Ramanujan's Lost Notebook

Part II

S. Ramanujan

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Springer

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By the kindness of heaven,
O lovely faced one,
You stand before me,
The darkness of delusion dispelled,
By recollection of that which was lost.

Verse 7.22 of Kalidasa's *Sakuntala*,
4th century A.D.

Preface

This is the second of approximately four volumes that the authors plan to write in their examination of all the claims made by S. Ramanujan in *The Lost Notebook and Other Unpublished Papers*. This volume, published by Narosa in 1988, contains the "Lost Notebook," which was discovered by the first author in the spring of 1976 at the library of Trinity College, Cambridge. Also included in this publication are other partial manuscripts, fragments, and letters that Ramanujan wrote to G.H. Hardy from nursing homes during 1917–1919. The authors have attempted to organize this disparate material in chapters. This second volume contains 16 chapters comprising 314 entries, including some duplications and examples, with chapter totals ranging from a high of fifty-four entries in Chapter 1 to a low of two entries in Chapter 12.

Contents

Introduction

This volume is the second of approximately four volumes that the authors plan to write on Ramanujan's lost notebook. We broadly interpret "lost notebook" to include all material published *with* Ramanujan's original lost notebook by Narosa in 1988 [244]. Thus, when we write that a certain entry is found in the lost notebook, it may not actually be located in the original lost notebook discovered by the first author in the spring of 1976 at Trinity College Library, Cambridge, but instead may be in a manuscript, fragment, or a letter of Ramanujan to G.H. Hardy published in [244]. We are attempting to arrange all this disparate material into chapters for each of the proposed volumes. For a history and general description of Ramanujan's lost notebook, readers are advised to read the introduction to our first book [31].

The Organization of Entries

With the statement of each entry from Ramanujan's lost notebook, we provide the page number(s) in the lost notebook on which the entry can be found. All of Ramanujan's claims are given the designation "Entry." Results in this volume named theorems, corollaries, and lemmas are (unless otherwise stated) not due to Ramanujan. We emphasize that Ramanujan's claims always have page numbers from the lost notebook attached to them. We remark that in Chapter 9, which is devoted to establishing Ramanujan's values for an analogue λ_n of the classical Ramanujan–Weber class invariant G_n, we have followed a slightly different convention. Indeed, we have listed all of Ramanujan's values for λ_n in Entry 9.1.1 with the page number indicated. Later, we establish these values as corollaries of theorems that we prove, and so we record Ramanujan's values of λ_n *again*, listing them as corollaries with page numbers in the lost notebook attached to emphasize that these corollaries are due to Ramanujan.

In view of the subject mentioned in the preceding paragraph, it may be prudent to make a remark here about Ramanujan's methods. As many read-

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ers are aware from the work of the authors and others who have attempted to prove Ramanujan's theorems, we frequently have few or no clues about Ramanujan's methods. Many of the proofs of the values for G_n that are given in [57] are almost certainly not those found by Ramanujan, for he would have needed knowledge of certain portions of mathematics that he likely did not know or that had not been discovered yet. Similar remarks can be made about our calculations of λ_n in Chapter 9. In the last half of the chapter, we employ ideas that Ramanujan would not have known.

So that readers can more readily find where a certain entry from the lost notebook is discussed, we place at the conclusion of each volume a *Location Guide* indicating where entries can be found in that particular volume. Thus, for example, if a reader wants to know whether a certain identity on page 1729 of the Narosa edition [244] can be found in a particular volume, she can turn to this index and determine where in that volume identities on page 1729 are discussed.

Following the Location Guide, we provide a *Provenance* indicating the sources from which we have drawn in preparing significant portions of the given chapters. We emphasize that in the Provenance we do not list all papers in which results from a given chapter are established. For example, in Chapter 3, Ramanujan's famous ${}_1\psi_1$ summation theorem, which is found in more than one version in the lost notebook, is discussed, but we do not refer to all papers on the ${}_1\psi_1$ summation formula in the Location Guide, although in Chapter 3 itself, we have attempted to cite all relevant proofs of this celebrated formula. On the other hand, most chapters contain previously unpublished material. For example, each of the first four chapters contains previously unpublished proofs.

This Volume on the Lost Notebook

Two primary themes permeate our second volume on the lost notebook, namely, q-series and Eisenstein series. The first seven chapters are devoted to q-series identities from the core of the original lost notebook. These chapters are followed by three chapters on identities for the classical theta functions or related functions. The last six chapters feature Eisenstein series, with much of the material originating in letters to Hardy that Ramanujan wrote from Fitzroy House and Matlock House during his last two years in England. We now briefly describe the contents of the sixteen chapters in this volume.

Heine's transformations have long been central to the theory of basic hypergeometric series. In Chapter 1, we examine several entries from the lost notebook that have their roots in Heine's first transformation or generalizations thereof. The Sears–Thomae transformation is also a staple in the theory of basic hypergeometric series, and consequences of it form the content of Chapter 2. In Chapter 3, we consider identities arising from certain bilateral series identities, in particular the renowned ${}_1\psi_1$ summation of Ramanujan and well-known identities due to W.N. Bailey. We have also placed in Chapter 3 some identities dependent upon the quintuple product identity. Watson's q-analogue of Whipple's theorem and two additional theorems of Bailey are the main ingredients for the proofs in Chapter 4 on well-poised series. Bailey's lemma is utilized to prove some identities in Chapter 5. Chapter 6, on partial theta functions, is one of the more difficult chapters in this volume. Chapter 7 contains entries from the lost notebook that are even more difficult to prove than those in Chapter 6. The entries in this chapter do not fall into any particular categories and bear further study, because several of them likely have yet-to-be discovered ramifications.

Theta functions frequently appear in identities in the first seven chapters. However, in Chapters 8–10, theta functions are the focus. Chapter 8 is devoted to theta function identities. Chapter 9 focuses on one page in the lost notebook on values of an analogue of the classical Ramanujan–Weber class invariants. The identities in Chapter 10 do not fit in any of the previous chapters and are among the most unusual identities we have seen in Ramanujan's work.

