of underwater acoustics engineering [28, 43, 58−63]. With the continuous progress of science and technology, the application fields and scope of parametric arrays are constantly expanding.

### *1.1.2 Multi-Beam Sound Field Research*

From the perspective of nonlinear acoustics, when there is a sound wave propagating in space, the original medium in space is disturbed by this sound wave. If the medium was originally uniform, the result of the disturbance makes it non-uniform in space. If there is another sound wave in this disturbed medium, then it will be scattered by the non-uniform medium, or it can be said that one sound wave scatters another sound wave, this phenomenon is called sound of scattering by sound [64–70]. The sound field formed in this way is called a multi-beam sound field. All about the nonlinear interaction of two beams (i.e., the phenomenon of sound of scattering by sound) started with Lighthill's theory [14, 71]. Lighthill gave the form of the wave equation

$$\frac{\partial p}{\partial t} + \frac{\partial\left(\rho u_j\right)}{\partial x_i} = 0 \tag{1.1}$$

$$\frac{\partial}{\partial t}\left(\rho u_j\right) + c_0^2 \frac{\partial p}{\partial x_i} = -\frac{\partial T_{ij}}{\partial x_j} \tag{1.2}$$

$$\frac{\partial^2 p}{\partial t^2} - c_0^2 \nabla^2 \rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \tag{1.3}$$

where $T_{ij}$ is the stress tensor, $\rho$. is the fluid density, $\nabla^2$ is the Laplace operator. In 1956, Ingard and Pridmore-Brown [72] first proposed the concept of sound of scattering by sound. Ignoring viscosity, they derived the expression for the scattered pressure field based on Lighthill's far-field expression. The result suggests that when two orthogonal quasi-straight beams interact, the scattered sound occurs outside the common region, and their experimental results also support their theoretical results. In 1957, Westervelt [19] published two papers expressing different views on the results of Ingard et al. He argued that the first-order field in the above theory does not satisfy the homogeneous wave equation required by the Lighthill equation. He chose a plane wave as the first-order field, and derived from the Lighthill equation that there is no second-order scattered wave outside the common region of two non-parallel waves. In 1960, Bellin and Beyer [73] experimentally verified Westervelt's theory. Dean [74] considered the interaction of two concentric cylindrical waves and spherical waves. Garrett et al. [34] used parabolic approximation to explain the interaction of two wave fields at small angles. Darvennes et al. [64] theoretically derived the interaction of two Gaussian beams at small angles. There are many related studies on this issue, and the books by Beyer [6] and Rudenko [38] specifically include a chapter on sound of scattering by sound. From the above literature, there is a dispute about whether there is scattered sound outside the common region. A new theory has been used to reinterpret the interaction of two beams, and a series of articles by Tjøtta et al. [75–80] have been published to derive the sound scattering in the case of arbitrary crossing angles, pointing out the reasons why related research may reach different conclusions. They illustrate this using the simplest model of rectangular sound source interaction. From the continuity equation and equation of state of a uniform non-viscous fluid, the equation for density $\rho$ is obtained

$$\Box^2\rho = \left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\rho = -\frac{1}{c_0^2}\sum_{i,j}\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \tag{1.4}$$

When the two interacting sound waves are plane waves, the above equation can be transformed into $\Box^2(\rho + f) = 0$ form, where $f$ is the quadratic function of the primary wave field, $\Box^2$ is the D'Alembert operator, that is, $\Box^2 = \nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}$. For collimated beam plane waves, this formula is valid in the common region, with $\rho_F = -f$ as a particular solution. Westervelt used this feature to believe that there is no scattered field outside the common region. However, $\rho_F$ is not the general solution of the equation, because any $\rho_H$ can always be added such that $\Box^2\rho_H = 0$ holds. Thus, $\rho = \rho_F + \rho_H$ provides another solution for the density equation in the common region. Outside the common region, the sound field must satisfy $\Box^2\rho_H = 0$, which depends on the boundary conditions. If the solution in the common region is not uniquely determined, it is impossible to determine the boundary conditions of the collimated plane wave. Tjøtta et al. believed that the reason why Westervelt did not find scattered sound outside the common region is that he ignored the effect of diffraction and used the approximation of a narrow beam. They also considered the effects of boundary conditions and sound absorption on the results of multi-beam nonlinear effects. Tjøtta's method can be seen as a generalization of small-angle multi-beam interaction. When in small-angle situations, this method is approximately consistent with the results obtained by Garrett et al. [34] Tjøtta's method is a general solution of quasi-linear approximation, which is valid at any distance, crossing angle, and frequency ratio, but their results contain multiple integrals, which are not easy to solve. Although they provided an approximate solution in the derivation process, this solution is only applicable to the far field and contains two integrals, which takes a lot of time. Yang's research group provided a fast algorithm for simplifying the calculation of the small-angle offset model using the Gaussian beam expansion method [81–83], and studied the experimental phenomenon of the difference frequency sound generated by two ultrasonic beams in the presence of an artificial head [84]. Qian studied the interaction between plane waves and plane pulse waves in a non-dispersive medium, pointing out that only second-order scattered waves exist in the common region of the two parallel waves [47]. Garner [85] used cascaded second-order nonlinear effects to simulate the third-order nonlinear action of non-collimated ultrasonic beams produced by two displaced piston sound sources.

### *1.1.3 Research on Parametric Arrays*

The basic principle of parametric array research is to modulate the audio frequency signal to be emitted with a high-frequency (ultrasonic frequency) carrier signal, and then emit it with ultrasonic transducers. Due to the nonlinear acoustic effect of the medium, the audio frequency signal in the sound wave will demodulate itself during propagation. Due to the super directivity of ultrasonic propagation, it is possible to achieve directional radiation of audio frequency signals, as shown in Fig. 1.1.

