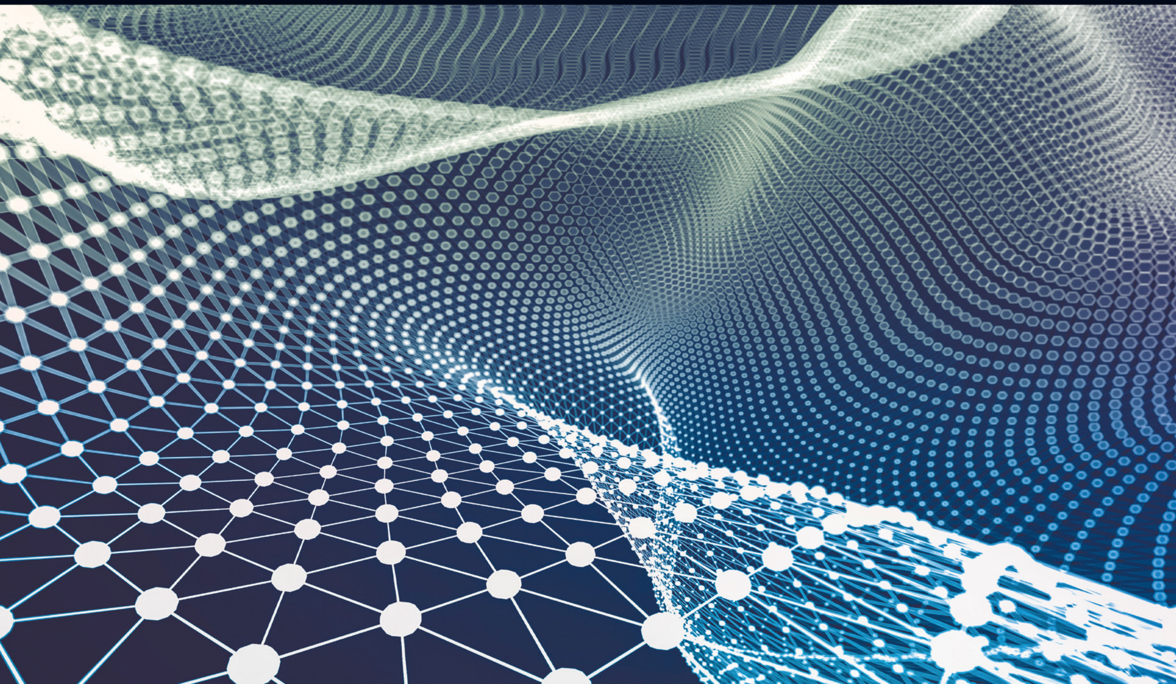
MECHANICAL ENGINEERING AND SOLID MECHANICS SERIES

Rheology, Physical and Mechanical Behavior of Materials 3

Rigidity and Resistance of Materials, Sizings, Pieces and Structures

Maurice Leroy



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Contents

Preface

In the case of relatively low loads, the deformation mechanisms for materials, parts, and structures are reversible, and the elastic deformations are proportional to the stresses (with E, Young's modulus of elasticity).

In the case of complex loads, Hooke's law is generalized into a three-dimensional relationship, and the linear nature of this law results in the following superposition principle: the stresses or deformations produced by the sum of several loading states on an elastic solid are equal to the sum of the stresses or deformations generated by each of the load states applied in isolation to the solid.

If the stress exceeds a certain value σ_e (or R_e, σ_0, Y), known as the elasticity limit stress, the phenomenon ceases to be reversible and linear, and the theory of elasticity can no longer be applied. This limit may be difficult to demonstrate experimentally. It is conventionally defined as being the stress that generates an irreversible deformation close to 0.2%.

For three-dimensional loads, different sets of criteria for yield strength will define the corresponding domain in the stress space. These include the Tresca and Von Mises criteria, while Hill's criteria are suitable for composites, and are often used in the calculations to determine the scale of parts and structures.

In the case that the elastic limit is crossed, slips will occur within the materials (dislocation in the crystals) and irreversible and permanent deformations can occur (in the plasticity domain).

The stress on metals at a temperature exceeding about one-third of the absolute melting temperature has the property of deforming even if the stress remains constant: this phenomenon is known as creep (untreated), which translates into a viscoplastic deformation.

In principle, the laws of elastic, plastic and viscoplastic behavior associated with the equations describing the mechanics of continuous mediums make it possible to calculate the stresses and deformations in parts and structures.

In many cases, it is sufficient to use the theory of elasticity, with the dimension criteria used to address safety concerns for the determination of the maximum permissible stress and/or maximum deformation.

The criteria for ruptures make use of other theories.

Structure of this book

Chapter 1 provides a review of the concepts of rigidity, resistance and elastic energy, as well as stress-strain relationships and domains. *Chapter 2* covers the use of scaling criteria for isotropic and anisotropic materials. *Chapters 3* and *4* address the elastic mechanics of parts and structures. *Chapter 5* covers elastic limit deflections and plastic hinges and *Chapter 6* covers shearing and shear force.

NOTE.– The full and detailed *Appendix* to this book provides diagrams of the 100 examples covered throughout the book and the relevant page numbers and is available to download from www.iste.co.uk/leroy/rheology3.zip.