As remarked above, the last six chapters in this volume feature Eisenstein series. Perhaps the most important chapter is Chapter 11, which contains proofs of results sent to Hardy from nursing homes, probably in 1918. In these letters, Ramanujan offered formulas for the coefficients of certain quotients of Eisenstein series that are analogous to the Hardy–Ramanujan–Rademacher series representation for the partition function $p(n)$. The claims in these letters continue the work found in Hardy and Ramanujan's last joint paper [177], [242, pp. 310–321]. Chapter 12 relates technical material on the number of terms that one needs to take from the aforementioned series in order to determine these coefficients precisely. In Chapter 13, the focus shifts to identities for Eisenstein series involving the Dedekind eta function. Chapter 14 gives formulas for certain series associated with the pentagonal number theorem in terms of Ramanujan's Eisenstein series P, Q, and R. These results are found on two pages of the lost notebook, and, although not deep, have recently generated several further papers. Chapter 15 is devoted primarily to a single page in the lost notebook demonstrating how Ramanujan employed Eisenstein series to approximate π. Three series for $1/\pi$ found in Ramanujan's epic paper [239], [242, pp. 23–39] are also found on page 370 of [244], and so it seems appropriate to prove them in this chapter, especially since, perhaps more so than other authors, we follow Ramanujan's hint in [239] and use Eisenstein series to establish these series representations for $1/\pi$. This volume concludes with a few miscellaneous results on Eisenstein series in Chapter 16.

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1

The Heine Transformation

1.1 Introduction

E. Heine [178], [179, pp. 97–125] was the first to generalize Gauss's hypergeometric series to q-hypergeometric series by defining, for $|q| < 1$,

$$
{}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, t\right) := \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} t^n, \tag{1.1.1}
$$

where $|t| < 1$ and where, for each nonnegative integer n,

$$
(a)_n = (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{1.1.2}
$$

with the convention that $(a)_0 = (a;q)_0 := 1$. If an entry and its proof involve only the base q and no confusion would arise, we use the notation at the left in (1.1.2) and (1.1.4) below. If more than one base occurs in an entry and/or its proof, e.g., both q and q^2 appear, then we use the second notation in (1.1.2) and (1.1.4). Ramanujan's central theorem is a transformation for this series, now known as the Heine transformation, namely [179, p. 106, equation (50)],

$$
{}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, t\right) = \frac{(b;q)_\infty (at;q)_\infty}{(c;q)_\infty (t;q)_\infty} {}_2\phi_1\left(\begin{matrix} c/b, t \\ at \end{matrix}; q, b\right), \tag{1.1.3}
$$

where $|t|, |b| < 1$ and where

$$
(a)_\infty = (a;q)_\infty = \lim_{n\to\infty} (a;q)_n, \qquad |q| < 1. \tag{1.1.4}
$$

His method of proof was surely known to Ramanujan, who recorded an equivalent formulation of (1.1.3) in Entry 6 of Chapter 16 in his second notebook [243], [54, p. 15]. Furthermore, numerous related identities can be proved using Heine's original idea.

In Section 1.2, we prove several basic formulas based on Heine's method. In the remainder of the chapter we deduce 53 formulas found in the lost

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notebook. In some instances, we call upon a result not listed in Section 1.2, but each identity that we prove relies primarily on results in Section 1.2.

In order to keep our proofs to manageable lengths, we invoke certain standard simplifications (usually without mentioning them explicitly), such as

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}, \tag{1.1.5}$$

$$(a;q)_n(-a;q)_n = (a^2;q^2)_n, \qquad 0 \le n < \infty, \tag{1.1.6}$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad -\infty < n < \infty. \tag{1.1.7}$$

The identity (1.1.5) is a famous theorem of Euler, which we invoke numerous times in this book. Identity (1.1.7) can be regarded as the definition of $(a;q)_n$ when n is a negative integer.

1.2 Heine's Method

In [6], Heine's method was encapsulated in a fundamental formula containing ten independent variables and a nontrivial root of unity. As a result, it is an almost unreadable formula. Consequently, we prove only special cases of this result here. In light of the fact that many of these results are not easily written in the notation (1.1.1) of q-hypergeometric series, we record all our results in terms of infinite series. For further work connected with that of Andrews in [6], see Z. Cao's thesis [97] and a paper by W. Chu and W. Zhang [131].

We begin with a slightly generalized version of Heine's transformation [6], [7].

Theorem 1.2.1. *If h is a positive integer, then, for $|t|, |b| < 1$,*

$$\sum_{m=0}^{\infty} \frac{(a;q^h)_m(b;q)_{hm}}{(q^h;q^h)_m(c;q)_{hm}} t^m = \frac{(b;q)_\infty(at;q^h)_\infty}{(c;q)_\infty(t;q^h)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m(t;q^h)_m}{(q;q)_m(at;q^h)_m} b^m. \tag{1.2.1}$$

Proof. We need the q-binomial theorem given by [54, p. 14, Entry 2], [18, p. 17, Theorem 2.1]

$$\sum_{m=0}^{\infty} \frac{(a/b;q)_m}{(q;q)_m} b^m = \frac{(a;q)_\infty}{(b;q)_\infty}, \tag{1.2.2}$$

where $|b| < 1$. Since we frequently need two special cases in the sequel, we state them here. If $a = 0$ in (1.2.2), then [18, p. 19, equation (2.2.5)]

$$\sum_{m=0}^{\infty} \frac{b^m}{(q;q)_m} = \frac{1}{(b;q)_\infty}. \tag{1.2.3}$$

Letting $b \to 0$ in (1.2.2), we find that [18, p. 19, equation (2.2.6)]

$$\sum_{m=0}^{\infty} \frac{(-a)^m q^{m(m-1)/2}}{(q;q)_m} = (a;q)_\infty. \tag{1.2.4}$$

Upon two applications of (1.2.2), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a;q^h)_n (b;q)_{hn}}{(q^h;q^h)_n (c;q)_{hn}} t^n &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^h)_n}{(q^h;q^h)_n} \frac{(cq^{hn};q)_\infty}{(bq^{hn};q)_\infty} t^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^h)_n}{(q^h;q^h)_n} t^n \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m q^{hmn} \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m \sum_{n=0}^{\infty} \frac{(a;q^h)_n}{(q^h;q^h)_n} (tq^{hm})^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m \frac{(atq^{hm};q^h)_\infty}{(tq^{hm};q^h)_\infty} \\
&= \frac{(b;q)_\infty (at;q^h)_\infty}{(c;q)_\infty (t;q^h)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m (t;q^h)_m}{(q;q)_m (at;q^h)_m} b^m,
\end{aligned}$$

which is (1.2.1). □

Heine's transformation is the case $h = 1$ of Theorem 1.2.1, and Theorem A_3 of [6] is the case $h = 2$. The complete result appears in [7, Lemma 1].

The next result is more intricate, but it is based again on Heine's idea; it is Theorem A_1 of [6].