Figure 1.2 describes the schematic diagram during propagation. In the figure, $f_1$ and $f_2$ are two ultrasonic signals with a small difference in frequency, called primary waves, assuming $f_1 > f_2$, $f_1 - f_2$ is the audio frequency signal. These two signals are emitted using the ultrasonic transducer. Due to the nonlinear effect of the medium, if only considering the second order, the difference frequency signal $f_1 - f_2$, the sum frequency signal $f_1 + f_2$, plus the two primary signals and their second harmonic signals will be obtained. According to the principle of sound absorption in the medium, the absorption is roughly proportional to the square of the frequency value of the sound wave, so the sum frequency signal $f_1 + f_2$ and $f_1$, $f_2$ and their harmonic signals will decay quickly, leaving mainly the audio frequency signal $f_1 - f_2$.

Generally speaking, parametric arrays are divided into three types of sound sources [62, 86, 87]: absorption-limited sound sources, diffraction-limited sound sources, and saturation-limited sound sources. The types of these sources are determined by three distances, namely the Rayleigh distance, the absorption distance, and

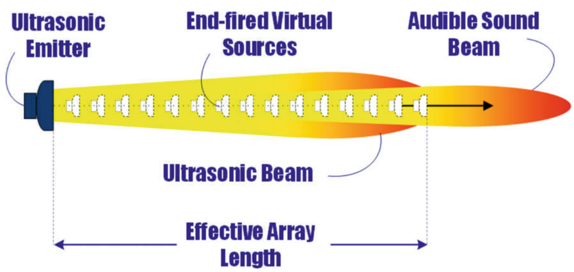


**Fig. 1.1** Schematic diagram of directional sound source formation by parametric array

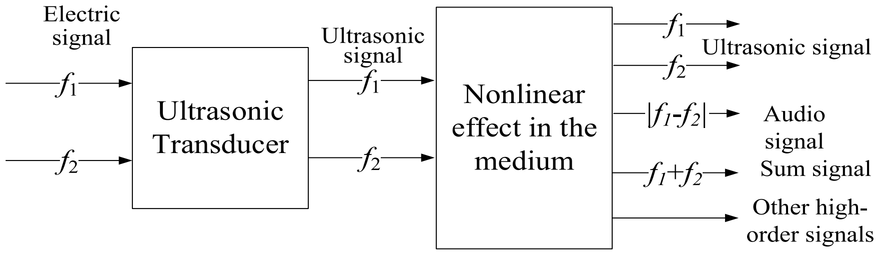


**Fig. 1.2** Schematic diagram of propagation in medium

the saturation distance. The Rayleigh distance is defined as the distance from the transducer to the transition between the near-field region and the far-field region. Therefore, the wavefront of the primary wave is a plane wave within the Rayleigh distance, but when it propagates beyond the Rayleigh distance, it will propagate spherically. The absorption distance determines the length of the primary wave before it stops generating secondary waves. Therefore, the absorption distance represents the length of the virtual source array. The saturation distance represents the area of saturation effect. When the amplitude of the primary wave is high enough, saturation effects will occur. In this case, harmonics and shock formations are generated in the primary wave beam.

When the absorption distance is shorter than the Rayleigh distance, it is a situation where absorption is limited, and the nonlinear effect terminates within the near field. In this case, the difference frequency sound is generated by the collimated beam of the primary wave. When the absorption distance is longer than the Rayleigh distance, the virtual source array extends to the far field, forming a diffraction-limited situation. In addition, if the absorption distance is much longer than the Rayleigh distance, the secondary waves generated in the near field can be ignored. Although the primary wave propagates spherically in the far field, as long as the primary wave beam is very narrow, the primary wave will still produce parametric array effect similar to the Westervelt equation. When the sound pressure level of the primary wave exceeds a certain limit, saturation effects occur within the near field of the source. This results in a saturation-limited situation, where the primary wave in the near field can be considered as a plane wave.

The parametric array can be seen as a special case of a multi-beam sound field, that is, two sound sources emit in the same center and direction. The study of parametric arrays is a multidisciplinary subject, which not only involves mechanics, acoustics, psychology, but also includes knowledge of circuits and signal processing. Therefore, the research on parametric arrays has been greatly developed with the progress of various disciplines and technologies, and has gradually been widely used in different aspects.

The initial understanding of the parametric array phenomenon was the Tartini tone discovered by Italian pianist Tartini in the mid-eighteenth century [88]. Since this is a form of beat, it was initially thought to be caused by the vibration of the human ear when perceiving sound. However, Helmholtz doubted it, and later he confirmed that this was caused by the nonlinear effect of air. He believed that this effect simultaneously produced the sum (sum frequency) and the absolute difference (difference frequency) of two frequency signals. But he did not give the theoretical analysis results. In the 1960s, Westervelt [21, 22], based on Lighthill's theory [14, 71], gave the relationship and specific theoretical derivation of the second-order field generated when two plane waves propagate in a non-uniform medium.

In 1965, Berktay [24] proposed a more accurate and complete explanation of the parametric array. He did not limit the analysis to the primary wave being a form of two frequencies, but further used the concept of envelope in modulation. This is very practical, because parametric arrays usually do not just emit a single frequency. The final demodulated signal (the signal of interest) is determined by this envelope, that is, the signal after the parametric array is demodulated is proportional to the second derivative of the square of the envelope with respect to time, that is $p_2(t) \propto \frac{\partial}{\partial t^2} E^2(t)$, where $p_2$ is the demodulated audio frequency signal, $E(t)$ is any envelope function. Since Berktay assumed that this solution was obtained in the case of being far from the sound source, ignoring the influence of ultrasound, this solution is called "Berktay's farfield solution". Although the primary wave signal, the sum frequency signal and the demodulated difference frequency signal coexist in the near-field, this solution is also applicable. This is the basic expression of the output of the parametric array. This conclusion was quickly verified in underwater acoustics [89, 90]. In 1981, Qian believed that Berktay ignored the amplitude characteristics of the near-field source, re-established the theoretical model of the nonlinear sound field calculation of the parametric array, and conducted experimental verification [42]. In the same year, Qian proposed a new method for calculating Huygens' integration, and used it to study the radiation field of the line source parametric array and the truncated parametric array [41]. In 2008, Zheng et al. improved the Berktay's model, obtained the difference frequency sound field in the axial and off-axis directions, and obtained the distribution of the difference frequency sound field more accurately [91].

