Research in Mathematics Education

Series Editors: Jinfa Cai · James A. Middleton

Keith Weber

Miloš Savić *Editors*

New Directions for Mathematics Education Research on Proving

Honoring the Legacy of John and Annie Selden

Springer

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Keith Weber • Miloš Savić
Editors

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Part I
Introductory Chapters

Chapter 1
The Legacy of John and Annie Selden on Proof and Proving

Keith Weber and Miloš Savić

Contents

Abstract This chapter celebrates the legacy of Annie Selden and John Selden on research in proof and proving. It provides our motivations for bringing together this book and shares the impact the Seldens had on us and the field of mathematics education. We finish the chapter with a preview of the rest of the book.

Keywords Mathematics education · Phenomenology · Proof · Proof as genre · Proof reading · Proving · Proving processes · Selden

In 2007, the first author attended a talk by John Selden. During this talk, John provided an illustration of a proof in real analysis that was easy to understand if the reader focused on symbols and logic, but difficult to understand in terms of a

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conceptual model of the real numbers. The following year, the first author approached John at a conference, informing him that he planned to build upon the content of John's talk in a future paper and asking John how his idea should be cited. This question made John visibly uncomfortable, and he could only respond, "you don't need to cite my talk. I don't care about things like that". The first author protested that he wanted to honor John's work, to which John immediately replied, "The best way to honor my work is by building on my ideas".

John Selden passed away on January 8, 2022. The impetus for this book came from our attendance at the 22nd Research in Undergraduate Mathematics Education (RUME) Conference in February of 2022. We were struck not only by the number of talks on proof, but the methodological and topical variety in the proof talks as well. In the first decade of this century, in most venues, educational research on proof largely consisted of analyzing students' epistemology on proof with a particular focus on what types of arguments students found convincing (see Harel & Sowder, 2007 for a review of this literature). At this conference, what we witnessed were thoughtful studies on how proofs were written and how proofs were read. There were analyses on the practicalities on the teaching of proof, in both traditional classroom settings and in research-based interventions. There were several papers about how students read proofs and how their understanding of those proofs could be measured. We saw that research on proof is dynamic and flourishing. In this opening chapter, we argue that this exciting research area exists because of the hard work of John Selden and Annie Selden.

The purpose of this volume is to follow John's directive for honoring his work. We want to give scholars the opportunity to build upon exciting novel directions in proof that emerged from John Selden and Annie Selden's work in this subject. Each chapter in this volume is a self-contained primer for a new direction for research on proof, containing a review of research that has been done in the area, illustrations of the authors' own work in the area, and open questions that remain. This volume is ideal for scholars who are beginning their research on proof. This includes doctoral students and professional mathematicians, as well as more seasoned mathematics educators who are looking to delve into a new research direction. However, we believe that even an experienced mathematics education researcher who has focused on proof can still learn about interesting topics, theories, and methodologies that will augment their research program.

1.1 John Selden and Annie Selden's Impact on Research on Proof

In mathematics education research, proof is widely accepted as an important topic of research. Indeed, there is a large and growing body of research on this topic. Research in proof at the undergraduate level is particularly flourishing (Stylianides et al., 2017). However, this was not always the case. In the early 1990's, some mathematics education researchers were questioning whether proof should be taught at all in the K-12 classroom (see Knuth, 2000, for a discussion of the controversy at the time). Further, during the early 1990's, outside of calculus, there was little research of any kind in undergraduate mathematics education.[1]

We believe that the reason that mathematics education researchers were reluctant to advocate teaching proof in K-12 mathematics was due to misconceptions about the nature of mathematical proof (see Hanna, 1989, and Schoenfeld, 1994, for discussion) and the influence of the theoretical perspective of radical constructivism that was widely practiced in this period (Inglis & Foster, 2018). Regarding misconceptions about the nature of proof, at the time, proof was often conceptualized by many mathematics teachers and mathematics education researchers as a very rigorous activity where the form of a proof was more important than the ideas contained in the proof. Frequently, in secondary mathematics, proofs were required to be presented in a highly regimented format, like the two-column proofs that were dominant in United States high school geometry classrooms. Some radical constructivists had little interest in engaging students in activities which required them to use institutional representation systems (as rigorous proof was perceived as requiring) without an emphasis on personal sense making. As a result, many mathematics education researchers encouraged students to engage in meaning-making activities such exploration, conjecturing, and argumentation in lieu of proof.