Theorem 1.2.2. *For* $|t|, |b| < 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a;q^2)_n (b;q)_n}{(q^2;q^2)_n (c;q)_n} t^n &= \frac{(b;q)_\infty (at;q^2)_\infty}{(c;q)_\infty (t;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(c/b;q)_{2n} (t;q^2)_n}{(q;q)_{2n} (at;q^2)_n} b^{2n} \\
&\quad + \frac{(b;q)_\infty (atq;q^2)_\infty}{(c;q)_\infty (tq;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(c/b;q)_{2n+1} (tq;q^2)_n}{(q;q)_{2n+1} (atq;q^2)_n} b^{2n+1}.
\end{aligned} \tag{1.2.5}$$

Proof. Using (1.2.2) twice, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a;q^2)_n (b;q)_n}{(q^2;q^2)_n (c;q)_n} t^n &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^2)_n}{(q^2;q^2)_n} \frac{(cq^n;q)_\infty}{(bq^n;q)_\infty} t^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^2)_n}{(q^2;q^2)_n} t^n \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m q^{mn} \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^2)_n}{(q^2;q^2)_n} t^n \Bigg\{ \sum_{m=0}^{\infty} \frac{(c/b;q)_{2m}}{(q;q)_{2m}} b^{2m} q^{2mn} \\
&\quad + \sum_{m=0}^{\infty} \frac{(c/b;q)_{2m+1}}{(q;q)_{2m+1}} b^{2m+1} q^{(2m+1)n} \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^\infty \frac{(c/b;q)_{2m}}{(q;q)_{2m}} b^{2m} \sum_{n=0}^\infty \frac{(a;q^2)_n}{(q^2;q^2)_n} (tq^{2m})^n \\
&\quad + \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^\infty \frac{(c/b;q)_{2m+1}}{(q;q)_{2m+1}} b^{2m+1} \sum_{n=0}^\infty \frac{(a;q^2)_n}{(q^2;q^2)_n} (tq^{2m+1})^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^\infty \frac{(c/b;q)_{2m}}{(q;q)_{2m}} b^{2m} \frac{(atq^{2m};q^2)_\infty}{(tq^{2m};q^2)_\infty} \\
&\quad + \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^\infty \frac{(c/b;q)_{2m+1}}{(q;q)_{2m+1}} b^{2m+1} \frac{(atq^{2m+1};q^2)_\infty}{(tq^{2m+1};q^2)_\infty} \\
&= \frac{(b;q)_\infty (at;q^2)_\infty}{(c;q)_\infty (t;q^2)_\infty} \sum_{m=0}^\infty \frac{(c/b;q)_{2m}(t;q^2)_m}{(q;q)_{2m}(at;q^2)_m} b^{2m} \\
&\quad + \frac{(b;q)_\infty (atq;q^2)_\infty}{(c;q)_\infty (tq;q^2)_\infty} \sum_{m=0}^\infty \frac{(c/b;q)_{2m+1}(tq;q^2)_m}{(q;q)_{2m+1}(atq;q^2)_m} b^{2m+1}.
\end{aligned}
$$

□

In addition to Theorems 1.2.1 and 1.2.2, we require two corollaries of Theorem 1.2.1. The first is also given in [7, equation (I5)].

Corollary 1.2.1. *For* $|t| < 1$,

$$
\sum_{n=0}^\infty \frac{(b;q)_{2n}}{(q^2;q^2)_n} t^{2n} = \frac{(-tb;q)_\infty}{(-t;q)_\infty} \sum_{n=0}^\infty \frac{(b;q)_n}{(q;q)_n(-tb;q)_n} t^n. \tag{1.2.6}
$$

Proof. By (1.2.1) with $h = 2$, $a = c = 0$, and t replaced by t^2, we see that

$$
\begin{aligned}
\sum_{n=0}^\infty \frac{(b;q)_{2n}}{(q^2;q^2)_n} t^{2n} &= \frac{(b;q)_\infty}{(t^2;q^2)_\infty} \sum_{n=0}^\infty \frac{(t^2;q^2)_n}{(q;q)_n} b^n \\
&= \frac{(b;q)_\infty}{(t^2;q^2)_\infty} \sum_{n=0}^\infty \frac{(t;q)_n(-t;q)_n}{(q;q)_n} b^n \\
&= \frac{(b;q)_\infty}{(t^2;q^2)_\infty} \frac{(t;q)_\infty(-tb;q)_\infty}{(b;q)_\infty} \sum_{n=0}^\infty \frac{(b;q)_n}{(q;q)_n(-tb;q)_n} t^n,
\end{aligned}
$$

by (1.2.1) with $t = b$ and then $h = 1$, $a = -t$, $b = t$, and $c = 0$. Upon simplification above, we deduce (1.2.6). □

The next result can be found in [7, equation (I6)].

Corollary 1.2.2. *For* $|b| < 1$,

$$
\sum_{n=0}^\infty \frac{(t;q^2)_n}{(q;q)_n} b^n = \frac{(btq;q^2)_\infty}{(bq;q^2)_\infty} \sum_{n=0}^\infty \frac{(t;q^2)_n}{(q^2;q^2)_n(btq;q^2)_n} b^n. \tag{1.2.7}
$$

Proof. By (1.2.1) with $h = 2$ and $a = c = 0$, we see that

$$\begin{aligned}\frac{(b;q)_\infty}{(t;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(t;q^2)_n}{(q;q)_n}b^n &= \sum_{n=0}^{\infty}\frac{(b;q)_{2n}}{(q^2;q^2)_n}t^n \\ &= \sum_{n=0}^{\infty}\frac{(bq;q^2)_n(b;q^2)_n}{(q^2;q^2)_n}t^n \\ &= \frac{(b;q^2)_\infty(btq;q^2)_\infty}{(t;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(t;q^2)_n}{(q^2;q^2)_n(btq;q^2)_n}b^n,\end{aligned}$$

where we applied (1.2.1) with q replaced by q^2, $h = 1$, $a = bq$, and $c = 0$. Upon simplification, we complete the proof. □

Our next result comes from [9, Theorem 7].