If the Berktay far-field solution is used to derive an output of a single-frequency double-sideband modulation method with an input of 2 kHz (assuming the modulation index $m$ is 1), two single-frequency signal outputs with the same amplitude are obtained, which is equivalent to 100% total harmonic distortion (THD). This shows that the double-sideband modulation method cannot provide the desired single-frequency output, so it is necessary to preprocess the audio frequency signal [92].

The parametric array was first widely used in underwater acoustics. Because the absorption coefficient of air is large and its nonlinear parameter is small, it is relatively difficult to produce this nonlinear effect in the air. It was not until 1975 that Bennett and Blackstock [93] realized the parametric array in the air.

Because the audio frequency obtained by the parametric array demodulation has very highly directivity, and this audio frequency decays much slower when propagating in the air than the sound emitted by traditional loudspeakers, making the development of directional sound sources possible. In recent years, with the development of parametric array technology research, the application of parametric arrays in the air, namely parametric array loudspeakers, is becoming more and more widespread.

The earliest recorded was Yoneyama et al. in Japan [94]. They formed a hexagonal array with 547 ultrasonic transducers in 1983, with a center frequency of 40 kHz. To overcome the frequency response slope of 12dB/octave proposed by Berktay, they suggested the input signal should be preprocessed before double-sideband modulation. In actual experiments, the audio frequency signal is simply processed with the transducer frequency response curve. They obtained a relatively flat frequency response between 1.5~7 kHz. At the International Nonlinear Acoustics Conference in 1984, an article on directional speakers [95] mentioned three issues: the optimal carrier frequency, distortion calibration, and the impact of ultrasound on listeners. Their view is that the smaller the carrier frequency, the better, but it cannot be too small, otherwise the beam will diverge, and a range of recommended carrier frequency values is given. They made an array with 581 ultrasonic transducers and found that the distortion was significantly improved after preprocessing. During this period, in-depth discussions were held on issues such as preprocessing methods, sound source shapes, conversion efficiency, etc. [96–102]. Steer's research group [103, 104] also calculated the third-order nonlinear sound field, and it was simplified using the Gaussian beam expansion method. The issue of low conversion efficiency of the parametric array has also attracted the attention of scholars [105], who have conducted research from the perspective of transient signal technology [97, 106–110] or transducer technology [111–113], to some extent alleviating the problem. For this issue, some scholars have also tried to solve it by using multiple parametric array loudspeakers or combining various modulation methods [114–117].

Similar to the end-fire array effect [21, 118], the parametric array can be regarded as a virtual source array composed of countless virtual sources. The parametric array is a cumulative array, and it needs a certain propagation distance to fully form. It is the aforementioned absorption distance, which can be referred to as the array length [119], where the primary waves no longer interact with each other. For some reasons, if the observation point is less than the array length before the propagation has reached the array length, this situation is the truncation of the parametric array, and the formed part of the parametric array sound source is called the truncated parametric array [41, 58, 59, 120]. This research is often used in practical applications in many fields such as detection and material acoustic properties.

### *1.1.4 Research on Parametric Array Loudspeakers*

Since the 21st century, with the launch of commercial products and its emergence as a research focus in audio engineering, parametric array loudspeakers have achieved substantial academic progress in both foundational theories and key technologies [119, 121–129], mainly reflected in the following directions.

#### 1.1.4.1 Preprocessing Algorithms

The parametric array loudspeaker involves nonlinear effects, so in order to achieve sound field reproduction, the input signal of the parametric array loudspeaker must be preprocessed to reduce harmonic distortion. Based on the Berktay far-field solution, various signal processing algorithms are applied to the parametric array loudspeaker. Yoneyama et al. [94] used the ordinary double sideband amplitude modulation (DSBAM) method to generate the envelope signal $E(t) = 1 + mg(t)$, where $g(t)$ is the input audio signal, $m$ is the modulation index. However, this simple processing method has caused significant harmonic distortion. In 1984, Kamakura et al. [95] proposed the square root double sideband modulation method (Square Root AM, SRAM), which generated the envelope signal $E(t) = \sqrt{1 + mg(t)}$. Compared with ordinary double sideband modulation, this method can effectively reduce harmonic distortion, but it requires an ultrasonic transducer with infinite bandwidth to generate harmonic signals formed by square root processing. Subsequently, they proposed the single-sideband amplitude modulation method (SSB-AM) [130], which only requires half the bandwidth compared to ordinary double sideband modulation. However, for complex input signals, single-sideband modulation is prone to errors. Therefore, Croft et al. [131, 132] proposed the recursive single-sideband modulation method (Recursive SSB-AM) to approximate the results of square root double sideband modulation, but this requires a high-speed processor to achieve real-time calculation. Wu et al. [133] adopted multi-rate processing technology and proposed an implementation method for recursive single-sideband modulation technology. Its feasibility was verified by experiments. Tan et al. [134] proposed adjustable single-sideband modulation (Modified AM, abbreviated as MAM) using the interaction of orthogonal signals, which can effectively reduce harmonic distortion and reduce computational complexity. Sakai and Kamakura [135] proposed a single-sideband modulation method with dynamic carrier (Dynamic SSB-AM), which dynamically adjusts the amplitude of the carrier signal according to the amplitude of the modulation signal to improve the linear relationship between the output signal and the modulation signal, thereby reducing harmonic distortion. Shi et al. [136] provided the experimental results of four preprocessing methods: DSBAM, SRAM, Lower sideband-AM (LSB-AM) and Upper sideband-AM (USB-AM), and based on DSB-AM and SRAM, respectively, they proposed two new improved preprocessing methods. Ikefuji et al. [137] combined the DSB-AM and the SSB-AM methods, using different preprocessing methods for different frequency bands, and proved the effectiveness of this method through subjective testing. Since the Berktay far-field solution is only an approximate solution of the parametric loudspeaker, it cannot truly reflect the actual situation of the sound field. Ji et al. [138] summarized various preprocessing algorithms and compared the above DSB-AM, SSB-AM, and MAM preprocessing algorithms based on the time-domain solutions of the KZK equation. All of the above are based on amplitude modulation, and frequency modulation methods have also been studied by scholars due to their low implementation costs [139, 140], some progress has been made, but the high harmonic distortion has not yet been well resolved [105], which is worth further in-depth study.

