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Mohamed El Kadiri
Bent Fuglede

Classical Fine Potential Theory

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To the memory of my parents Ahmed and Fatna,
To my wife Malika,
To my daughters: Asmae, Soumaya, Nadia, Meryem, and Zineb.
To the memory of my co-author Bent Fuglede,
To Dorthea and Einar.
Mohamed El Kadiri

Bent Fuglede (8.10.1925–7.12.2023)

Preface

By classical potential theory, we mean potential theory connected to the Laplace operator for open sets in Euclidean space. By contrast, modern potential theory deals with such things as harmonic spaces (axiomatic theories), potentials on Riemannian manifolds, probabilistic potential theory, potential theory of Markov process, and nonlinear potential theory or complex potential theory (pluripotential theory).

The present book deals with the fine potential theory issued from classical potential theory and founded, in the axiomatic frame, by Fuglede (see [Fu8]) in the beginning of the 1970s. The word "fine" in the title of this book refers to the *fine topology* on $\mathbb{R}^N$, introduced by Henri Cartan in a letter to Marcel Brelot in December 1939. Brelot introduced and studied the concept of a set $E \subset \mathbb{R}^N$ being *thin* at a point $x \in \mathbb{R}^N$.[1] The typical example of thinness of a set E at a point x was $E = \mathbb{R}^N \setminus U$, where U is a domain in $\mathbb{R}^N$ and x is an irregular boundary point for U in the sense of the Dirichlet problem. Cartan noticed that a set $E \subset \mathbb{R}^N$ is thin at point $x \notin E$ if and only if $\mathbb{R}^N \setminus E$ is a neighborhood of x in a certain topology, namely the smallest topology on $\mathbb{R}^N$ in which all subharmonic functions are continuous. This new topology is strictly finer than the Euclidean topology on $\mathbb{R}^N$, and Cartan called it the *fine topology*.

Doob has pointed out in [Do4] that for several years after its appearance, the fine topology was viewed "merely as a tool for phrasing results elegantly". However, it has gradually become a powerful instrument in potential theory, and many results are better understood when formulated in terms of the fine topology.

The fine topology was extended to axiomatic potential theory by Brelot, Doob, and Bauer (harmonic spaces) and later by Boboc, Constantinescu, and Cornea (H-cones). The fine topology, say on $\mathbb{R}^N$, admits a well-known characterization in probabilistic terms (Brownian motion), and many results have been established in the literature by probabilistic arguments. In the present book we only use analytic methods. For the probabilistic setup, see in particular [Do4], [BH3], and [Ng6].

After having shown that the fine topology is locally connected [Fu7], the second named author of the present book felt that perhaps there is a good notion of harmonic and sub/superharmonic functions relative to the fine topology, and this turned out to

[1] The English word "thin" serves as a (somewhat rough) translation of the French "effilé" (thready).

be the case, see [Fu8], and later works by several authors. Again, much of this work was framed in suitable axiomatic settings, especially with $\mathbb{R}^N$ replaced by a harmonic space in the sense of Brelot, often required to satisfy the axiom of domination and sometimes also further axioms. Many results in the beginning of the theory also hold in more general harmonic spaces and without assuming the axiom of domination; see in particular [BBC2], [BC], [LM1], and [LMZ].

The methods of studying the fine topology extend to a certain degree to a topology on $\mathbb{C}^n$ called the *pluri-fine topology*, defined as the smallest topology in which all plurisubharmonic functions on subsets of $\mathbb{C}^n$ are continuous. The pluri-fine topology was first studied thoroughly by Bedford and Taylor [BT3] in 1987, and they applied it to the Monge–Ampère operator on an open subset of $\mathbb{C}^n$. El Marzguioui and Wiegerinck [EMW1] proved the local connectedness of the pluri-fine topology. The trace of the pluri-fine topology on each complex line in $\mathbb{C}^n$ is identical with the Cartan fine topology on that line (identified with $\mathbb{C} \cong \mathbb{R}^2$). This fact led El Kadiri, Fuglede, and Wiegerinck [EKFW] to formulate and establish the main properties of two kinds of plurifinely plurisubharmonic functions (weak and strong) on a plurifinely open subset of $\mathbb{C}^n$. On a Euclidean open subset of $\mathbb{C}^n$ these two kinds of functions are both identical with subharmonicity.

Fine potential theory has led to solutions of classical problems (for example, regarding asymptotic paths for subharmonic functions), or has provided the language to answer natural questions (for example, regarding uniform limits of harmonic functions on compact sets), or has allowed an elegant extension of classical holomorphic functions theory to a wider class of "domains" (namely, finely holomorphic functions), in addition to more recent developments.

The present book does not deal with axiomatic settings, but exclusively with the case of functions (and measures) defined on subsets of $\mathbb{R}^N$, resp. $\mathbb{C}^n$. The book contains five chapters. The first four chapters are about fine potential theory (including a section on finely holomorphic functions), Chap. 1 being preparatory. Chapter 5 deals with pluri-fine pluripotential theory.