Corollary 1.2.3. *For* $|t| < 1$,

$$\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q^2)_n}{(q;q)_n(abt;q^2)_n}t^n = \frac{(at;q^2)_\infty(bt;q^2)_\infty}{(t;q^2)_\infty(abt;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(a;q^2)_n(b;q^2)_n}{(q^2;q^2)_n(bt;q^2)_n}(tq)^n. \tag{1.2.8}$$

Proof. In (1.2.1), set $h = 2$, interchange t with b, replace a by at, and then replace c by at. Upon simplification, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q^2)_n}{(q;q)_n(abt;q^2)_n}t^n &= \frac{(at;q)_\infty(b;q^2)_\infty}{(t;q)_\infty(abt;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(at;q^2)_n(t;q)_{2n}}{(q^2;q^2)_n(at;q)_{2n}}b^n \\ &= \frac{(at;q)_\infty(b;q^2)_\infty}{(t;q)_\infty(abt;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(t;q^2)_n(tq;q^2)_n}{(q^2;q^2)_n(atq;q^2)_n}b^n \\ &= \frac{(at;q)_\infty(b;q^2)_\infty(tq;q^2)_\infty(bt;q^2)_\infty}{(t;q)_\infty(abt;q^2)_\infty(atq;q^2)_\infty(b;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(a;q^2)_n(b;q^2)_n}{(q^2;q^2)_n(bt;q^2)_n}(tq)^n,\end{aligned}$$

where we invoked (1.2.1) with $h = 1$, q replaced by q^2, and the variables a, b, c, and t replaced by t, tq, atq, and b, respectively. Upon simplifying above, we deduce (1.2.8) to complete the proof. □

We also require the direct iteration of (1.2.1) with $h = 1$ [9, Theorem 8]. This is often called the second Heine transformation.

Corollary 1.2.4. *For* $|t|, |c/b| < 1$,

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(q)_n(c)_n}t^n = \frac{(c/b)_\infty(bt)_\infty}{(c)_\infty(t)_\infty}\sum_{n=0}^{\infty}\frac{(abt/c)_n(b)_n}{(q)_n(bt)_n}\left(\frac{c}{b}\right)^n. \tag{1.2.9}$$

Proof. By two applications of Theorem 1.2.1 with $h = 1$, the second with a, b, c, and t replaced by t, c/b, at, and b, respectively, we find that

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(q)_n(c)_n} t^n = \frac{(b)_\infty(at)_\infty}{(c)_\infty(t)_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_n(t)_n}{(q)_n(at)_n} b^n$$
$$= \frac{(b)_\infty(at)_\infty}{(c)_\infty(t)_\infty} \frac{(c/b)_\infty(bt)_\infty}{(at)_\infty(b)_\infty} \sum_{n=0}^{\infty} \frac{(abt/c)_n(b)_n}{(q)_n(bt)_n} \left(\frac{c}{b}\right)^n,$$

which is the desired result. □

Finally, we need one more iteration of (1.2.1) with $h = 1$ [18, p. 39, equation (3.3.13)]. This is often called the q-analogue of Euler's transformation.

Corollary 1.2.5. *For* $|t|, |abt/c| < 1$,

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(q)_n(c)_n} t^n = \frac{(abt/c)_\infty}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(c/a)_n(c/b)_n}{(q)_n(c)_n} \left(\frac{abt}{c}\right)^n. \tag{1.2.10}$$

Proof. Apply (1.2.1) with $h = 1$ and a, b, c, and t replaced by b, abt/c, bt, and c/b, respectively. Consequently,

$$\sum_{n=0}^{\infty} \frac{(abt/c)_n(b)_n}{(q)_n(bt)_n} \left(\frac{c}{b}\right)^n = \frac{(abt/c)_\infty(c)_\infty}{(bt)_\infty(c/b)_\infty} \sum_{n=0}^{\infty} \frac{(c/a)_n(c/b)_n}{(q)_n(c)_n} \left(\frac{abt}{c}\right)^n. \tag{1.2.11}$$

Substituting the right-hand side of (1.2.11) for the sum on the right-hand side of (1.2.9) and simplifying yields (1.2.10). □

1.3 Ramanujan's Proof of the q-Gauss Summation Theorem

On pages 268–269 in his lost notebook, Ramanujan sketches his proof of the q-Gauss summation theorem, normally given in the form

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(q)_n} \left(\frac{c}{ab}\right)^n = \frac{(c/a)_\infty(c/b)_\infty}{(c)_\infty(c/(ab))_\infty}. \tag{1.3.1}$$

This theorem was first discovered in 1847 by Heine [178], whose proof, which is the most frequently encountered proof in the literature, is based on Heine's transformation, Theorem 1.2.1, with $h = 1$. This proof can be found in the texts of Andrews [18, p. 20, Corollary 2.4], Andrews, R. Askey, and R. Roy [30, p. 522], and G. Gasper and M. Rahman [151, p. 10]. A second proof employs the q-analogue of Saalschütz's theorem and can be read in the texts of W.N. Bailey [44, p. 68] and L.J. Slater [263, p. 97]. Ramanujan's proof is different from these two proofs and was first published in full in a paper by Berndt and A.J. Yee [79]. Ramanujan's proof encompasses Lemma 1.3.1, Lemma 1.3.2, and Entry 1.3.1 below. After giving Ramanujan's proof, we prove a corollary of (1.3.1), which is found on page 370 in Ramanujan's lost notebook.

Before providing Ramanujan's argument, we derive the q-analogue of the Chu–Vandermonde theorem and record a special case that will be used in Chapter 6. If we set $b = q^{-N}$, where N is a nonnegative integer, in (1.3.1) and simplify, we find that

$$ {}_2\phi_1(a, q^{-N}; c; q, cq^N/a) = \frac{(c/a)_N}{(c)_N}, \tag{1.3.2} $$

which is the q-analogue of the Chu–Vandermonde theorem. If we reverse the order of summation on the left-hand side of (1.3.2), we deduce an alternative form of the q-Chu–Vandermonde theorem, namely,

$$ {}_2\phi_1(a, q^{-N}; c; q, q) = \frac{(c/a)_N}{(c)_N} a^N. \tag{1.3.3} $$

Setting $a = q^{-M}$ and $c = q^{-M-N}$, where M is a nonnegative integer, in (1.3.3) yields

$$ \begin{aligned} {}_2\phi_1(q^{-M}, q^{-N}; q^{-M-N}; q, q) &= \frac{(q^{-N})_N}{(q^{-M-N})_N} q^{-MN} = \frac{(q^{-M})_M (q^{-N})_N}{(q^{-M-N})_{M+N}} q^{-MN} \\ &= \frac{(q)_M (q)_N q^{-M(M+1)/2 - N(N+1)/2}}{(q)_{M+N} q^{-(M+N)(M+N+1)/2}} q^{-MN} \\ &= \frac{(q)_M (q)_N}{(q)_{M+N}}. \end{aligned} \tag{1.3.4} $$

In this chapter, we are providing analytic proofs of many of Ramanujan's theorems on basic hypergeometric series. Another approach uses combinatorial arguments. In [78], Berndt and Yee provided partition-theoretic proofs of several identities in the lost notebook arising from the Rogers–Fine identity; a few of these proofs were reproduced in [31, Chapter 12]. In [79], the same authors gave a combinatorial proof of the q-Gauss summation theorem. Other combinatorial proofs of this theorem based on overpartitions have been given by S. Corteel and J. Lovejoy [144], Corteel [143], and Yee [285].