Due to the limitations of the Berktay far-field solution, the preprocessing methods based on the Berktay far-field solution cannot completely eliminate the nonlinear distortion in the parametric array loudspeaker system, and in recent years, the preprocessing method based on the Volterra filter has been introduced. In 2002, Lee et al. introduced the Volterra filter (VF) into the research of parametric array loudspeakers [141], based on the Berktay far-field solution model. They established a 2nd-order VF model, which simulated the nonlinearity of the parametric array loudspeaker well. Later, based on the KZK equation, Ji et al. obtained a more accurate 2nd-order VF model, and on the basis of the VF model, designed a 2nd-order inverse filter, which can effectively reduce the harmonic distortion of the parametric array loudspeaker [142]. Yang's research group introduced the diagonal Volterra filter with lower complexity into the research of parametric array loudspeakers, and theoretically demonstrated the feasibility of using a one-dimensional Volterra filter (ODVF) to compensate for the nonlinear distortion of the loudspeaker instead of the traditional VF. They proposed a solution based on ODVF to compensate for the harmonic and intermodulation distortion of the parametric array loudspeaker [143–146]. Shi et al. proposed an ultrasonic-ultrasonic VF [147], and applied the sparse normalized least mean square algorithm to the identification of the first-order Volterra filter and acoustic delay [148]. For the parametric array loudspeaker system based on frequency modulation, Hatano et al. designed a parallel cascade structure of VF and its inverse system [140].

The above-mentioned preprocessing methods are mainly to solve the harmonic distortion problem generated in the parametric array loudspeaker system, and designers can therefore choose the appropriate preprocessing technology according to the specific system requirements, such as analog or digital implementation, available computing resources, bandwidth limitations of ultrasonic transmitters, etc. To solve the problem of poor low-frequency response of the parametric array loudspeaker, Shi et al. [149] proposed the "missing fundamental" concept widely used in traditional speaker virtual bass enhancement to improve the bass quality of the parametric array loudspeaker. This psychoacoustic preprocessing technique splits the audio input with a low-pass and a high-pass filter under a cutoff frequency of 500 Hz. The high-pass audio content uses general preprocessing techniques, while the low-pass bass is processed by the function $F(x)$, which serves as the harmonic generator of the parametric array loudspeaker. Subjective tests showed that this preprocessing method can promote the enhancement of low volume while maintaining a good auditory experience.

#### 1.1.4.2 Beam Control

Beam control, based on the principle of parametric array effect, the difference frequency sound of parametric array loudspeakers has high directivity. This high directivity can make the difference frequency sound only emitted to the target area, without polluting the sound environment outside the target area. However, to ensure that the beam of the difference frequency sound to follow the target area for mobile tracking, Woodward et al. [150] in 1994 applied phased array technology to underwater parametric arrays for seabed mapping, which can meet the beamsteering requirements within $\pm 20°$. In 2005, Olszewski et al. [151] used a mechanical motor to rotate each channel's transducer, and then compensated the phase of each channel by delaying, eliminating the sound path difference caused by rotation, thereby achieving the purpose of beamsteering. However, this method has a slow beamsteering speed and requires motor assistance, which adds extra equipment. The use of digital signal processing to achieve beamsteering is low cost, energy saving, fast, and does not require additional auxiliary equipment, which is the trend of future development. The delay and sum algorithm is a common method for array beamforming. This method is simple and practical, and can conveniently and quickly realize the beamsteering of the parametric array loudspeaker. In 2006, Yang et al. [152] proposed to use the zero-order Bessel function weighting method for parametric array difference frequency sound beamsteering. In addition, Yang et al. [153–155] used the delay and sum method, used the Chebyshev weighting method combined with SSB modulation method, weighted the carrier and sideband frequencies separately, and obtained a sound beam with a constant beam width. In actual implementation, the angle of the parametric array beamsteering is limited by the sampling rate of the digital-to-analog converter. In the case of low sampling rate, if integer sampling period delay is used, although the parametric array beam can be deflected, the beam has a minimum steering angle, and the steering angle of the beam is only distributed at a few discrete angles. In 2006, Gan et al. [156] pointed out that at a sampling rate of 192 kHz, the minimum steering angle is about 26 degrees. For this reason, Gan et al. achieved a very small steering angle with the smallest amount of calculation by delaying the carrier and sideband signals separately without increasing the sampling frequency, but this method requires the generation of an omnidirectional carrier beam, so the demodulated difference frequency sound energy is severely reduced. In 2010, Takeoka et al. [157] first used $\Sigma - \Delta$ high-pass filter to upsample and then integer delay, and then replaced the power amplifier to drive the ultrasonic transducer to emit sound waves with an inverter. Although this method solves the problem of delay accuracy, the signal-to-noise ratio of this method depends heavily on the stability of the clock, and secondly, the output dynamic range is small, resulting in a small sound pressure level of the difference frequency sound. The research group led by Gan at Nanyang Technological University has done a lot of theoretical and experimental work in this area [158–163]. The research group led by Yang [164] applied the Farrow structure fractional delay algorithm to the parametric array loudspeaker, and designed a phased parametric array loudspeaker system to achieve arbitrary beamsteering with fixed filter coefficients.