Maurice LEROY
October 2024

1

Elasticity, Rigidity

1.1. Elasticity and rigidity tensors

1.1.1. *Hooke's law*

In the domain of low elastic deformations, the deformation is proportional to the stress:

$$\varepsilon = S\,\sigma$$

where S is called elasticity. Similarly:

$$\sigma = C\,\varepsilon$$

with $C = S^{-1}$, where C is the rigidity.

Hooke's law consists of nine equations with nine terms, totaling 81 S_{ijkl} coefficients.

Through taking into account the physical phenomena, we can reduce the number of these coefficients to 36.

σ_{ij} is a symmetric tensor, thus $S_{ijkl} = S_{ijlk}$, and ε_{ij} is a symmetric tensor, thus, $S_{ijkl} = S_{ijlk}$.

1.1.2. *Matrix notation*

The tensors $[\sigma_{ij}]$ and $[\varepsilon_{ij}]$ are symmetrical. The notation can be simplified by adopting the following equivalences:

– For $[\sigma_{ij}]$:

	xx	yy	zz	yz, zy	zx, xz	xy, yx
Tensor notation	11	22	33	23.32	31.13	12.21
Matrix notation	1	2	3	4	5	6

Table 1.1. *Tensor and matrix notations of* $[\sigma_{ij}]$

In other words, six coefficients are given as:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \longleftrightarrow \begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{bmatrix}$$

– For $[\varepsilon_{ij}]$ in relation to the strain tensor $[e_{ij}]$, we obtain:

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \longleftrightarrow \begin{bmatrix} \varepsilon_1 & \frac{1}{2}\varepsilon_6 & \frac{1}{2}\varepsilon_5 \\ \frac{1}{2}\varepsilon_6 & \varepsilon_2 & \frac{1}{2}\varepsilon_4 \\ \frac{1}{2}\varepsilon_5 & \frac{1}{2}\varepsilon_4 & \varepsilon_3 \end{bmatrix}$$

In S_{ijkl}, the first two indices are contained in a single variant from 1 to 6 and the same is done for the last two indices. That is, $S_{ijkl} = S_{mn}$. The factors 1, 2 and 4 are introduced at the same time as follows:

– if m and n have the values of 1, 2, 3, $S_{ijkl} = S_{mn}$;

– if m or n have the values of 4, 5, 6, $S_{ijkl} = \frac{1}{2} S_{mn}$;

– if m and n have the values of 4, 5, 6, $S_{ijkl} = \frac{1}{4} S_{mn}$.

1.1.3. *Relationships between stresses and strains for isotropic bodies*

In the classic works on elasticity, the relationships between the stresses and the strains are expressed as a function of different quantities of the coefficients S_{ij} or C_{ij}.

These are as follows:

– Young's modulus: E: $\frac{\Delta l}{l_0} = \frac{1}{E}\frac{F}{S}$. This equation gives the relative elongation of a rod of section S subjected to an axial force F.

– Poisson's ratio ν. At the same time as a cylinder lengthens, it also narrows: $\frac{\Delta a}{a_0} = \nu \frac{\Delta l}{l_0}$.

– The rigidity modulus, G: $G = \frac{E}{2\,(1+\nu)}$.

– The Lamé coefficients, λ and μ: $\lambda = \frac{\nu}{(1+\nu)(1-2\nu)} E$, $\mu = \frac{1}{2}\frac{E}{(1+\nu)}$.

To establish the relationships between the coefficients S_{ij} and C_{ij} and the classic coefficients, we will develop the matrices that we have obtained for the isotropic solids and compare them with the classic equations.

Classic expressions	**Matrix notations**
$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu\,(\sigma_2 + \sigma_3)]$	$\varepsilon_1 = S_{11}\,\sigma_1 + S_{12}\,\sigma_2 + S_{13}\sigma_3$
$\varepsilon_2 = \frac{1}{E}[\sigma_2 - \nu\,(\sigma_3 + \sigma_1)]$	$\varepsilon_2 = S_{12}\,\sigma_1 + S_{11}\,\sigma_2 + S_{12}\sigma_3$
$\varepsilon_3 = \frac{1}{E}[\sigma_3 - \nu\,(\sigma_1 + \sigma_2)]$	$\varepsilon_3 = S_{12}\,\sigma_1 + S_{12}\,\sigma_2 + S_{11}\sigma_3$
$\varepsilon_4 = \frac{1}{G}\sigma_4$	$\varepsilon_4 = 2\,(S_{11} - S_{12})\,\sigma_4$
$\varepsilon_5 = \frac{1}{G}\sigma_5$	$\varepsilon_5 = 2\,(S_{11} - S_{12})\,\sigma_5$
$\varepsilon_6 = \frac{1}{G}\sigma_6$	$\varepsilon_6 = 2\,(S_{11} - S_{12})\,\sigma_6$

Table 1.2. *Classical expressions and notations matrix of deformations*

The comparison of the coefficients gives:

$$S_{11} = \frac{1}{E} \qquad S_{12} = -\frac{\nu}{E}$$

and:

$$2\,(S_{11} - S_{12}) = \frac{1}{G}$$

and thus the equation $G = E / [2\,(1+\nu)]$.

We express the stresses as a function of the strains (Table 1.3).