The work of Schoenfeld (1994), Hanna (1989, 1995), Dreyfus (1999), and others rejuvenated interest in proof. These scholars argued that, contrary to popular belief, proof, at least when done properly, is a sense-making tool that can allow students to *know* mathematics. Proof was not defined by its rigor, but as the codification of a convincing argument (Schoenfeld, 1994). With this perspective of proof in mind, radical constructivists could investigate students' conceptions of proof as their frames for what constitutes a good mathematical argument. In particular, a students' proof frame could be interpreted as the type of argument would provide them with certainty that a mathematical claim was true (e.g., Coe & Ruthven, 1994). This perspective was crystallized in Harel and Sowder's (1998) seminal research on proof schemes. The focus on students' conceptions of conviction and certainty rejuvenated mathematics education researchers' interest on proof and led to a decade of research on what types of justifications students produced and preferred. In the years after Harel and Sowder's (1998) proof schemes paper, most educational research on proof was of this type and focused on students' epistemology regarding proof (Harel & Sowder, 2007).

John Selden and Annie Selden did not approach proof as mathematics education researchers steeped in constructivist theory. Rather, their interest in proof was based on their decades of experience as mathematicians who were helping undergraduate students cope with proofs. Their questions tended to be more prosaic: What should a teacher do when a student is writing a proof and does not know what to do next? Why do students find a particular step in a proof to be hard? And what can a teacher do in real time to alleviate these difficulties?

[1] In 1987, Annie Selden and John Selden published a conference proceedings on frequent errors that undergraduates made when writing proofs in their collegiate mathematic courses (Selden & Selden, 1987). This manuscript was remarkably ahead of its time.

Their practitioner-oriented lens led to a number of insights. For instance, in their classic “proof framework” paper, Selden and Selden (1995) observed that what one assumes when writing a proof and what one must conclude depends greatly on the form of the theorem to be proven. If an individual is trying to prove a conditional statement of the form, “if P, then Q”, the individual may choose a direct proof, assuming P and then deducing Q. This initial structuring of the proof can be done without any knowledge of what P or Q means or how the proof might proceed. Similarly, if one is trying a straightforward proof of a universally quantified statement of the form “for all x, P(x)”, one may begin by choosing x to be arbitrary and concluding that P(x) must hold. The notion that setting up a proof—or in Selden and Selden’s (1995) words, choosing a proof framework—can (and often should) be done without attention to the content being proven was a tremendous insight, and went against the constructivist disposition to focus on conceptual meanings. A further insight from Selden and Selden (1995) is that the everyday language in which theorems are expressed can obscure the logical form of the theorem, masking the proof framework that one should employ to prove the theorem. For instance, the theorem that “differentiable functions are continuous” is actually a universally quantified statement, even though the universal quantification is implicit. One such equivalent statement to the theorem might be “for all real-valued functions *f*, if *f* is a differentiable function, then *f* is a continuous function”. When a theorem is expressed in this more verbose way, the proof framework may be obvious to the student, but the way that theorems are casually stated can obfuscate the proof framework that should be used. Selden and Selden (1995) demonstrated that undergraduate students are often unable to translate theorems into logical forms that suggest proof frameworks. The insights from Selden and Selden (1995) continue to shape the field today. Indeed, in the Dawkins and Vroom chapter in this volume, Dawkins and Vroom show that the language used to express existential statements influences how students engage with these statements. However, Selden and Selden’s insights were not obtained by applying deep theory to the nature of proof. Rather, they were obtained by two mathematicians puzzling why their students could not begin proofs of some theorems, even though what could be assumed and what should be proven were ostensibly mechanical and obvious.

To avoid misinterpretation, our intent in this section is not to cast shade on the more epistemologically-oriented research on proof, but to highlight how John Selden and Annie Selden’s experience as mathematicians and college teachers provided different types of insights that continue to shape the field. That difference in vision was a main driving force for their notions on mentoring newcomers (like the authors of this chapter) to tertiary mathematics education.

1.2 John Selden and Annie Selden's Impact on Mentoring

At least at conferences in undergraduate mathematics education in the United States, one cannot run into a colleague in proof research without discussing the mentoring impact John and Annie Selden had on their careers. This mentoring happened in many various ways—through questions in presentations at conferences, through sitting down and discussing concepts and ideas, and, occasionally, through virtual means via emails and video calls. John and Annie Selden received the inaugural Excellence in Mentoring and Service award from the 20th National RUME Conference. In the presentation, the authors of this chapter detailed how their interactions with John and Annie Selden shaped their careers –preparing a job application, writing and rewriting articles, handling reviews from disgruntled reviewers, and networking with the rest of the international RUME community. They meant so much to their success in the field of mathematics education.

In an article about John Selden's impact in mentoring (Dreyfus et al., 2022), Hauk detailed the amount of care John and Annie Selden had for the community. Whether it was philosophical conversations about proof research or directed conversations about methods of data collection, they always wanted what was best for the early and mid-career researcher. As you can see in the book, many invited authors are either graduate students or early-career faculty, thus continuing the essence of John and Annie Selden opening doors for others in their mentoring process.