#### 1.1.4.3 New Transducers

The new transducer, the ultrasonic transducer, as the emission source of the parametric array loudspeaker, has a significant impact on its performance. Factors such as the frequency response, phase, and amplitude consistency of the transducer directly affect the radiation power, output sound pressure level, and distortion of the parametric array loudspeaker. Early parametric array loudspeakers used traditional piezoelectric ceramic ultrasonic transducers (PZT), generally using a parallel method to form a transducer array sound source with multiple ultrasonic transducer units. This method is simple in structure and easy to implement. However, PZT has a large mechanical impedance, resulting in a small bandwidth and low radiation efficiency [165]. Polyvinylidene Difluoride (PVDF), as a new type of piezoelectric high polymer material, has a high piezoelectric constant, light weight, good flexibility, can be bent into any shape, and has a flat frequency response in a wide frequency band. Toda conducted a detailed research on PVDF film, using various different structures (cylindrical, arc, and corrugated, etc.), and carried out a theoretical and experimental research on its resonance frequency, directivity, radiation sound pressure, driving voltage, etc. [166–171], and used PVDF film to make a parametric array loudspeaker for actual measurement [172]. However, the dielectric constant of PVDF film is small, it is difficult to produce high-intensity ultrasonic signals, and it requires a high driving voltage, which puts higher requirements on the power amplifier. In recent years, with the development of MEMS technology, new types of ultrasonic transducers have appeared. Capacitive Micromachined Ultrasonic Transducers (CMUTs), fabricated utilizing CMOS technology, exhibit high sensitivity and a broad bandwidth. They are characterized by their compact form factor and low intrinsic noise. The compatibility of their manufacturing process with CMOS technology enables the integration of the pre-stage driving circuit, preamplifier, signal processing circuit onto a single silicon chip, thereby achieving electronic integration and minimizing parasitic effects. These attributes render CMUTs highly suitable for a range of applications, including ultrasonic non-destructive testing and medical imaging, etc. [173–177]. Wygant et al. [112] designed CMUT transducers with two different membrane thicknesses (40 and 60 μm), with resonant frequencies of 46 kHz and 55 kHz, respectively. When supplied with a 200 VDC bias and 380 V and 350 VAC signals, the transducers were capable of generating 135 dB and 129 dB ultrasonic signals at their surfaces. When used to produce a 5 kHz difference frequency signal, the sound pressure level measured at a distance of 3 m was 58 dB, with a 6 dB bandwidth of 8.7°. Zhang et al. [178] designed and manufactured an air-coupled hexagonal CMUT array for the application of the parametric array loudspeaker. The piezoelectric Micro-machined Ultrasonic Transducer (pMUT) is composed of a vibrating membrane made of a certain material and a piezoelectric thin film with upper and lower electrodes. The mechanical vibration of the vibrating membrane and the piezoelectric effect of the piezoelectric thin film can interact to realize the mutual conversion of sound energy and electrical energy. Since the reception and transmission of silicon micro piezoelectric ultrasonic transducers are both in the bending vibration mode, their working frequency is mainly determined by the first-order vibration frequency of the bending vibration of the vibrating membrane. Therefore, the resonance frequency of the transducer can be changed by changing the thickness and area of its vibrating membrane. In addition, silicon micro piezoelectric ultrasonic transducers also have the advantages of low power consumption, small volume, and easy integration [179–183]. Lee et al. [184] developed a pMUT transducer with a size of $35 \times 33$ mm$^2$, with primary wave frequencies of 95 and 135 kHz, producing a difference frequency sound of 40 kHz. The experimental measurement at 0.8 m showed a 6 dB beam angle of 5° and a maximum axial sound pressure level of 85.4 dB. The development of new ultrasonic transducers has greatly promoted the development of parametric array loudspeakers. Je et al. [185] designed a more efficient (up to 70%) and wider bandwidth (15 kHz) pMUT array. Ahn et al. [186] used a pMUT array to design a compact parametric array loudspeaker.

#### 1.1.4.4 Measurement Issue

Measurement problems arise because parametric array loudspeakers are different from ordinary loudspeakers, and ultrasonic primary waves coexist with the demodulated audio frequency wave, which puts higher requirements on the accurate measurement of parametric array loudspeakers. When two ultrasonic waves with frequencies $f_1$ and $f_2$ ($f_1 > f_2$) are emitted by the ultrasonic transducer, due to the nonlinear effect in the air, multiple different frequency sound waves can be reproduced, that is, the difference of these two frequencies ($f_1 - f_2$) and the sum of the two frequencies ($f_1 + f_2$), as well as higher-order harmonics. The sound wave attenuation coefficient is almost directly proportional to the square of the frequency, causing the high-frequency components to attenuate faster during propagation. Therefore, after a certain distance of propagation, only the difference frequency sound wave remains, forming the audio frequency. However, in the near-field of the parametric loudspeaker, there are both ultrasonic waves $p_1$ and $p_2$ and the audio frequency $p_d$. Because the sound pressure level of the ultrasonic waves $p_1$ and $p_2$ is high, especially in the near field, it is generally above 120 dB, so when measuring directly with a loudspeaker in this area, spurious sound will be also generated. The spurious sound comes from two aspects, one is due to the nonlinear effect of the microphone pre-amplification circuit, and the other is due to the radiated sound pressure on the surface of the loudspeaker. The frequency values of the above spurious sound generated is the same as the demodulated audio frequency wave. The amplitude of the spurious sound is proportional to the product of the amplitudes of the ultrasound waves $p_1$ and $p_2$, that is, $p_n \propto p_1 p_2$. It is apparent that the spurious sound greatly interferes with the measurement of the actual demodulated audio frequency. Especially in the near field area, the spurious sound is significantly larger than the sound frequency generated by the parametric loudspeaker. Therefore, it is necessary to design a sound filter to filter out the spurious sound and obtain the amplitude of the actual demodulated audio frequency.