Classic expressions	Matrix notations
$\sigma_1 = (2\mu + \lambda)\,\varepsilon_1 + \lambda\,\varepsilon_2 + \lambda\,\varepsilon_3$	$\sigma_1 = C_{11}\,\varepsilon_1 + C_{12}\,\varepsilon_2 + C_{12}\,\varepsilon_3$
$\sigma_2 = \lambda\,\varepsilon_1 + (2\mu + \lambda)\,\varepsilon_2 + \lambda\,\varepsilon_3$	$\sigma_2 = C_{12}\,\varepsilon_1 + C_{11}\,\varepsilon_2 + C_{12}\,\varepsilon_3$
$\sigma_3 = \lambda\,\varepsilon_1 + \lambda\,\varepsilon_2 + (2\,\mu + \lambda)\,\varepsilon_3$	$\sigma_3 = C_{12}\,\varepsilon_1 + C_{12}\,\varepsilon_2 + C_{11}\,\varepsilon_3$
$\sigma_4 = \mu\,\varepsilon_4$	$\sigma_4 = \frac{1}{2}(C_{11} - C_{12})\,\varepsilon_4$
$\sigma_5 = \mu\,\varepsilon_5$	$\sigma_5 = \frac{1}{2}(C_{11} - C_{12})\,\varepsilon_5$
$\sigma_6 = \mu\,\varepsilon_6$	$\sigma_6 = \frac{1}{2}(C_{11} - C_{12})\,\varepsilon_6$

Table 1.3. *Classical expressions and matrix notations of strains*

According to the results of Table 1.3, we obtain:

$$C_{11} = 2\,\mu + \lambda \quad \text{and} \quad C_{12} = \lambda$$

1.1.4. *Tensors [σ] and [ε] and deviators*

The values of the traces in stresses and strains are as follows:

– tr [σ] = $-$ 3 p or $-$ p = σ_m, with σ_m, the average stress;

– tr [ε] = Θ dilatation with $\Theta/3 = \varepsilon_m$, the average strain.

In the primary reference area, we have:

$$\underset{\text{tr} = -\,3p}{\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_1 - \sigma_m & 0 & 0 \\ 0 & \sigma_{II} - \sigma_m & 0 \\ 0 & 0 & \sigma_{III} - \sigma_m \end{bmatrix}$$

or:

$$\underset{\text{tr} = 0}{\begin{bmatrix} S_I & 0 & 0 \\ 0 & S_{II} & 0 \\ 0 & 0 & S_{III} \end{bmatrix}} \Longrightarrow \text{Notation for the deviator [S]}$$

$$\underset{\text{tr} = \Theta}{\begin{bmatrix} \varepsilon_I & 0 & 0 \\ 0 & \varepsilon_{II} & 0 \\ 0 & 0 & \varepsilon_{III} \end{bmatrix}} = \begin{bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{bmatrix} + \begin{bmatrix} \varepsilon_1 - \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_{II} - \varepsilon_m & 0 \\ 0 & 0 & \sigma_{III} - \sigma_m \end{bmatrix}$$

or:

$$\underset{tr=0}{\begin{bmatrix} e_I & 0 & 0 \\ 0 & e_{II} & 0 \\ 0 & 0 & e_{III} \end{bmatrix}} \Longrightarrow \text{ Deviator } [e]$$

1.1.4.1. *The case of isotropic materials in elasticity in the main reference area*

$$\begin{bmatrix} \sigma_I \\ \sigma_{II} \\ \sigma_{III} \end{bmatrix} = \underbrace{\begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}}_{\textit{Elastic stiffness matrix for isotropes}} \begin{bmatrix} \varepsilon_I \\ \varepsilon_{II} \\ \varepsilon_{III} \end{bmatrix}$$

Thus, we have:

$$\sigma_I = a\, \varepsilon_I + b\, (\varepsilon_{II} + \varepsilon_{III})$$

or:

$$\sigma_I = (a - b)\ \varepsilon_I + b\, \underbrace{(\varepsilon_I + \varepsilon_{II} + \varepsilon_{III})}_{\Theta}$$

$$\sigma_I = (a - b)\ \varepsilon_I + b\, \Theta$$

We will establish that $\begin{cases} a - b = 2G \\ b = \lambda \end{cases}$.

We obtain the Lamé coefficients in the stresses:

$$\begin{cases} \sigma_I = 2G\ \varepsilon_I + \lambda\, \Theta = 2G\ \varepsilon_I + 3\ \lambda\, \varepsilon_m \\ \sigma_{II} = 2G\ \varepsilon_{II} + \lambda\, \Theta = 2G\ \varepsilon_{II} + 3\ \lambda\, \varepsilon_m \\ \sigma_{III} = 2G\ \varepsilon_{III} + \lambda\, \Theta = 2G\ \varepsilon_{III} + 3\ \lambda\, \varepsilon_m \end{cases} \quad [1.1]$$

$$\rightarrow \underbrace{(\sigma_I + \sigma_{II} + \sigma_{III})}_{tr\,[\sigma]} = 2G\, \Theta + 3\, \lambda\, \Theta = (2G + 3\, \lambda)\ \Theta \rightarrow\ tr\,[\varepsilon]$$

and thus:

$$\cancel{3}\,\sigma_m = (2\,G + 3\lambda)\,\cancel{3}\,\varepsilon_m$$

$$\sigma_m = (2\,G + 3\lambda)\,\varepsilon_m \qquad [1.2]$$

$$[1.1] \rightarrow \ \sigma_I = 2\,G\,\varepsilon_I + 3\lambda\,\varepsilon_m$$

$$[1.1] - [1.2] \rightarrow \ \sigma_I - \sigma_m = 2G\,\varepsilon_I + 3\lambda\,\varepsilon_m - 2G\,\varepsilon_m - 3\lambda\,\varepsilon_m$$