Lemma 1.3.1. *If n is any nonnegative integer, then*

$$ (a)_n = \sum_{k=0}^{n} (-1)^k \frac{(q^{n+1-k})_k}{(q)_k} q^{k(k-1)/2} a^k. \tag{1.3.5} $$

Lemma 1.3.1 is a restatement of the q-binomial theorem (1.2.2) and can be found in [54, p. 24, Lemma 12.1] or [18, p. 36, Theorem 3.3]. We now use Lemma 1.3.1 to establish Lemma 1.3.2 below along the lines indicated by Ramanujan. Alternatively, Lemma 1.3.2 can be deduced from [151, p. 11, equation (1.5.3)] by setting $c = 0$ and replacing q by $1/q$ there.

Lemma 1.3.2. *If $c \neq 0$ and n is any nonnegative integer, then*

$$c^n = \sum_{j=0}^{n} \frac{c^j (1/c)_j (q^{n+1-j})_j}{(q)_j}. \tag{1.3.6}$$

Proof. Denote the right side of (1.3.6) by $g(c)$ and apply (1.3.5) with $a = 1/c$ and $n = j$ in the definition of $g(c)$ to find that

$$g(c) = \sum_{j=0}^{n} \sum_{k=0}^{j} (-1)^k \frac{(q^{j+1-k})_k (q^{n+1-j})_j}{(q)_j (q)_k} q^{k(k-1)/2} c^{j-k} =: \sum_{r=0}^{n} a_r c^r.$$

The coefficient of $c^r, 0 \leq r \leq n$, above is

$$a_r = \sum_{k=0}^{n-r} (-1)^k \frac{(q^{r+1})_k (q^{n+1-r-k})_{r+k}}{(q)_{r+k} (q)_k} q^{k(k-1)/2}. \tag{1.3.7}$$

Now we can easily verify that

$$\frac{(q^{r+1})_k}{(q)_{r+k}} = \frac{1}{(q)_r}$$

and

$$(q^{n+1-r-k})_{r+k} = (q^{n+1-r-k})_k (q^{n+1-r})_r.$$

Using these last two equalities in (1.3.7), we find that

$$\begin{aligned} a_r &= \frac{(q^{n+1-r})_r}{(q)_r} \sum_{k=0}^{n-r} (-1)^k \frac{(q^{n-r+1-k})_k}{(q)_k} q^{k(k-1)/2} \\ &= \frac{(q^{n+1-r})_r}{(q)_r} (1)_{n-r} = \begin{cases} 1, & \text{if } r = n, \\ 0, & \text{otherwise}, \end{cases} \end{aligned}$$

by (1.3.5). This therefore completes our proof of Lemma 1.3.2. □

Entry 1.3.1 (pp. 268–269, q-Gauss Summation Theorem). *If $|abc| < 1$ and $bc \neq 0$, then*

$$\frac{(ac)_\infty}{(abc)_\infty} = \frac{(a)_\infty}{(ab)_\infty} \sum_{n=0}^{\infty} \frac{(1/b)_n (1/c)_n}{(a)_n (q)_n} (abc)^n. \tag{1.3.8}$$

In Entry 4 of Chapter 16 in his second notebook [243], [54, p. 14], Ramanujan states the q-Gauss summation theorem in precisely the same form as that given in (1.3.8).

Proof. We rewrite the right side of (1.3.8) in the form

$$\sum_{j=0}^{\infty} \frac{(aq^j)_\infty}{(ab)_\infty} \frac{(1/b)_j(1/c)_j}{(q)_j}(abc)^j \tag{1.3.9}$$

and examine the coefficient of a^n, $n \geq 0$, on each side of (1.3.8). From (1.2.2), with b replaced by ab and a replaced by aq^j, we find that

$$\frac{(aq^j)_\infty}{(ab)_\infty} = \sum_{k=0}^{\infty} \frac{(q^j/b)_k}{(q)_k}(ab)^k. \tag{1.3.10}$$

The coefficient of a^{n-j} in (1.3.10) is

$$\frac{(q^j/b)_{n-j}}{(q)_{n-j}} b^{n-j},$$

and so the coefficient of a^n in (1.3.9) equals

$$\sum_{j=0}^{n} \frac{(1/b)_j(1/c)_j(q^j/b)_{n-j}}{(q)_j(q)_{n-j}} b^n c^j$$
$$= \frac{(1/b)_n b^n}{(q)_n} \sum_{j=0}^{n} \frac{c^j(1/c)_j(q^{n+1-j})_j}{(q)_j} = \frac{(1/b)_n b^n}{(q)_n} c^n, \tag{1.3.11}$$

by Lemma 1.3.2. But by (1.2.2), with b replaced by abc and a replaced by ac,

$$\frac{(ac)_\infty}{(abc)_\infty} = \sum_{n=0}^{\infty} \frac{(1/b)_n}{(q)_n}(abc)^n. \tag{1.3.12}$$

So, the coefficient of a^n in (1.3.12) is precisely that on the right side of (1.3.11). Hence, (1.3.8) immediately follows, since the coefficients of a^n, $n \geq 0$, on both sides of (1.3.8) are equal. The proof of Entry 1.3.1 is therefore complete. □

Entry 1.3.2 (p. 370). *For any complex numbers a and b,*

$$\frac{(-aq)_\infty}{(bq)_\infty} = \sum_{n=0}^{\infty} \frac{(-b/a)_n a^n q^{n(n+1)/2}}{(q)_n(bq)_n}. \tag{1.3.13}$$

Proof. In (1.3.8), replace a by bq, c by $-a/b$, and b by t to find that

$$\frac{(bqt)_\infty(-aq)_\infty}{(bq)_\infty(-aqt)_\infty} = \sum_{n=0}^{\infty} \frac{(1/t)_n(-b/a)_n}{(q)_n(bq)_n}(-aqt)^n. \tag{1.3.14}$$

If we let $t \to 0$ in (1.3.14), we immediately arrive at (1.3.13) to complete the proof. □

A combinatorial proof of Entry 1.3.2 in the case $b = 1$ has been given by S. Corteel and J. Lovejoy [145], but it can easily be extended to give a proof of Entry 1.3.2 in full generality. Another combinatorial proof can be found in a paper by Berndt, B. Kim, and A.J. Yee [73].