In 1975, Bennett and Blackstock [93] first realized the experiment of a parametric loudspeaker in the air, using a hemispherical sound filter made of 0.06 mm thick plastic glass paper, placed at the front end of the measurement transducer, attenuating the difference frequency sound (5 kHz) by 3.5 dB, but attenuating the ultrasonic primary wave (18.6 and 23.5 kHz) by 20 dB, achieving low-pass characteristics, thus effectively reducing spurious sound. Moffett et al. [187] studied the nonlinear problem of measuring hydrophones in water in 1982. Toda [172] designed a tubular sound filter with a four-layer polymer film structure, where the spacing between the four films is half the carrier wavelength (carrier frequency is 34 kHz), and the maximum attenuation of the ultrasonic carrier is up to 30 dB. However, this sound filter structure is too complex and not conducive to practical application. Wygant et al. [112] used a layer of Celen synthetic fiber as a sound filter when using CMUT transducers as the emission source for parametric array loudspeakers, but did not provide a detailed description of the structure and performance of the filter. Kamakura et al. [188, 189] used two sets of primary waves with opposite phases as the driving signals for the parametric array loudspeaker to reduce the sound pressure level of the primary wave, and it has small effect on the generated difference frequency sound. This method can also be used to reduce spurious sound. Yang's research group used this phase cancellation method to measure the sound pressure on the axis [190]. Cylindrical filters [191] and filters based on phononic crystals [192, 193] were used to reduce the impact of spurious sound on measurements. On this basis, the effects of four main parameters on spurious sound, namely the intensity of the primary wave, the difference frequency, the radius of the sound source, and the observation distance, were investigated [194].

#### 1.1.4.5 Research on Applications

As a new type of directional sound source, parametric array loudspeakers have a wide range of application prospects, and there are already many parametric array loudspeaker products and their related applications on the market. Nakashima et al. [195, 196] installed two small parametric array loudspeakers on a mobile phone to form a dual-channel system. The audio frequency at 50 cm reached 70 dB in experimental measurements, and the sound pressure level difference at both ears was 15 dB using a head and torso simulator (HATS), indicating a good directivity. Nakadai and Tsujino [197] installed a parametric array loudspeaker in the mouth of a humanoid robot, achieving human–robot interactive voice communication. When a person and the humanoid robot speak at the same time, the robot can pick up voice signals through the loudspeaker without being disturbed by its own voice signals, realizing the function of "listening while speaking". Johannes et al. [198] used a parametric array loudspeaker to achieve 3D sound reproduction, reducing interaural crosstalk. Phanomchoeng et al. used the high-directivity characteristics of the parametric array at low-frequency to warn specific lanes on highways without affecting the normal driving of vehicles in other directions, ensuring the smooth progress of construction on that lane [199]. Castagnede et al. [200,201] used a parametric array loudspeaker as a sound source to measure the absorption coefficient of materials using a single transducer pulse reflection method. Haupt and Rolt from the Lincoln Laboratory in the United States [202] used a parametric array loudspeaker from Holosonics to detect landmines buried at different depths below the ground, and discussed the indicators that a parametric array loudspeaker suitable for landmine detection should have from aspects such as power, frequency, and volume. Calicchia et al. [203] used a parametric array loudspeaker to achieve non-destructive testing of Italian Renaissance murals.

Gibson et al. [204] designed and built a parametric array loudspeaker consisting of an ultrasonic transducer array, forming a high-intensity, collimated low-frequency sound to achieve sound penetration of small areas/small targets and induce their vibration. Achanta et al. [205] and Kaduchak et al. [206] used a parametric array loudspeaker to achieve long-distance portable hidden weapon detection. Sayin et al. took a different approach and explored the feasibility of using a parametric array loudspeaker with 750 ultrasonic transducer units to construct an omnidirectional sound source [207]. They found that its performance at high frequencies is better than traditional dodecahedron sound sources, and it is smaller and lighter. Arnela et al. further improved the omnidirectional parametric array loudspeaker based on the theoretical model, making it easier to manufacture and install [208].