$$= 2G\,(\varepsilon_I - \varepsilon_m)$$

and thus:

$$S_I = 2\,G\,e_I \qquad [1.3]$$

The same is attained by permutation for S_{II} and S_{III}, hence:

$$[S] = 2\,G\,[e] \qquad [1.4]$$

1.1.4.1.1. Relationship between stress and strain deviators

The relationship between the stress and strain deviators is independent of the reference area in use. Thus, for any reference area x, y, z, we obtain:

$$\begin{bmatrix} (\sigma_{xx} - \sigma_m) & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & (\sigma_{yy} - \sigma_m) & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & (\sigma_{zz} - \sigma_m) \end{bmatrix} = 2G \begin{bmatrix} (\varepsilon_{xx} - \varepsilon_m) & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & (\varepsilon_{yy} - \varepsilon_m) & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & (\varepsilon_{zz} - \varepsilon_m) \end{bmatrix}$$

and thus, for example:

a) $\sigma_{xx} - \sigma_m = 2G\,(\varepsilon_{xx} - \varepsilon_m)$

but according to [1.2]:

$$\sigma_{xx} = 2\,G\,\varepsilon_{xx} - 2\,G\,\varepsilon_m + \sigma_m$$

$$= 2\,G\,\varepsilon_{xx} \cancel{-2\,G}\,\varepsilon_m \cancel{+2\,G}\,\varepsilon_m + 3\,\lambda\,\varepsilon_m$$

$$\Rightarrow \sigma_{xx} = 2\,G\,\varepsilon_{xx} + 3\,\lambda\,\varepsilon_m$$

and two other relationships via permutation:

b) $\sigma_{xy} = 2\,G\,\varepsilon_{xy}$

and thus:

$$\sigma_{ij} = 2\,G\,\varepsilon_{ij} + \delta_{ij}\,3\,\lambda\,\varepsilon_m \qquad [1.5]$$

$$i = j \Leftrightarrow x, y, z \quad \text{and} \quad \begin{cases} \delta_{ij} = 0 \text{ for } i \neq j \\ \delta_{ij} = 1 \text{ for } i = j \end{cases}$$

– From [1.5]: shear strain for i ≠ j equal to:

$$2\,\varepsilon_{xy} = \frac{\sigma_{xy}}{G} = \gamma_{xy}$$

$$2\,\varepsilon_{yz} = \frac{\sigma_{yz}}{G} = \gamma_{yz}$$

$$2\,\varepsilon_{zx} = \frac{\sigma_{zx}}{G} = \gamma_{zx}$$

with G, the elastic modulus for shearing (or transverse) (or μ) in Pascals.

– From [1.5]: the elongation strain is equal to:

$$\begin{cases} +\text{: traction} \\ -\text{: compression} \end{cases}$$

EXAMPLE 1.1.–

We have that i = j:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{2\,G} - \frac{3\,\lambda\,\varepsilon_m}{2\,G}$$

but [1.2] gives:

$$3\,\varepsilon_m = 3\,\frac{\sigma_m}{2G + 3\lambda} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{2G + 3\lambda}$$

and thus:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{2\,G} - \frac{3\,\lambda}{2\,G}\,\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{2G + 3\lambda}$$

or:

$$\varepsilon_{xx} = \underbrace{\frac{\lambda + G}{G\,(2G + 3\lambda)}}_{1/E}\ \sigma_{xx} - \underbrace{\frac{\lambda}{2G\,(2G + 3\lambda)}}_{\nu/E}\left(\sigma_{yy} + \sigma_{zz}\right)$$

With:

– Values that can be measured by tests:

$E = \frac{G\,(2G + 3\lambda)}{\lambda + G}$, longitudinal elastic modulus (in Pa)

$\nu = \frac{\lambda}{2\,(\lambda + G)}$, Poisson's ratio (without unit) [1.6]

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\,(\sigma_{yy} + \sigma_{zz}\right]$$

– Values that can be measured by permutation:

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\,(\sigma_{zz} + \sigma_{xx}\right] \quad [1.7]$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\,(\sigma_{xx} + \sigma_{yy}\right]$$

REMARK.–

$$G = \frac{E}{2\,(1 + \nu)} \qquad \lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \quad [1.8]$$

↓

With:

$2G = a - b$ or b

1.1.4.1.2. A few values: λ, E, G in daN/mm²

	$\lambda\ 10^{-3}$	$E\ 10^{-3}$	$G\ 10^{-3}$	ν
Steel	9–13	20–22	7.9–8.4	0.27–0.31
Brass	8.5	11	4.1	0.33
Copper	9–13	13	4.8	0.33–0.38
Lead	3.5	1.6	0.56	0.43
Glass	2.7–3	6	2.38	0.26

Table 1.4. *Elasticity coefficients of materials*

The following relationships are obtained in the main reference area:

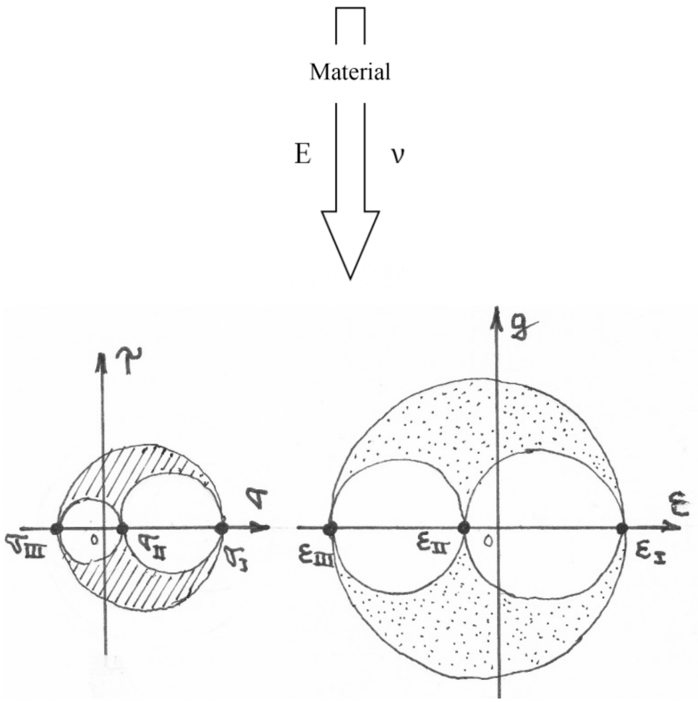
$$\varepsilon_{I} = \frac{1}{E}[\sigma_{I} - \nu(\sigma_{II} + \sigma_{III})]$$

$$\varepsilon_{II} = \frac{1}{E}[\sigma_{II} - \nu(\sigma_{III} + \sigma_{I})]$$

$$\varepsilon_{III} = \frac{1}{E}[\sigma_{III} - \nu(\sigma_{I} + \sigma_{II})]$$

(note: $\sigma > 0$ traction; $\sigma < 0$ compression)

and thus the shift to the stress and deformation states for elastic isotropic material in Figure 1.1.



a) Stresses b) Strains

Figure 1.1. *Representation of Mohr domains from stress and strain states*

Material	E in GPs	Material	E in GPs
Diamond	1000	Niobium and alloys	80–110
Tungsten carbide WC	450–650	Silicon	107
Osmium	551	Zirconium and alloys	96
Cobalt/tungsten carbide cements	400–530	Silica glass and SiO_2 (quartz)	94
Ti, Zr, and Hf borides	500	Zinc and alloys	43–96
Silicon carbide siC	450	Gold	82
Boron	441	Calcite (marble, limestone)	81
Tungsten	406	Aluminum	69
Alumina Al_2O_3	390	Aluminum alloys	69–79
Beryl BeO	380	Silver	76
Tungsten carbide TiC	379	Sodium glass	69
Molybdenum and alloys	320–365	Alkali metal halides (NaCl, Lif…)	15–68
Tantalum carbide TaC		Granite	62
Niobium carbide, NbC		Tin and alloys	41–53
Silicon nitride, Si_3N_4		Concrete, cement	45–50
Chrome	289	Fiberglass/epoxide composite	35–45
Beryllium and alloys	200–289	Magnesium and alloys	41–45
Magnesia MgO	250	GFRP	7–45
Cobalt and alloys	200–248	Calcite	31
Zirconia ZrO	160–241	Graphite	27
Nickel	214	Alkides	20
Nickel alloys	130–234	Shale (bituminous)	18
CFRP	70–200	Common wood (// with fibers)	9–16
Iron	196	Lead and alloys	14
Iron-based superalloys	193–214	H_2O Ice	9.1
Ferric or weak steels	200–207	Melamines	6–7
Alloys	190–200	Polyimides	3–5
Mild steel	196	Polyesters	1–5
Cast Irons	170–190	Acrylic resins	1.6–3.4
Tantalum and alloys	150–186	Nylon	2–4
Platinum	172	PMMA	3.4
Uranium	172	Plysyrene	3–3.4
Boron/epoxide composites	125	Polycarbonate	2.6
Copper	124	Epoxy resins	3
Copper alloys	120–150	Common wood (⊥ with fibers)	0.6–1.0
Mullite	145	Polypropylene	0.9
Zirconia ZrO_2	145	High density polyethylene	0.7
Vanadium	130	Polyurethane foam	0.01–0.06
Titanium	116	Low density polyethylene	0.2
Titanium alloys	80–130	Rubbers	0.01–0.1
Palladium	124	PVC	0.003–0.01
Brass and bronze	103–124	Expanded polymers	0.0001–0.01

Table 1.5. *Young's moduli values (Ashby and Jones 1980)*

	E Young's modulus in 10^9 Pascal	**ν** Poisson's Number	**ρ** Density in 10^3 kg/m^3	**$\frac{\alpha}{3}$** Linear expansion coefficient in 10^{-6}/°C
Aluminum	71	0.34	2.6	23
AU4G Alloy	75	0.33	2.8	23.5
Structural steel	210	0.285	7.8	13
Spring steel	220	0.29	7.8	13
Stainless steel 18–10	203	0.29	7.9	16.5
Invar	140	0.29	8.7	0.9
Common gray cast iron	90 to 120	0.29	7.1 to 7.2	9 to 11
Malleable cast iron	170 to 190	0.17	7.2 to 7.4	9 to 11
Commercial zinc	78	0.21	7.15	30
Copper	100	0.33	8.9	17
Beryllium	300	0.05	1.85	12
Beryllium bronze	130	0.34	8.25	17
Titanium	105	0.34	4.5	9
Granite	60	0.27	2.3 to 3	20
Marble	26	0.3	2.8	8 to 8.5
Glass	60	0.2 to 0.3	2.5 to 2.9	3.4 to 5.9
Plexiglass	2.9	0.4	1.8	80 to 90
Rubber	0.02	0.5	1	160
Concrete under compression	at 200 kg Cement by m^3: 10 at 300 kg Cement by m^3: 11 at 400 kg Cement by m^3: 13	0.15	2 to 2.4	14