1.4 Corollaries of (1.2.1) and (1.2.5)

Entry 1.4.1 (p. 3). *For* $0 < |aq|, |k| < 1$,

$$\frac{(aq;q)_\infty(cq;q^2)_\infty}{(-bq;q)_\infty(kq^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(kq^2;q^2)_n(-bq/a;q)_n}{(cq;q^2)_{n+1}(q;q)_n}a^nq^n = \sum_{n=0}^{\infty}\frac{(cq/k;q^2)_n(aq;q)_{2n}}{(q^2;q^2)_n(-bq;q)_{2n+1}}k^nq^{2n}. \tag{1.4.1}$$

Proof. In (1.2.1), set $h = 2$ and $t = kq^2$, and replace c by $-bq^2$, a by cq/k, and b by aq. The resulting identity is equivalent to (1.4.1). □

We note that no generality has been lost by the substitutions above; so Ramanujan had (1.2.1) in full generality for $h = 2$. Padmavathamma [225] has also given a proof of (1.4.1).

Entry 1.4.2 (p. 3). *For* $|bq| < 1$,

$$(q;q^2)_\infty(aq;q^2)_\infty\sum_{n=0}^{\infty}\frac{(-q;q)_n(-bq;q)_n}{(aq;q^2)_{n+1}}q^n = (-bq;q)_\infty\sum_{n=0}^{\infty}\frac{(q;q^2)_n(aq;q^2)_n}{(-bq;q)_{2n+1}}q^{2n}.$$

Proof. In (1.2.1), set $h = 2$, $b = q$, and $t = q^2$, and replace a by aq and c by $-bq^2$. The result then reduces to the identity above upon simplification. □

Entry 1.4.3 (p. 12). *For* $|aq|, |b| < 1$,

$$\sum_{n=0}^{\infty}\frac{a^nq^n}{(q;q)_n(bq;q^2)_n} = \frac{1}{(aq;q)_\infty(bq;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(aq;q)_{2n}b^nq^{n^2}}{(q^2;q^2)_n}.$$

Proof. In (1.2.1), set $h = 2$, $c = 0$, and $t = \tau$, and replace a by bq/τ and b by aq. Then let $\tau \to 0$. The result easily simplifies to the identity above. □

Entry 1.4.4 (p. 12). *For* $|a|, |b| < 1$,

$$\sum_{n=0}^{\infty}\frac{a^nq^{2n}}{(q^2;q^2)_n(bq;q)_{2n}} = \frac{1}{(aq^2;q^2)_\infty(bq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(aq^2;q^2)_nb^nq^{n(n+1)/2}}{(q;q)_n}.$$

Proof. In (1.2.1), set $h = 2$ and $a = 0$, let $b \to 0$, and then replace t by aq^2 and c by bq. □

The previous two entries were also established by Padmavathamma [225]. The next result is a corrected version of Ramanujan's claim.

Entry 1.4.5 (p. 15, corrected). *For any complex number a,*

$$\sum_{n=0}^{\infty}(-aq;q)_n(-q;q)_nq^n = (-q;q)_\infty(-aq;q)_\infty \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(-aq;q)_{2n+1}}.$$

Proof. In (1.2.1), set $h=2$, $a=0$, $b=q$, $c=-aq^2$, and $t=q^2$. Simplification yields Ramanujan's assertion. □

The next two entries specialize to instances of identities for fifth-order mock theta functions, as we shall see in our fourth volume on the lost notebook [33]. The first is a corrected version of Ramanujan's claim.

Entry 1.4.6 (p. 16, corrected). *For any complex number a,*

$$\begin{aligned}\frac{(-aq;q)_\infty}{(-q;q)_\infty}&\sum_{n=0}^{\infty}\frac{(aq;q)_nq^{n^2}}{(q^2;q^2)_n}\\ &=\sum_{n=0}^{\infty}\frac{(-1)^n(a^2q^2;q^2)_{2n}q^{2n^2}}{(q^4;q^4)_n} - a\sum_{n=1}^{\infty}\frac{(a^2q^2;q^2)_{n-1}(-q)^{n(n+1)/2}}{(-q;-q)_{n-1}}.\end{aligned}$$

Proof. The proof of this result is rather more intricate than the proofs of the previous entries in this section. In (1.2.5), replace t by $-q/a$ and let $a\to\infty$ to deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(b;q)_nq^{n^2}}{(q^2;q^2)_n(c;q)_n} &= \frac{(b;q)_\infty(-q;q^2)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(c/b)_{2n}}{(q;q)_{2n}(-q;q^2)_n}b^{2n}\\ &\quad+\frac{(b;q)_\infty(-q^2;q^2)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(c/b)_{2n+1}}{(q;q)_{2n+1}(-q^2;q^2)_n}b^{2n+1}.\end{aligned}$$

Now set $c=0$ and $b=aq$. If we multiply both sides of the resulting identity by $(-aq;q)_\infty/(-q;q)_\infty$, we arrive at

$$\begin{aligned}\frac{(-aq;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(aq;q)_nq^{n^2}}{(q^2;q^2)_n} &= \frac{(a^2q^2;q^2)_\infty}{(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{a^{2n}q^{2n}}{(q;q)_{2n}(-q;q^2)_n}\\ &\quad+\frac{(a^2q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{a^{2n+1}q^{2n+1}}{(q;q)_{2n+1}(-q^2;q^2)_n}\\ &=: T_1+T_2. \qquad (1.4.2)\end{aligned}$$

Next, in (1.2.1) with $h=2$, replace q by q^2, set $a=q^2/t$, $b=a^2q^2$, and $c=0$, and let $t\to 0$. Noting that $(-q^2;q^2)_\infty = 1/(q^2;q^4)_\infty$, we deduce that

$$T_1=\sum_{n=0}^{\infty}\frac{(-1)^n(a^2q^2;q^2)_{2n}q^{2n^2}}{(q^4;q^4)_n}. \qquad (1.4.3)$$