It was their knowledge of the field that made them wonderful mentors. John and Annie Selden were people that made connections, sometimes seemingly unrelated, to proof research. One of their last forays into proof involved connecting system 1 and 2 cognition from the economist and cognitive psychologist Daniel Kahneman (2011) to their formal-rhetorical and problem-centered aspects of proving (Selden & Selden, 2013a, pp. 308–309). They were voracious readers and had knowledge of papers like a walking encyclopedia, so when they watched a presentation, they immediately had another paper or two in mind for you to include in your research. They were curious about what students were thinking and how researchers would investigate that thinking, so they asked questions of you that made you pause, reflect, and either incorporate or refute, but consider. Ultimately, they geared all their mentoring towards growth of the field of mathematics education.

It is an honor to have Annie Selden write the opening chapter in this book (Chap. 2). This chapter gives a large overview of the evolving history of proof research, ending with her thoughts on future research directions, which, again, shows how focused she is about growth in the field. After this chapter, we discuss four aspects of proof research that the chapters of the book are organized: proof writing, proof reading, proof as a genre, and affective aspects of proof.

1.3 Building on John Selden and Annie Selden's Work: The Contributions of this Volume

1.3.1 Proof Writing

Research on students' epistemology of proving has tended to focus on what *types* of arguments students produce when they are asked to justify a statement or write a proof (Stylianides et al., 2017). For instance, there are many studies demonstrating that when students are asked to justify a universal statement, they will do so by verifying the statement holds with a small number of specific examples (e.g., Recio & Godino, 2001), which of course is not an acceptable method of proof. John Selden and Annie Selden's focus was different. If students want to produce a deductive justification, how can they do so and why is this difficult? Their pioneering work distinguished between how proofs were structured by identifying what is assumed and what is proved (c.f., Selden & Selden, 1995) and the complex problem-solving that individuals must engage in to deductively link the assumptions to the conclusions (Selden & Selden, 2013a).

In Chap. 3, Daniel Sommerhoff, Ingo Kollar, and Stefan Ufer present a methodological advance to studying the competencies students need to write proofs. Prior research has primarily identified these competencies in small-scale qualitative studies, illustrating how the possession of a specific competency enables a student or mathematician to write a proof and lacking a competency prevents a student from doing so (e.g., Weber, 2001). However, while this research is useful for generating hypotheses, researchers employing this methodology often cannot justify generalizable conclusions, both because their work employs small sample size and because causal claims linking the presence or absence of a competency to performance are inherently speculative. Sommerhoff, Kollar, and Ufer present a study showing how hypotheses about the competencies needed for proving can be tested at scale using quantitative methodologies; other scholars can emulate this approach to put our understanding of proving competencies on a more secure empirical footing.

In Chap. 4, Elise Lockwood and John Caughman follow Dawkins and Karunakaran's (2016) call to look at domain-specific proving processes. Lockwood and Caughman argue that although transition-to-proof classes teach students general proving methods (e.g., proof by contradiction), these methods are used in specific mathematical domains in unique ways (e.g., proof by contradiction using the pigeonhole principle for counting arguments in combinatorics). In their careful analysis of discrete mathematics, Lockwood and Caughman identify particular competencies and understandings that students would need to successfully write (and understand) proofs in these domains. Their work also is an example that one might follow in other mathematical domains such as real analysis, abstract algebra, or topology.

In Chap. 5, Amanda Lake Heath and Sarah Bleiler-Baxter discuss theoretical frameworks for investigating collaborative problem solving. Most research on students' proving processes has conceptualized proving as an individual activity.

However, increasingly in inquiry-oriented classrooms, proving is a collaborative and social activity. The shift toward collaboration creates interesting issues, such as which group members have the authority to decide which statements are acceptable and which inferences are correct (e.g., Bleiler-Baxter et al., 2023). Heath and Bleiler-Baxter discuss how theoretical frameworks can be chosen to address these crucial issues.

In Chap. 6, Yaomingxen Lu explores how students handle impasses, or "stuck points", when proving. Lu's approach is innovative in that she does not view stuck points as avoidable obstacles to be overcome but as necessary parts of the proving process that are learning opportunities. Her work looks at general approaches that students might take that can account for success or failure. Hopefully, other researchers can follow in Lu's path to address one of the most challenging aspects of proving.

1.3.2 Proof Reading

In 2003, Annie Selden and John Selden published a paper that explored undergraduates' ability to "validate" proofs—i.e., to check a purported proof for logical correctness. This paper was highly influential in several respects, perhaps most notably in calling attention to how students read proofs. Prior to this paper, mathematics education researchers used proof reading as a lens into students' epistemology on proving (e.g., Martin & Harel, 1989). Researchers would ask students to evaluate different *types* of justifications to see what types of justifications students found convincing or believed would qualify as a proof. Selden and Selden's (2003) insight is that it is not enough for students to (only) gain certainty from deductive justifications. In fact, students should only gain certainty from *correct* deductive justification. A proof with a serious logical error is no proof at all. As Selden and Selden (2003) illustrated, checking a proof for correctness is a hard cognitive task for students.