Table 1.6. *Elastic characteristics of various materials (Bellet and Barrau, elasticity course)*

1.1.4.2. *A solid of stiffness [K] under shear*

A homogeneous and isotropic solid is stressed within its elastic domain of a stiffness [K]:

$$[K] = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix} \qquad \text{a, b: consts “elastic”}$$

The stress is "shear-type" stress with the value A (a real number), that is,

$$[\sigma]_{x,y,z} = \begin{bmatrix} 0 & A & A \\ A & 0 & A \\ A & A & 0 \end{bmatrix} \text{(MPa)}$$

Questions

1) On the basis of A, plot the domain of the stress states (σ, τ) that all the facets (F) of the material undergo (note: the process used here is to search for the primary data values). What particular details do you notice?

2) On the basis of A, determine the spherical tensors and deviators of the stress states that the facets (F) undergo. What particular details do you notice?

3) On the basis of A, a and b, plot the domain of the strain states of the solid. What specific observations can you make?

4) Determine the stresses and strains of the facets of equal inclination (having the same directional cosines) in the main reference area.

5) Given that a = 25 GPa and b = 10 GPa, and that the elastic tension limit of the material is equal to σ_E = 0.5 GPa, deduce the intensity value of A for the facets with the same inclination.

6) Provide the representation of these facets in the fields of stresses and deformations.

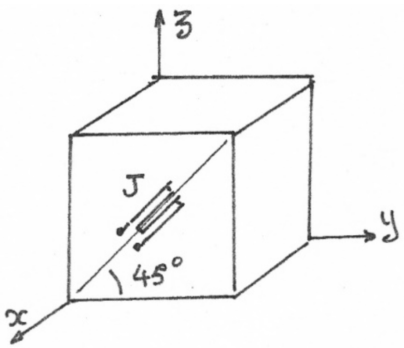


Figure 1.2. *Position gauge J (J: a gauge glued onto one side ⊥ at x; direction at 45°/y)*

– Based on A, a, b, determine the response of the gauge.

– Calculate this response (in microdeformations) for the intensity A of the question (5) and determine the value (in degrees) by sliding the solid.

Solutions

The characteristic equation is:

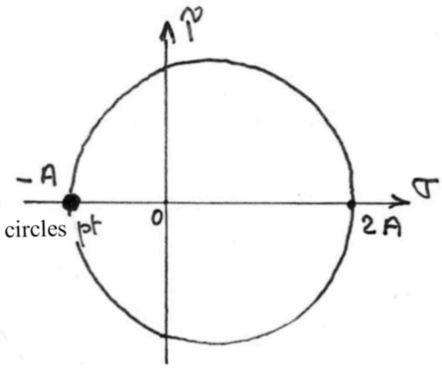
$$[\sigma]_{x,y,z} = \begin{bmatrix} 0 & A & A \\ A & 0 & A \\ A & A & 0 \end{bmatrix}$$

$$(-\sigma)(-\sigma)(-\sigma) + AAA + AAA - (-\sigma)AA - (-\sigma)AA - (-\sigma)AA = 0$$

$$-\sigma^3 + 2A^3 + \sigma A^2 + \sigma A^2 + \sigma A^2 = 0$$

$$-\sigma^3 + 2A^3 + 3\sigma A^2 = 0$$

1) $[\sigma] = \begin{bmatrix} 2A & 0 & 0 \\ 0 & -A & 0 \\ 0 & 0 & -A \end{bmatrix}$.



with $R = \frac{3A}{2\,(a-b)}$

Figure 1.3. *Mohr stress circle*

2) $[\sigma] = [P] + [S]$ gives $[P] = [0]$, $\mathrm{tr}\,[\sigma] = 0$

$$\begin{bmatrix} 0 & A & A \\ A & 0 & A \\ A & A & 0 \end{bmatrix} = \underset{[P]}{\begin{bmatrix} 0 \end{bmatrix}} + \underset{[S]}{\begin{bmatrix} 0 & A & A \\ A & 0 & A \\ A & A & 0 \end{bmatrix}}$$

where:

$$\begin{bmatrix} 2A & 0 & 0 \\ 0 & -A & 0 \\ 0 & 0 & -A \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 2A & 0 & 0 \\ 0 & -A & 0 \\ 0 & 0 & -A \end{bmatrix}$$