Compared to foreign countries, domestic research on parametric array loudspeakers started relatively late, and is mostly focused on the development and experimental research of parametric array loudspeakers [110, 209–219], with relatively fewer reports on application research [220–222]. The research group led by Yang at Institute of Acoustics, Chinese Academy of Sciences, has done a lot of research work on the theoretical foundation, measurement, engineering development, and application research of parametric array loudspeakers, and has achieved fruitful results. In theoretical research, a new calculation method of Gaussian beam expansion coefficients was proposed [223], and the sound field of multiple beams crossing at small angles was calculated using the Gaussian beam expansion method [82, 83], and various preprocessing algorithms for parametric array loudspeakers were discussed in detail [133, 138, 143–146, 224, 225]. In terms of measurement, several methods to remove or reduce spurious sound were proposed [190–192], and the parameters affecting spurious sound were systematically studied [194]. In terms of engineering development, research was conducted on the new ultrasonic transducer PVDF film [226], multi-path parametric array loudspeakers [227], and impedance matching of transducer arrays [228, 229], and several prototypes were successively introduced. In terms of application research, field measurement research on the sound absorption coefficient of materials [230] and active noise reduction control research [231, 232] were conducted using parametric array loudspeakers. The research group led by Xu at the University of Electronic Science and Technology conducted research on parametric array loudspeakers from the aspects of preprocessing algorithms [233], ultrasonic transducers [234], and hardware implementation [235, 236]. In addition, the research group led by Zhao at Shandong University of Science and Technology provided a design scheme for implementing parametric array loudspeakers using DSP [237, 238]. Yi et al. at the Jiaxing Engineering Center of Institute of Acoustics, Chinese Academy of Sciences [239] also conducted system design and testing of parametric array loudspeakers.

## 1.2 Fundamentals of Nonlinear Acoustics

The basics of nonlinear acoustics will be discussed in the following sections, focusing on the equations used in nonlinear acoustics. The review will start with the propagation of acoustic waves in lossless fluids, which provides the basic principles of nonlinear acoustics.

### *1.2.1 Propagation of Finite Amplitude Waves in Non-Attenuating Fluids*

This section first introduces the basic part of nonlinear acoustics, that is, the propagation of finite amplitude waves in non-attenuating fluids.

Currently, there are four main equations used to describe the general motion of viscous heat-conducting fluids: ① mass conservation equation, ② momentum conservation equation, ③ entropy balance equation, and ④ thermodynamic state equation. To derive the wave equation for sound propagation in fluids, it is assumed that the fluid is homogeneous and the effects of viscous and thermal conductivity coefficients caused by sound wave disturbances are ignored.

The mass conservation equation is as follows

$$\frac{D\rho}{Dt} + \rho\nabla \cdot u = 0 \tag{1.5}$$

where $\rho$ is the medium density, $u$ is the fluid velocity vector, $\nabla$ is the gradient operator, and $D/Dt = \partial/\partial t + u \cdot \nabla$.

The momentum conservation equation can be written as

$$\rho\frac{Du}{Dt} + \nabla P = \mu\nabla^2 u + \left(u_B + \frac{1}{3}\mu\right)\nabla(\nabla \cdot u) \tag{1.6}$$

where $P$ is the thermodynamic pressure, $\mu$ is the shear viscosity coefficient, $\nabla^2$ is the Laplace operator, $u_B$ is the bulk viscosity coefficient. The shear viscosity coefficient refers to the momentum diffusion between adjacent fluid elements with different speeds. The bulk viscosity coefficient provides an effective approximation at low frequencies, that is, the imbalance deviation between the actual local pressure and the thermodynamic pressure. Here, for convenience, it is assumed that all relaxation times are much smaller than the time scale of sound disturbances.

The entropy equation can be expressed as

$$\rho T\frac{Ds}{Dt} = \kappa\nabla^2 T + \mu_B(\nabla \cdot u)^2 + \frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{1}{2}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)^2 \tag{1.7}$$

where $T$ is the absolute time, $s$ is the specific entropy, $\kappa$ is the thermal conductivity, $\delta_{ij}$ is the Kronecker delta function, that is, when $i = j$, it is 1, and 0 in all other cases.

The equation of state is

$$P = P(\rho, s) \tag{1.8}$$

For an ideal gas, $P/\rho T$ and the heat capacity ratio are constants, the above equation can be written as

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \exp\left(\frac{s - s_0}{c_v}\right) \tag{1.9}$$

where $P_0$, $\rho_0$ and $s_0$ are the corresponding equilibrium values, $\gamma$ is the ratio of specific heat at constant pressure $c_p$ to that at constant volume $c_v$. To derive the sound propagation formula, the above equation uses the Taylor formula to expand at the point $(\rho_0, s_0)$.

If the fluid under consideration is lossless, $\mu$, $\mu_B$ and $\kappa$ are all zero. This allows us to obtain the exact solutions of the above formulas, which are applicable to the propagation of plane waves in lossless fluids. Specifically, for the problem of plane wave propagation in an ideal isentropic gas, a simple formula can be derived as follows

$$\frac{\partial u}{\partial t} + (c_0 + \beta u)\frac{\partial u}{\partial x} = 0 \tag{1.10}$$

where $u(x, t)$ is the velocity of the gas, $x$ is the spatial variable, $t$ is the time, $c_0 = (\gamma P_0/\rho)^{1/2}$ is the speed of small signal sound, $\beta$ is a nonlinear coefficient.

To obtain the sound field under a specific excitation at the sound source $u = f(x)$, that is, the signal at the origin is

$$u(0, t) = f(t) \tag{1.11}$$

The implicit solution satisfying the above equation is obtained by Poisson as follows

$$u = f(t - x/(c_0 + \beta u)) \tag{1.12}$$

The above equation is known as the Poisson solution. Another implicit solution is obtained by Earnshaw. Assume that the sound wave is excited by a piston with the finite amplitude. If the piston displacement is specified as $X(t)$, the sound source condition is: when $x = X(t)$, $u = \dot{X}(t)$, where $\dot{X}(t) = \frac{dX}{dt}$. At the moment $\phi$, the piston is at $x = X(\phi)$, and the speed is $\dot{X}(\phi)$. This speed is applied to the fluid causing disturbance, that is, the fluid particle speed at this wave point is $u = \dot{X}(\phi)$. Thus, the Earnshaw solution is obtained

$$u = \dot{X}(\phi), \quad \phi = t - \frac{x - X(\phi)}{c_0 + \beta \dot{X}(\phi)} \tag{1.13}$$