In Chap. 7, Pablo Mejía-Ramos and his colleagues describe the utility and the psychometric features of their proof comprehension tests. This chapter highlights the recent trend in mathematics education on students' proof reading, which focuses on students' understanding of a proof. In their chapter, Mejía-Ramos et al. describe work updating their widely used model of proof comprehension at the undergraduate level (Mejia-Ramos et al., 2012) and provide an instrument that can be used at scale by other researchers to address research questions on students' proof reading and understanding.

In Chap. 8, Ben Davies provides another assessment tool to measure students' proof comprehension. He illustrates how one can use comparative judgments (e.g., Jones & Inglis, 2015) on students' proof summaries to assess proof comprehension. Davies demonstrates that his methodology can produce valid and reliable assessments of students' proof understanding. Davies' comparative judgments has an important advantage over the Mejía-Ramos et al. (2017) approach to proof comprehension. Specifically, one can use Davies' approach to measure students' understanding of *any* proof, not only the proofs for which validated proof comprehension tests have been laboriously constructed.

In Chap. 9, Kristen Amman reviews self-explanation as a pedagogical tool to improve students' proof comprehension. In a prior seminal study, Hodds et al. (2014) demonstrated that asking students to engage in self-explanation (i.e., asking and answering particular types of comprehension questions about the proofs that they are reading as they read them, see Chi et al., 1994) improves students' comprehension of the proofs that they read. Amman reviews this literature and offers several important research directions for building upon Hodds et al.'s work. In particular, there is the question of which types of self-explanation questions are effective and why.

In chap. 10, Eyob Demeke also focuses on how students can comprehend the mathematical proofs that they read. Whereas Amman focuses on leveraging the specific proof reading strategy of self-explanation, Demeke identifies other strategies that expert proof readers use to understand proofs better, and that students can be taught to use to improve their proof comprehension. This chapter offers many productive research directions for better understanding the teaching and learning of proof comprehension.

1.3.3 Proof as a Genre

In their proof validation paper, Selden and Selden (2003) noted that exclusive reliance on deductive justification was not the only characteristic of proofs that appeared in mathematical journals and undergraduate mathematics textbooks. Proofs usually contained other features that were not common in other academic texts, such as the strict avoidance of redundancy and not including a description of the process used to generate a proof. To Selden and Selden (2003), proof was a type of academic genre. In Selden and Selden (2013b), the authors gave a detailed discussion of features that were unique to proof. Three chapters in this volume build upon thinking about proof as genre.

In Chap. 11, Kristen Lew builds on her prior research (Lew & Mejía-Ramos, 2019) in which she investigated students' understanding of the norms associated with the genre of proof. In this chapter, Lew considers the issue of how students cope with the genre of proof in their educational experience. She raises several stances that students might take with respect to the genre of proof. This work is innovative, as it is one of the first studies to attend to student cognition and emotion with respect to the genre of proof. It is generative in that it suggests a research program investigating on how students adapt, or fail to adapt, to writing proofs in accordance with the norms of the genre. Lew also raises the ethical question of whether we should compel students to write proofs in a particular genre.

In Chap. 12, Stacy Brown takes a critical stance toward the norms of proof writing. She does not treat proof merely as a genre, but as a power discourse. Brown uses proof as a lens to think about the tensions of managing inquiry-oriented instruction and equity-oriented instruction. A key point that Brown raises is that larger societal issues influence classroom hierarchies and participation regardless of how a particular inquiry-oriented course is arranged. Brown's work represents an important line of research that integrates proof with broader themes of equity.

In Chap. 13, Paul Dawkins and Kristen Vroom explore research opportunities inherent in the genre itself. In the chapter, Dawkins and Vroom discuss new norms in the genre of proving that advance Selden and Selden's (2013b) work on the subject, including the use of imperatives in proofs (Weber & Tanswell, 2022) and the treatment of quantification. Further, the chapter analyzes the dialectic interaction between the norms of proof as a genre and the mathematical practices that they support and invite. The research programs go beyond identifying what the genre of proof is and how students understand it, but opens up research questions about the affordances of this genre.

1.3.4 Affective Aspects of Proving

There has been little research in mathematics education looking at the intersection between affect and proving. Perhaps this is because proving is typically viewed as a logical and rational activity, and indeed proofs are written to be impersonal (e.g., Dawkins & Weber, 2017). In an innovative paper, Selden et al. (2010) explored the role of affect in proving. Their report was based on John Selden and Annie Selden's experiences working with struggling students, where they noted that students' feelings of rightness, familiarity, trepidation, or confusion guided their students' behavior. We agree with Selden, McKee, and Selden that students' affective states are inseparable from their cognition as they work on proofs. Two chapters further this work on affect and proof.

In Chap. 14, Rani Satyam emphasizes the role that affect plays in students' persistence in studying proof. This is a point often overlooked in mathematics education teaching and research. If undergraduates do not *enjoy* studying proof, they are unlikely to seriously engage with their advanced mathematical content. Satyam presents her novel approach for measuring students' affect during the proving process, a methodological tool that can be used by others to address a wide range of questions about the relationship between affect and proof.