Finally, in (1.2.1), set $h=2$, $a=0$, and $c=-q^2$, and let $b\to 0$. Then set $t=a^2q^2$ and multiply both sides of the resulting equality by $1/(1+q)$. We therefore find that

$$\sum_{n=0}^{\infty}\frac{a^{2n}q^{2n}}{(q^2;q^2)_n(-q;q)_{2n+1}}=\frac{1}{(-q;q)_\infty(a^2q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(a^2q^2;q^2)_nq^{n(n+3)/2}}{(q;q)_n}.$$

Upon multiplying both sides of this last identity by $aq(-q;q)_\infty(a^2q^2;q^2)_\infty$ and noting that $(-q;q)_\infty=1/(q;q^2)_\infty$, we obtain, after replacing q by $-q$ and replacing n by $n-1$ on the right-hand side,

$$T_2=-a\sum_{n=1}^{\infty}\frac{(a^2q^2;q^2)_{n-1}(-q)^{n(n+1)/2}}{(-q;-q)_{n-1}}. \tag{1.4.4}$$

If we substitute (1.4.3) and (1.4.4) into (1.4.2), we obtain our desired identity to complete the proof. □

Entry 1.4.7 (p. 16). *If a is any complex number, then*

$$\frac{(-aq;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(aq;q)_nq^{n(n+1)}}{(q^2;q^2)_n}=\sum_{n=0}^{\infty}\frac{(a^2q^2;q^2)_n(-q)^{n(n+1)/2}}{(-q;-q)_n}$$
$$+a\sum_{n=0}^{\infty}\frac{(-1)^n(a^2q^2;q^2)_{2n}q^{2n^2+4n+1}}{(q^4;q^4)_n}.$$

Proof. In (1.2.5), let $t=-q^2/a$ and $c=0$. After letting $a\to\infty$, set $b=aq$. Multiplying both sides of the resulting identity by $(-aq;q)_\infty/(-q;q)_\infty$, we find that

$$\begin{aligned}\frac{(-aq;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(aq;q)_nq^{n(n+1)}}{(q^2;q^2)_n}&=\frac{(a^2q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(aq)^{2n}}{(q;q)_{2n}(-q^2;q^2)_n}\\&\quad+\frac{(a^2q^2;q^2)_\infty}{(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(aq)^{2n+1}}{(q;q)_{2n+1}(-q;q^2)_{n+1}}\\&=:S_1+S_2.\end{aligned} \tag{1.4.5}$$

Now in (1.2.1) with $h=2$, set $a=0$ and $c=-q$, and let b tend to 0. Then set $t=a^2q^2$. The result, after replacing q by $-q$ and simplifying, is given by

$$\sum_{n=0}^{\infty}\frac{(a^2q^2;q^2)_n(-q)^{n(n+1)/2}}{(-q;-q)_n}=\frac{(a^2q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(aq)^{2n}}{(q;q)_{2n}(-q^2;q^2)_n}=S_1. \tag{1.4.6}$$

Next, in (1.2.1), set $h=2$, replace q by q^2, and then set $b=a^2q^2$, $t=q^6/a$, and $c=0$. After letting $a\to\infty$ and substantially simplifying, we find that

$$a\sum_{n=0}^{\infty}\frac{(-1)^n(a^2q^2;q^2)_{2n}q^{2n^2+4n+1}}{(q^4;q^4)_n}$$
$$=\frac{(a^2q^2;q^2)_\infty}{(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(aq)^{2n+1}}{(q;q)_{2n+1}(-q;q^2)_{n+1}}=S_2. \tag{1.4.7}$$

If we substitute (1.4.7) and (1.4.6) into (1.4.5), we obtain the desired identity for this entry. □

Entry 1.4.8 (p. 16). *For arbitrary complex numbers a and b,*

$$\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(aq;q)_n b^n q^{n^2}}{(q^2;q^2)_n}=(-bq;q^2)_\infty\sum_{n=0}^{\infty}\frac{(aq)^{2n}}{(q;q)_{2n}(-bq;q^2)_n}$$
$$+(-bq^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{(aq)^{2n+1}}{(q;q)_{2n+1}(-bq^2;q^2)_n}.$$

Proof. This entry is a further special case of (1.2.5); replace a by $-bq/t$, set $c=0$ and $b=aq$, and let $t\to 0$. □

In her thesis [225], Padmavathamma also proved Entry 1.4.8. For a combinatorial proof of Entry 1.4.8, see the paper by Berndt, Kim, and Yee [73].

The next entry is the first of several identities in this chapter that provide representations of theta functions or quotients of theta functions by basic hypergeometric series. We therefore review here Ramanujan's notations for theta functions and some basic facts about theta functions.

Recall that the Jacobi triple product identity [18, p. 21, Theorem 2.8], [54, p. 35, Entry 19] is given, for $|ab|<1$, by

$$f(a,b):=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.4.8}$$

Deducible from (1.4.8) are the product representations of the classical theta functions [18, p. 23, Corollary 2.10], [54, pp. 36–37, Entry 22, equation (22.4)],

$$\varphi(-q):=f(-q,-q)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}=\frac{(q)_\infty}{(-q)_\infty}, \tag{1.4.9}$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.4.10}$$

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}=(q;q)_\infty, \tag{1.4.11}$$

where we have employed the notation used by Ramanujan throughout his notebooks. The last equality in (1.4.11) is known as Euler's pentagonal number theorem. We also need the elementary result [54, p. 34, Entry 18(iii)]

$$f(-1,a)=0, \tag{1.4.12}$$

for any complex number a with $|a|<1$. Later, we need the fundamental property [54, p. 34]: For $|ab|<1$ and each integer n,

$$f(a,b)=a^{n(n+1)/2}b^{n(n-1)/2}f\left(a(ab)^n,b(ab)^{-n}\right). \tag{1.4.13}$$

Entry 1.4.9 (p. 10). *Let $\varphi(-q)$ be defined by* (1.4.9) *above. Then*

$$\varphi(-q)\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q)_n^2}=\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{(q^2;q^2)_n}. \tag{1.4.14}$$

First Proof of Entry 1.4.9. In (1.2.1), we set $h=1$, $a=-q/\tau$, $b=\tau$, $c=q$, and $t=\tau$. Letting τ tend to 0, we find that

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q)_n^2}=\frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{(q)_n(-q)_n}. \tag{1.4.15}$$

The desired result follows once we invoke the well-known product representation for $\varphi(-q)$ in (1.4.9). □

Second Proof of Entry 1.4.9. Our second proof is taken from the paper [73] by Berndt, Kim, and Yee.