For example, let the sound source excitation be a sine wave, i.e., $u_0 \sin\omega t$, where $u_0$ is the amplitude, $\omega$ is the angular frequency. If a sinusoidal excitation is applied at the origin, $u(0, t) = u_0 \sin\omega t$, then the Poisson solution is

$$u = u_0 \sin\omega\left(t - \frac{x}{c_0 + \beta u}\right) \tag{1.14}$$

When the sound source is a vibrating piston, the boundary condition is: $u = u_0$, at $x = \frac{u_0}{\omega}(1 - \cos\omega t)$

As solved by Earnshaw

$$u = u_0 \sin\omega\phi, \quad \phi = t - \frac{x - u_0/[\omega(1 - \cos\omega\phi)]}{c_0 + \beta u_0 \sin\omega\phi} \tag{1.15}$$

Once the particle velocity is obtained, the sound pressure $p$ can be calculated using the characteristic impedance relationship as follows

$$\frac{dP}{du} = \frac{dp}{du} = \rho c \tag{1.16}$$

where $P$ is the total pressure of the fluid. For small signal waves, the right side of the above equation has a constant value $\rho_0 c_0$, and $p = \rho_0 c_0 u$ is obtained. For finite amplitude waves, the relationship is not simple. For adiabatic gas, after integration and a series expansion, the following equation can be obtained

$$p = \rho_0 c_0^2 \left[\frac{u}{c_0} + \frac{\beta}{2}\left(\frac{u}{c_0}\right)^2 + \frac{\beta}{6}\left(\frac{u}{c_0}\right)^3 + \ldots\right] \tag{1.17}$$

## 1.2.2 Approximation of Thermoviscous Fluids

### 1.2.2.1 Second-Order Approximation Theory

The second-order approximation theory of the thermal viscous fluid approximation finite amplitude is affected by two different nonlinear effects: cumulative and local effects. In the case of traveling waves, the cumulative effect is generally dominant, that is, it accumulates with the increase of propagation distance, leading to the steepening of the waveform. The distortion that occurs at a location is based on all previous distortions and continues to expand. The local nonlinear effect, on the other hand, produces distortion that does not increase with the propagation distance. An example of the latter is the nonlinearity of the characteristic impedance relationship. If the particle velocity waveform has been calculated, the pressure waveform is obtained by Eq. (1.17). The linear term in the formula can obtain a pressure waveform with the same cumulative distortion as the particle velocity waveform. Since the remaining terms in Eq. (1.17) are at least smaller than the linear term by $O(\varepsilon)$ (when a term is determined to be $O(\varepsilon^n)$ or smaller to $\varepsilon^n$, for very small $\varepsilon$, its size is $\alpha\varepsilon^n$, where $\alpha$ is a finite positive number), so the additional distortion it introduces is very small. Moreover, since the additional distortion only depends on the local waveform, it often remains unchanged and does not increase with the increase of propagation distance. Therefore, the only place where local distortion dominates is near the sound source, where the cumulative distortion is still very small.

Local effects make the analysis of traveling wave propagation more complex, but their impact on the solution is generally negligible because the distortion they cause is usually much smaller than the distortion resulting from cumulative effects. Therefore, many simplifications can be achieved at a small cost by ignoring local effects. When dissipation must be considered, a certain small value of the second-order approximation is usually introduced to obtain simplified wave equations and solutions. Two small order parameters are usually used in the approximation. One is the acoustic Mach number $\varepsilon = u_0/c_0$ (where $u_0$ is the typical sound speed amplitude). This value is 154 dB in the air (reference value $2 \times 10^{-5}$ Pa), and 264 dB in water (reference value $1 \times 10^{-6}$ Pa). The other small parameter is $\eta = \mu\omega/\rho_0 c_0^2$. In physics, $\eta$ is an important measure of the significance of viscous stress in plane traveling sound waves, relative to the fluctuating pressure. Under standard conditions, $\eta = 10^{-6}$ for 1 kHz in the air and 1 MHz in water can be obtained.

In the analysis of the second-order approximation, expand Eqs. (1.5) to (1.8), and discard all $O(\eta^2\varepsilon)$, $O(\eta\varepsilon^2)$ and $O(\varepsilon^3)$ terms. The resulting model retains the order of $\varepsilon$, $\eta\varepsilon$ and $\varepsilon^2$ terms, describing the small signal sound as the leading order of $\eta$. More importantly, it is expected to explain the combined effects of nonlinearity and dissipation on weak nonlinear three-dimensional sound waves.

The basis of the second-order approximation theory is two assumptions. First, the intensity of the wave is not large, that is, $|u| \ll c_0$ (or $\varepsilon \ll 1$). On this basis, although the $O(\varepsilon^2)$ term must be retained, the $O(\varepsilon^3)$ term can be discarded. Secondly, distortion is mainly cumulative. That is to say, the observation point is not close to the sound source. For periodic waves, "not close to the source" means $x \gg \lambda/(2\pi\beta)$, where $\lambda$ is the wavelength of the fundamental frequency component. For most fluids, this means that $x$ is greater than about one wavelength. The direct consequences of these two assumptions are as follows.

The relationship of linear characteristic impedance can be used. The finite displacement of the source from its rest position may be ignored, that is, for a sinusoidal source, the equation is sufficient to describe the sound source conditions. The difference between material characterization (Lagrangian) and spatial characterization (Eulerian) can be ignored. $O(\varepsilon^2)$ term and $O(\varepsilon)$ factors can be replaced by equivalent traveling waves $O(\varepsilon)$, which will introduce error term $O(\varepsilon^3)$. Substitute $\rho = \rho_0 + \rho'$ into Eq. (1.5), move $O(\varepsilon)$ term to the left side of the equation, and move $O(\varepsilon^2)$ term to the right side of the equation, the equation can be rewritten as