In Chap. 15, Sheena Tan and Nathalie Sinclair open up an investigation into the aesthetics of proving. Tan and Sinclair note that in mathematical practice, proofs are not only evaluated for their correctness. They are also evaluated on aesthetic qualities, such as beauty, elegance, simplicity, and efficiency. Tan and Sinclair propose a framework for analyzing proof's aesthetic qualities, by thinking about the ways that the aesthetics are dependent (i.e., depends on the ways in which a reader interprets and engages with the proof text) and autonomous (i.e., dependent upon the inner logic of mathematics). They also propose a program in which the teaching of proof can be framed as an enculturation into aesthetics, which can give much more focus on the aesthetics of proof in mathematics education research that are typically ignored by purely cognitive approaches.

The volume concludes with Chap. 16, where two experts in proving and undergraduate mathematics education, Lara Alcock and Matthew Inglis, offer their perspective on the chapters in this volume.

1.4 Conclusion

As the opening chapter of this volume illustrates, mathematics education research on proof is a thriving field. Researchers have built upon Selden and Selden's classic papers that established proof writing, proof reading, proof as a genre, and affect aspects of proving to provide new directions for educational research on proof. We are honoring their work in this volume by inviting other scholars to continue to build on John Selden and Annie Selden's research.

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Chapter 2
My Take: Proof Research at the Undergraduate Level: How it Has Evolved

Annie Selden

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Abstract This is my personal perspective on, and review of, the mathematics education research literature on proof at the undergraduate level from the early days of the late1980s to the present. Much of the initial research was on what students could not do in proving. Later in the early 2000s, the research broadened to include proof validation, proof comprehension, affect during proving, and the teaching of proof. From approximately 2010 onward, research expanded further to include such topics as mathematicians' grading of proofs, the IBL teaching of proving, expert versus novice reading of proofs, the use of examples in proving, the genre of proof, and attempts to help university students learn how to read and write proofs. Lastly, there is a brief speculation on future directions.

Keywords Literature review · Mathematics education · Proof · Proof comprehension · Proof validation · Proof construction · Proving processes

2.1 Early Days: The Beginnings of Mathematics Education Research on Proof at the Undergraduate Level

Although there had been mathematics education research on proof more generally (e.g., Bell, 1976; Fischbein, 1982), there was little mathematics education research on proof at the undergraduate level until about the mid-1980s. John and my first foray into such research began in the mid-70 s when we were teaching at the University of the Bosphorus in Istanbul, Turkey, and submitted an article on logical reasoning errors that students made in an abstract algebra course taught Moore Method, published in a local journal (Selden & Selden, 1978). This article was later recast in terms of misconceptions research and presented at a Cornell University conference (Selden & Selden, 1987), and is sometimes cited today. At about the same time, unbeknownst to us, Ed Dubinsky was researching topics on undergraduates' understandings of logic (Dubinsky, 1987, 1989; Dubinsky et al., 1988). As far as I can tell, many of these publications were based on personal observations of teaching undergraduates. Research on, and interest in undergraduate students' knowledge of, and use of, logic in proving has continued to the present time (e.g., Dawkins & Norton, 2022; Durand-Guerrier et al., 2012; Savic, 2012).

At about the same time, Gila Hanna, whose mathematics education research has often focused on the more theoretical and philosophical aspects of proof and proving, made the now well-known and influential distinction between "proofs that only proof and proofs that explain" (Hanna, 1989, 1990). However, Hanna's later research has sometimes been empirical and focused on undergraduates (e.g., Yan et al., 2019). Also, Michael de Villiers, whose mathematics education research has often focused on the teaching and learning of geometry, early on proposed five functions of proof which are still quoted today; namely, proof as means of: (1) *verification/conviction*; (2) *explanation*; (3) *systematization*; (4) *discovery*; and (5) *communication* (de Villiers, 1990). De Villiers states that this analysis was based on epistemological considerations and personal testimonies of practicing mathematicians. Still, a reading of the article suggests it was also based on insightful observations of his teaching and reading of the available literature (e.g., Alibert, 1988; Freudenthal, 1973; Hanna, 1989; Lakatos, 1976; Wilder, 1944).

Another early foray into mathematics education research at the undergraduate level came with the formation in 1985 of the Advanced Mathematical Thinking Working Group of the International Group for the Psychology of Mathematics Education (PME). The idea behind this Working Group was to focus on "advanced mathematical thinking", in contrast to much of the prior work of PME, which had concentrated on "elementary mathematical thinking" (Harel et al., 2006, p. 147). Because the members of this Working Group came from a variety of countries, for convenience, it was decided that the term "advanced mathematical thinking" would be used for education beyond the compulsory stage, which at that time often concluded at age 16 and certainly included mathematics education research at the undergraduate level. A major undertaking of this Working Group was the writing of a book on advanced mathematical thinking (Tall, 1991); relevant to early research on proof, there were book chapters specifically on proof (Hanna; Alibert & Thomas), on the role of definitions, including a short discussion of concept image and concept definition (Vinner), and on research in the teaching and learning of mathematics at an advanced level (Robert & Schwarenberger). Other researchers have since provided their own definitions/descriptions of advanced mathematical thinking (e.g., in a special issue of *Mathematical Thinking and Learning* edited by Selden & Selden, 2005).