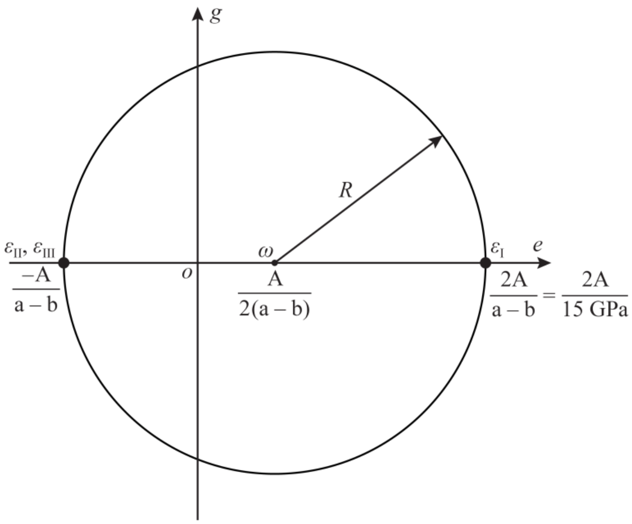
3) $3\ \sigma_{average} = \sigma_I + \sigma_{II} + \sigma_{III} = 0 = (2G + 3\lambda)\ \varepsilon_{average}$

$$= [a - b + 3b]\ \varepsilon_{average} = [a + 2b]\ \varepsilon_{average}$$

$\rightarrow \varepsilon_{average} = 0 \implies \Theta = 0$

$\sigma_I = 2\ G\ \varepsilon_I + \lambda\theta \ \rightarrow \varepsilon_I = \frac{\sigma_I}{2G} = \frac{\sigma_I}{a-b} = \frac{2A}{a-b}$

$\sigma_{II} = 2\ G\ \varepsilon_{II} + 0 \ \rightarrow \varepsilon_{II} = \frac{-A}{a-b} = \varepsilon_{III}$



with $R = \dfrac{3A}{2(a-b)}$

Figure 1.4. *Mohr strain circle*

4) Von Mises facet (VM) $a = b = c = \frac{1}{\sqrt{3}} \rightarrow \begin{cases} \sigma_{VM} = \dfrac{\sigma_I + \sigma_{II} + \sigma_{III}}{3} = 0 \\ \tau_{UM} = \dfrac{1}{3}\sqrt{9\ A^2 + 9\ A^2} = A\sqrt{2} \end{cases}$

$$\text{VM}\begin{cases}\varepsilon_{VM} = 0\\[2ex] g_{VM} = \dfrac{A\sqrt{2}}{a-b}\end{cases}$$

5) $\frac{\sqrt{2}}{3}\ \sigma_E = 0.5\ \text{GPa}\ \frac{\sqrt{2}}{3} = A\sqrt{2}\ \rightarrow\ A = \frac{\sigma_E}{3} = \frac{0.5}{3} = 0.166\ \text{GPa} = 166\ \text{MPa}.$

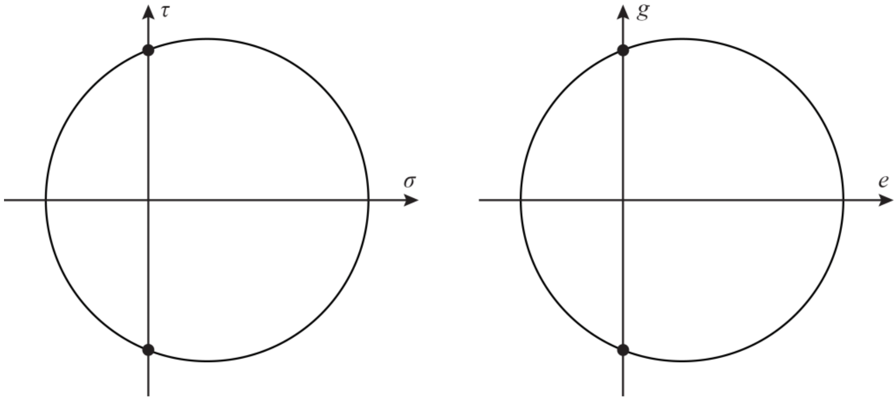
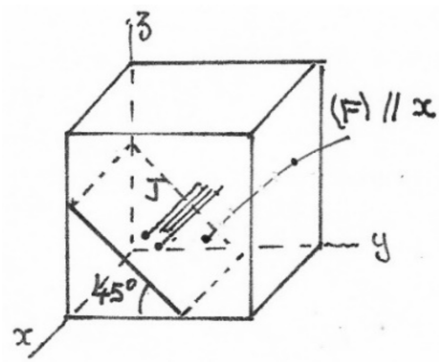


Figure 1.5. *Comparison of Mohr's circles*

6)



$$\vec{n}\begin{cases}a = 0\\ b = c\end{cases}$$

$$b^2 + c^2 = 2b^2 = 1 \rightarrow b = \frac{1}{\sqrt{2}}$$

Figure 1.6. *Facet (F) at 45° of the element*

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{2G} = \frac{A}{a-b} = \varepsilon_{yz} = \varepsilon_{zx}$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{2G} - \frac{3\lambda\,\varepsilon_m}{2G} = 0 = \varepsilon_{yy} = \varepsilon_{zz}$$

$$\left[\varepsilon_{x,y,z}\right] = \begin{bmatrix} 0 & A & A \\ A & 0 & A \\ A & A & 0 \end{bmatrix} \frac{1}{a-b} \quad \left| K = \frac{A}{a-b} \right.$$

$$\vec{\phi} = [\varepsilon]\,.\,\vec{n} = \begin{bmatrix} 0 & A & A \\ A & 0 & A \\ A & A & 0 \end{bmatrix} \frac{1}{a-b} \quad \begin{vmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{vmatrix}$$

with $K = \frac{A}{a-b}$;

$$\vec{\phi} \begin{cases} X = \frac{K}{\sqrt{2}} + \frac{K}{\sqrt{2}} = \frac{2K}{\sqrt{2}} = \sqrt{2}\,K \\ Y = \frac{K}{\sqrt{2}} \\ Z = \frac{K}{\sqrt{2}} \end{cases}$$