Multiplying both sides of (1.4.15) by $(q)_\infty$, we obtain the equivalent identity

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q)_n}(q^{n+1};q)_\infty=\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{(q)_n}(-q^{n+1};q)_\infty, \tag{1.4.16}$$

since $(q^2;q^2)_\infty=(-q;q)_\infty(q;q)_\infty$. The left side of (1.4.16) is a generating function for the pair of partitions (π,ν), where π is a partition into n distinct parts and ν is a partition into distinct parts that are strictly larger than n and where the exponent of (-1) is the number of parts in ν. For a given partition pair (π,ν) generated by the left side of (1.4.16), let k be the number of parts in ν. Detach n from the each part of ν and attach k to each part of π. Then we obtain partition pairs (σ,λ), such that σ is a partition into k distinct parts and λ is a partition into distinct parts that are strictly larger than k, and the exponent of (-1) is the number of parts in σ. These partitions are generated by the right side of (1.4.16). Since this process is easily reversible, our proof is complete. □

The series on the left-hand sides of (1.4.14) and (1.4.18) below are the generating functions for the enumeration of gradual stacks with summits and stacks with summits, respectively [23]. Another generating function for gradual stacks with summits was found by Watson [279, p. 59], [75, p. 328], who showed that

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q^2;q^2)_n}, \tag{1.4.17}$$

which is implicit in the work of Ramanujan in his lost notebook [244]. An elegant generalization of the concept of gradual stacks with summits has been devised by Yee, with her generating function generalizing that on the right-hand side of (1.4.17) [286, Theorem 5.2]. See Entry 6.3.1 for a significant generalization of Entry 1.4.10 involving two additional parameters.

Entry 1.4.10 (p. 10).

$$\sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2} = \frac{1}{(q)_\infty^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}. \tag{1.4.18}$$

Proof. In (1.2.1), set $h = 1$, $t = c = q$, and $a = 0$, and then let $b \to 0$. Entry 1.4.10 follows immediately. □

Entry 1.4.11 (p. 10).

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q)_n^2} = \frac{1}{(q)_\infty^2} \left(1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n(n+1)/2}\right). \tag{1.4.19}$$

Proof. In (1.2.1), set $h = 1$, $a = 0$, $c = q$, and $t = q^2$. Now let $b \to 0$ to deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{2n}}{(q)_n^2} &= \frac{1}{(q)_\infty^2}(1-q) \sum_{n=0}^{\infty} (-1)^n \frac{1-q^{n+1}}{1-q} q^{n(n+1)/2} \\ &= \frac{1}{(q)_\infty^2} \left(\sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} + \sum_{n=0}^{\infty} (-1)^{n+1} q^{(n+1)(n+2)/2} \right) \\ &= \frac{1}{(q)_\infty^2} \left(1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n(n+1)/2}\right).\end{aligned}$$

□

Observe that the sum on the right sides in Entries 1.4.10 and 1.4.11 is a false theta function in the sense of L.J. Rogers. Several other entries in the lost notebook involve this false theta function; see [31, pp. 227–232] for some of these entries. In providing a combinatorial proof of Entry 1.4.11, Kim [189] was led to a generalization for which he supplied a combinatorial proof.

The following entry has been combinatorially proved by Berndt, Kim, and Yee [73].

Entry 1.4.12 (p. 10). *For $|a|, |b| < 1$ and any positive integer n,*

$$(-bq^n;q^n)_\infty \sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)/2}}{(q;q)_m(-bq^n;q^n)_m} = (-aq;q)_\infty \sum_{m=0}^{\infty} \frac{b^m q^{nm(m+1)/2}}{(q^n;q^n)_m(-aq;q)_{nm}}. \tag{1.4.20}$$

Proof. In (1.2.1), set $h = n$ and let b tend to 0. Then set $t = -bq^n/a$ and let a tend to ∞. Finally, replace c by $-aq$. □

Entry 1.4.13 (p. 11). *For* $|a| < 1$,

$$\sum_{n=0}^{\infty} \frac{(aq)^n}{(q^2;q^2)_n (bq;q)_n} = \frac{1}{(aq;q^2)_\infty (bq;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq;q^2)_n b^{2n} q^{2n^2+n}}{(q;q)_{2n}} - \frac{1}{(aq^2;q^2)_\infty (bq;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq^2;q^2)_n b^{2n+1} q^{2n^2+3n+1}}{(q;q)_{2n+1}}.$$

Proof. In (1.2.5), let $a = 0$ and let $b \to 0$. Then replace t by aq and c by bq. □

In her doctoral dissertation [225], Padmavathamma gave another proof of Entry 1.4.13, and gave proofs of the following two entries as well.

Entry 1.4.14 (p. 11). *For any complex number* a,

$$(q^2;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^2;q^2)_n q^{n(n+1)/2}}{(q;q)_n} = (aq^4;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^2;q^4)_n q^{4n^2}}{(q^2;q^2)_{2n}} + (aq^2;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^4;q^4)_n q^{4n^2+4n+1}}{(q^2;q^2)_{2n+1}}.$$

Proof. In Entry 1.4.13, replace q by q^2 and set $b = -1/q$. This yields

$$\sum_{n=0}^{\infty} \frac{a^n q^{2n}}{(q^4;q^4)_n (-q;q^2)_n} = \frac{1}{(aq^2;q^4)_\infty (-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(aq^2;q^4)_n q^{4n^2}}{(q^2;q^2)_{2n}} + \frac{1}{(aq^4;q^4)_\infty (-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(aq^4;q^4)_n q^{4n^2+4n+1}}{(q^2;q^2)_{2n+1}}.$$

Consequently, in order to prove the desired result, we must show that

$$(aq^2;q^4)_\infty (-q;q^2)_\infty (aq^4;q^4)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{2n}}{(q^4;q^4)_n (-q;q^2)_n} = (q^2;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^2;q^2)_n q^{n(n+1)/2}}{(q;q)_n}, \qquad (1.4.21)$$

and this follows from (1.2.1). More precisely, let $h = 2$, $c = -q$, and $a = 0$, and let b tend to 0. Then put $t = aq^2$ and simplify. □

Entry 1.4.15 (p. 11). *If* a *is any complex number, then*

$$(q^2;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^2;q^2)_n q^{(n+1)(n+2)/2}}{(q;q)_n} = (aq^4;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^2;q^4)_n q^{4n^2+4n+1}}{(q^2;q^2)_{2n}} + (aq^2;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^4;q^4)_n q^{4n^2+8n+4}}{(q^2;q^2)_{2n+1}}.$$