2.1.1 Experiencing Proof and Proving in the Classroom, the Method of Scientific Debate, and Structured Proofs

While not about proof at the undergraduate level, Balacheff (1988), in an empirical study of French secondary pupils' proving, categorized their arguments into four different types and argued that these represented four increasingly sophisticated levels of thinking: (1) *Naïve empiricism*, in which an individual arrives at a conclusion on the validity of an assertion based on a small number of confirming cases; (2) *Crucial experiment*, in which an individual considers the possibility of generalization by examining a case that is not very specific; (3) *Generic example*; in which the proof rests upon properties which are a generalization of a class of examples; and

(4) *Thought experiment*, in which an individual can distance themselves from specific actions and make logical deductions based only on knowledge of the properties and relationships characteristic of the situation. These categorizations are sometimes still referred to today (e.g., Varghese, 2011).

The work of Alibert and Thomas (1991) on the *method of scientific debate* in French university classrooms was an attempt to get undergraduates to understand proofs, because they had observed that the students had difficulty understanding proofs when they read through textbook proofs in their strict formal, linear order. They wanted to "enable students to see proof as a necessary part of the scientific process of advancing knowledge, rather than just a formal exercise to be done for the teacher," (Alibert & Thomas, 1991, p. 224). They described three steps in generating a scientific deate: (1) The teacher initiates and organizes the production of scientific statements [conjectures] by the students. (2) The students then provide support for the conjecture, or not, by scientific argument, proof, refutation, or counter-example. (3) The statements that can be confirmed by a full demonstration become theorems accepted by the class. (Alibert & Thomas, 1991, p. 225).

Another attempt to get away from the strict linear order of presenting proofs to university mathematics students was a structural method suggested by Uri Leron (1983). He wrote.

> The method, triggered by recent ideas from computer science, is intended to increase the comprehensibility of mathematical presentations while retaining their rigor. The basic idea underlying the structural method is to arrange the proof in levels, proceeding from the top down; the levels themselves consist of short autonomous "modules," each embodying one major idea of the proof. (Leron, 1983, p. 174).

While this structural presentation method was long thought to be a plausible one, it was "put to the test" empirically much later in a qualitative study that presented structured proo8fs to university mathematics students to see how they "read and perceived this type of proof presentation. Although some students valued the summaries contained in structured proofs, many complained that structured proofs 'jumped around' and required them to scan different parts of the proof to coordinate information." (Fuller et al., 2014, p. 1).

2.2 Into the 1990s: Further Developments in Mathematics Education Research on Proof at the Undergraduate Level

The 1990s saw a gradual flowering of mathematics education research at the undergraduate level. While much of this research concentrated on the teaching/learning of individual subjects (e.g., calculus, see Selden et al., 1989; linear algebra, see Dorier, 1998) or on mathematical concept acquisition more generally (e.g., functions, see Dubinksy & Harel, 1992), some of it was devoted to proof and proving at the undergraduate level.

2.2.1 *Undergraduate Students' Difficulties in Constructing Proofs*

For example, John and I published research on undergraduate students' understanding of, and ability to unpack, logical statements (Selden & Selden, 1995). That study focused on transition-to-proof course students' ability to unpack informally written mathematical statements into the language of predicate calculus. For "simplified informal calculus statements, just 8.5% of unpacking attempts were successful; for actual statements from calculus texts, this dropped to 5%." We inferred that "these students would be unable to reliably relate informally stated theorems with the top-level logical structure of their proofs and hence could not be expected to construct proofs or validate them, i.e., determine their correctness." (Selden & Selden, 1995, p. 123).

Like much of the research on proof and proving in this time period, our research focused on what undergraduate students couldn't do (Selden & Selden, 1987, 1995). Perhaps the underlying motivation for doing such research was the idea that research should obtain some baseline information on what students were currently learning, and could do in the way of proving and problem solving, with then current university teaching. For example, Robert Moore did his dissertation research at the University of Georgia on transition-to-proof course students' understandings of formal proof. He found that "An inductive analysis of the data revealed three major sources of the students' difficulties: (a) concept understanding, (b) mathematical language and notation, and (c) getting started on a proof." (Moore, 1994, p. 249). Somewhat later in the decade, Keith Weber conducted his dissertation study at Carnegie-Mellon University on undergraduate and doctoral abstract algebra students' proving—he documented that the undergraduates in the study were unable to apply facts they knew to prove theorems on groups. He hypothesized that the undergraduates failed "to construct a proof because they could not use the syntactic knowledge that they had." In contrast, the doctoral students "appeared to know the powerful proof techniques in abstract algebra, which theorems are most important, when particular facts and theorems are likely to be useful, and when one should or should not try and prove theorems using symbol manipulation." (Weber, 2001, p. 101).