and:

$$\begin{cases} e = \frac{\Delta\ell}{\ell} = \vec{\phi}\,.\,\vec{n} = \frac{K}{2} + \frac{K}{2} = K = \frac{A}{a-b} \\ g = \sqrt{\phi^2 - e^2} = K\sqrt{2} = \frac{A\sqrt{2}}{a-b} \end{cases}$$

at the elastic limit of VM:

$$e = \frac{0.166}{25-10} = \frac{0.166}{15} = 0.01106 \text{ or } 11.066\ \mu D$$

$$g = e\sqrt{2} = 15{,}649\ \mu \text{ radians or } 0.89°$$

1.1.4.3. *Plate shearing*

A central load P is applied to a square plate on the side ℓ with a thickness of H. The isotropic material is deformed in its elastic domain.

Two extension gauges, A and B, are placed as shown in Figure 1.7, with a 90° angle formed between the two gauges.

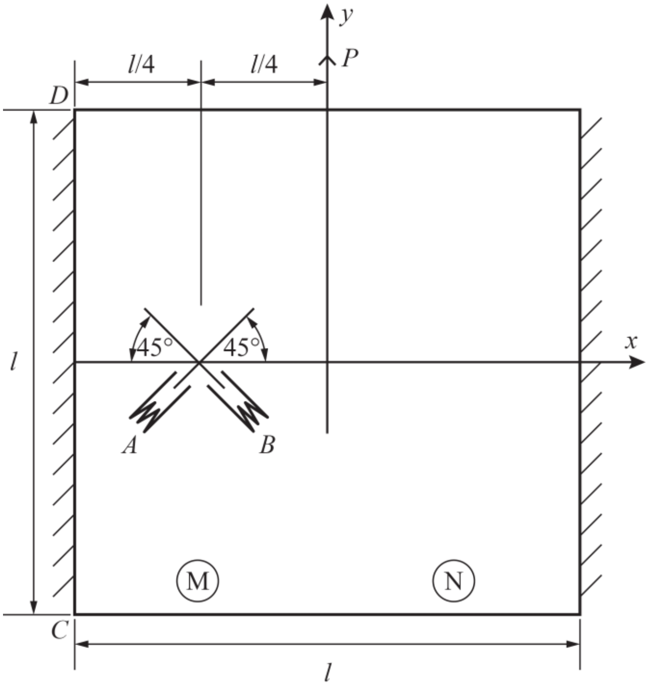


Figure 1.7. *Plate under stress from a central force P (the "rail shear stress" type test)*

Questions

1) On the basis of P, ℓ and h, determine the shear stress τ experienced by the half-element M according to CD. Plot the Mohr circles of the stress states and place the points A and B that represent the stresses for the facets perpendicular to the gauges. What can you observe about the strains?

2) Determine the equation, giving the difference of the elastic constants a − b in terms of P, ℓ, h and e_A, e_B (answers from the gauges).

3) Numerical application: Calculate the shear modulus G (in G Pa) for the following load:

P = 10^3 daN

with:

$e_A = 500\ \mu D \qquad (\mu D = 10^{-6})$;

$e_B = -\ 500\ \mu D$ microdeformation;

$\ell = 10$ cm;

h = 1 mm.

Solutions

1) $\tau = \frac{P/2}{S} = \frac{P}{2\ell h}$

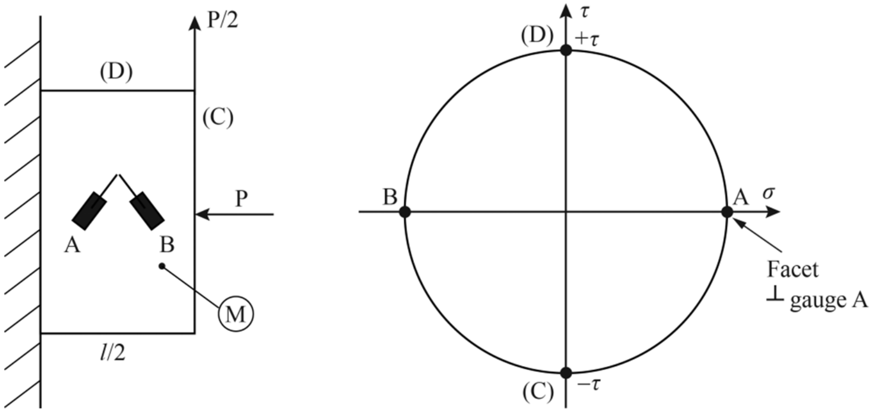


Figure 1.8. *Mohr stress circle*

$\varepsilon_A = \frac{1}{E}[\sigma_I - \nu(\sigma_{II} + 0)]$ with $\sigma_I = \tau$ and $\sigma_{II} = -\tau$

$\varepsilon_A = \frac{\tau}{E}(1 + \nu)$ and $\varepsilon_B = -\frac{\tau}{E}(1 + \nu)$

We thus obtain $\varepsilon_A = -\ \varepsilon_B$.

2) a – b = 2G and $\frac{\tau}{\gamma} = G$ with $\gamma = \varepsilon_A - \varepsilon_B$ and thus with $\tau = \frac{P}{2\ell h}$, we obtain:

$$2\ G = a - b = \frac{P}{\ell h\,(e_A - e_B)}$$