2.2.2 *Undergraduate Students' Proof Schemes*

There was also work in the 1990s on undergraduate students' ideas of proof and proving, which included work on how they construct proofs and what they see as constituting a proof. In particular, Harel and Sowder (1998) classified students' *proof schemes*. By *proving* they meant "*the process employed by an individual to remove or create doubts about the truth of an observation* [conjecture]." They described the process of proving as consisting of two subprocesess. (1) "*Ascertaining is the process employed by an individual to remove or create doubts about the truth of an observation* [conjecture]." (2) "*Persuading is the process an individual* *employs to remove others' doubts about the truth of an observation* [conjecture]." (Harel & Sowder, 1998, p. 241, italics in the original). They described, and gave examples of, three overarching proof scheme categories: (1) *External conviction proof schemes*—basically it's a proof if someone knowledgeable or a textbook tells you it's a proof; (2) *Empirical proof schemes*—checking enough cases to convince you that the conjecture is true; and (3) *Analytical proof schemes*—broken down further into *Transformational proof schemes* and *Axiomatic proof schemes*—basically proofs that would be acceptable to the mathematical community. The notion of proof schemes has been, and is sometimes still being, used (e.g., Erikson & Lockwood, 2021; Housman & Porter, 2003).

2.2.3 Establishment of RUME Organizations

By 1998, there was enough research on the teaching and learning of university mathematics to convene an ICMI Study Conference in Singapore, with the actual book published a bit later (Holton, 2003). While there are book chapters on research into the teaching/learning of calculus and linear algebra, on the secondary-tertiary transition, on APOS (Action, Process, Object, Schema) theory, and 51 entries on proof in the index, no single chapter was devoted to proof at the undergraduate level, suggesting that the organizers did not yet see, or know about, significant research on proof and proving at the undergraduate level.

However, in the USA, the beginnings of research in undergraduate mathematics education (RUME) were emerging, as indicated by the three conferences on RUME in 1996, '97, and '98, organized by the RUMEC (Research in Undergraduate Mathematics Education Community) group under the leadership of Ed Dubinsky. This was followed in 2001 by the formation of the first Special Interest Group of the Mathematical Association of America on Research on Undergraduate Mathematics Education (SIGMAA on RUME). The SIGMAA on RUME continues the tradition of holding annual RUME Conferences, at which there are always presentations on proof, proving, or proof comprehension (e.g., Moore et al., 2016).

2.3 In the Early 2000s, Mathematics Education Research on Proof Started to Broaden to Consider Topics Such as Proof Validation, the Teaching of Proof, Affect during Proving, and Proof Comprehension

2.3.1 Early Validation Studies

Our early exploratory study of eight transition-to-proof course students' ability to validate, that is, to determine the correctness of, proofs investigated how these students "read and reflected on four student-generated arguments purported to be proofs of a single [elementary number theory] theorem." (Selden & Selden, 2003, p. 4). This was not our very first foray into validation—we had provided a sample, hypothetical proof validation of a calculus theorem in Appendix 1 of our "unpacking paper" (Selden & Selden, 1995, pp. 143–147). In this study of transition-to-proof course students' ability to correctly validate other similar students' proof attempts, we found it to be at chance level. Subsequently, a number of other mathematics education researchers took up proof validation studies (e.g., Alcock & Weber, 2005). Building on this research, further studies examined how mathematicians validate proofs (Weber, 2008) and how mathematicians read proofs (Weber & Mejia-Ramos, 2011). Even later, proof validation research was subsumed by some under the more general heading of proof comprehension research—how individuals read, comprehend, and learn from presented proofs. However, validating a proof attempt, either one's own or another person's, seems to require somewhat different skills than comprehending a presented proof, whether in a text or a lecture, although there is some overlap. (Selden & Selden, 2015, p. 343).

2.3.2 Early Research on the Teaching of Proof at the Undergraduate Level

The teaching of proof at the undergraduate level, while not much researched up to this time, was investigated by Keith Weber (2004) in his study of the entirety of one professor's introductory real analysis course. He identified three separate teaching styles that the professor used: (1) a *logico-structural teaching style* in the case of sets and functions; (2) a *procedural teaching style* in the case of limits of sequences; and (3) a *semantic teaching style* in the case of elementary topological concepts like interior point of a set. This study revealed, amongst other things, that the popular, stereotypical idea that the teaching of university advanced mathematics courses, such as real analysis, consisted of "entirely of definition, theorem, proof, definition, theorem, proof, in solemn and unrelieved concatenation" (Davis & Hersh, 1981, p. 151) was too narrow and needed further investigation.

Additional studies of university mathematics teaching were not taken up until after it was pointed out that the teaching of proof-based university mathematics courses was in need of investigation (Speer et al., 2010). A recent literature review of 104 published papers reporting research on the teaching of proof-based mathematics courses at university considered both lecture-based and student-centered pedagogies. For each type of instruction, the authors described the instruction, instructor beliefs and rationales, and the relationship between instruction and students' learning. (Melhuish, Fukawa-Connelly, et al., 2022b). The authors noted that often the studies were hard to compare due to the use of different theoretical frameworks. They also observed that there are still too few studies today that attempt to link instructors' teaching of proof to university students' learning.

2.3.3 *The Beginnings of Incorporating Affect into Research on Undergraduates' Proving*

In this decade, we began thinking about how affect, in particular consciousness and non-emotional cognitive feelings, might be involved in the proving process (e.g., Selden et al., 2008, 2010). Previously affect had often been viewed as separate from, but related to cognition; affect had also been divided into considerations of *beliefs*, *attitudes*, and *emotions*. These were described as being of increasing intensity and decreasing stability, with emotions the most intense (McLeod, 1989). At this time, we were not the only ones who were thinking, and writing about the role of affect and its relation to cognition; for example, DeBellis and Goldin (2006) considered the relation of affect to mathematical problem solving, adding a fourth component, namely, *values*.

2.3.3.1 Our Take on Affect in Proving

In our paper, we focused on "a particular kind of affect—*nonemotional cognitive feelings*—and on the implementation of actions via behavioural schemas." (Selden et al., 2010, p. 199). Some examples of such nonemotional cognitive feelings are "feelings of knowing, of caution, of familiarity, of confusion, of not knowing what to do next, of rightness/appropriateness, of rightness/direction or of rightness/summation." (p. 202). Feelings can provide information, which can be positive or negative. Indeed "the feeling in constructing a proof that one is 'on the right track' is a cognitive feeling." (p. 203). Such feelings can lead to actions via *behavioural schemas*, a notion for which we provided a six-point theoretical sketch:

1. Behavioural schemas are immediately available. They do not normally have to be searched for, which distinguishes them from most conceptual knowledge and episodic and declarative memory.
2. Simple behavioural schemas operate outside of consciousness. One is not aware of doing anything immediately prior to the resulting action—one just does it. They are not under conscious control.
3. Behavioural schemas tend to produce immediate action—one becomes conscious of the action as it occurs or immediately after it occurs
4. Behavioural schemas cannot 'chained together'—they function entirely outside of consciousness and one is consciousness of only the final action.
5. An action due to a behavioural schema depends on conscious input, at least in large part.
6. Behavioural schemas are acquired through, possibly tacit, practice, that is, to acquire a beneficial schema a person should actually carry it a number of times—not just understand its appropriateness. Changing a detrimental behavioural schema requires similar, perhaps longer, practice. (This is a summary of Selden et al., 2010, pp. 205–206).

Following this description, we illustrated a number of actual examples of behavioural schemas that we had observed in undergraduate proving situations.

While there has been much research on what one might call "hot" emotions, such as mathematical anxiety (e.g., Rozgonjuk et al., 2020), there has been less research on more useful and calm emotions. One recent example of such research concerns undergraduate transition-to-proof course students' satisfying moments, including "understanding, overcoming challenges, and accomplishments without struggle". (Satyam, 2020, abstract).

Also, while there has been a great deal of international interest on affect in mathematics education research (e.g., see the ICME-13 monograph edited by Hannula et al., 2019), most research deals with affect during problem solving, rather than proving, although constructing a proof can be seen as a kind of problem solving.

2.3.4 Observing that Proof Comprehension Is under Researched

At the 19th ICMI Study Conference on Proof and Proving in 2009, Juan Pablo Mejia-Ramos and Matthew Inglis presented the results of a bibliographic study of the "different argumentative activities associated with the notion of mathematical proof" that had been done up to that time. In their sample of 131 empirical research articles (pared down from an original 641), they found that "of those articles in our sample that discussed specific tasks, the majority (82 papers) addressed students' construction of novel arguments, some (24 papers) involved students' reading of given arguments and none focused on the presentation of a given argument. In particular, only 3 articles addressed tasks related to the comprehension of a given argument and none of the articles discussed tasks directly focussed on the presentation of an argument to demonstrate students' understanding of it." (Mejía-Ramos & Inglis, 2009, p. 91).

This observation subsequently led to a burgeoning of research on proof comprehension in the next decade, beginning around 2010, much of it done by the Proof Comprehension Research Group at Rutgers University.

2.3.5 An Assessment Model for Proof Comprehension Research

Having established that proof comprehension was under researched (Mejía-Ramos & Inglis, 2009), Juan Pablo Meija-Ramos and others embarked on proof comprehension research, beginning with the development of a multi-dimensional assessment model for proof comprehension at the undergraduate level (Mejía Ramos et al., 2012). The authors contended that.