An overview on further developments and works on classical, axiomatic, or probabilistic fine potential theory and the use of fine topology in both real and complex analyses are given at the end of the book.

The presentation is largely self-contained. The book will serve as an essential reference for students and researchers alike, and to all with an interest in classical potential theory and its applications. It will be useful both in courses, and for self-study.

In preparing this book, we have benefitted from works of earlier authors. In particular, we acknowledge our indebtedness to Doob [Do4], Armitage and Gardiner [AG], Landkoff [Lan], and Fuglede [Fu8]. We are also grateful to Profs. Christian Berg, Palle E. Jorgensen, and Jan Wiegerinck for reading the book and making helpful suggestions, and for helping to publish it . We would like to extend our thanks to El Kadiri Asmae, El Kadiri Soumaya, El Kadiri Nadia, El Kadiri Meryem, and El Kadiri Zineb for their careful reading and the corrections they provided. We also express our appreciation to the staff of Springer Nature India.

Rabat, Morocco — Mohamed El Kadiri
Copenhagen, Denmark — Bent Fuglede

The Life and Work of Bent Fuglede

A little more than a month after the submission to Springer Nature of the manuscript to the present book, Bent Fuglede passed away.

With the death of Bent Fuglede, Denmark has lost one of its best mathematicians and the Department of Mathematics at the University of Copenhagen has lost a highly valued teacher, researcher, and colleague. He is survived by his son Einar and daughter-in-law Dorthea and two grandsons. His wife Ólafia Einarsdóttir of more than 60 years died in 2017.

Bent Fuglede was born on October 8, 1925, at Frederiksberg, a community within the Copenhagen area. His parents were both working in a bank. He received a high school diploma from Skt. Jørgens Gymnasium in 1943, and he graduated from the University of Copenhagen in 1948. The first two years of study comprised mathematics, physics, astronomy, and chemistry, and in the last part of his studies he specialized in mathematics, but followed also lectures in physics. His advisor in mathematics was Børge Jessen, who proposed him to specialize in the theory of Hilbert spaces and their operators, a subject that was relevant for his interest in quantum mechanics.

Bent Fuglede obtained a grant which made it possible for him to spend two years in the United States, and he arrived by ship to New York at the end of July 1949. He stayed there for a month before traveling West to participate in an AMS conference in Boulder. From there he went on to Stanford, where he spent a year.

During the month in New York, he visited the Institute for Advanced Study in Princeton for some days, curious about the place where he should stay for the academic year 1950–1951. It was during this short stay that he solved a problem, which had been on his mind since his study of Spektraldarstellung linearer Transformationen des Hilbertschen Raumes by Béla v. Sz. Nagy, published in 1942. In the book, there was a claim that it is easy to see that if a bounded operator commutes with a normal operator on a Hilbert space, then it also commutes with the adjoint operator. The claim was not easy to prove and at a meeting between Jessen and Nagy, the latter had admitted that he had overlooked the difficulty, and that he did not know a proof. Fuglede presented his proof at the Boulder meeting, and the participants agreed that it was correct. During the visit at Stanford, Fuglede submitted the proof to John von Neumann, and it was published in the *Proceedings of the National Academy of Sciences* in January 1950. Von Neumann had discussed the problem without solving it in a paper in Portugalia Math. 3 (1942); see Section 30.

During the year at Stanford, he followed lectures by Polya and Szegő, and he was invited to give lectures for a semester about operators on Hilbert space.

During the summer of 1950, he drove across the country to Princeton, where he stayed for the next year. He participated in the International Congress of Mathematicians (ICM) at Harvard, the first ICM after WW2.

Back in Denmark, he became scientific assistant at the Polytechnical Highschool (today Technical University of Denmark, DTU), and later he was appointed associate professor at the Department of Mathematics at the University of Copenhagen. Having defended the dissertation "Extremal Length and Closed Extensions of Partial Differential Operators", he was appointed professor at DTU in 1960. He returned to the University of Copenhagen as professor in 1965. He started by preparing excellent lecture notes for the major course in functional analysis for third year students of mathematics. In the years that followed, he taught at all levels, always with a clear and well-worked presentation and with an easy-to-read blackboard script. His accompanying lecture notes often contained new results and elegant proofs. In 1992, he allowed himself to retire, but continued as an active researcher and professor emeritus at the Department of Mathematics and he was active until shortly before his death.

During the long stay at the Institute for Advanced Study, he started collaboration with the American mathematician Richard V. Kadison on, among other things, determinant theory in infinite dimension. This led to Kadison coming to Copenhagen, and he gained a lifelong connection to Denmark, because he met his wife there. The Fuglede–Kadison determinant continues to be used, e.g. by Lück in his development of L^2-cohomology and in the construction of the Brown measure for non-normal operators.

Subsequently, Fuglede studied partial differential operators, which led to the aforementioned doctoral dissertation, and also to research on potential theory. This theory deals with the solutions to Laplace's equation, i.e. harmonic functions and Newton potentials. In this subject, he quickly became a leading figure worldwide, and he often gave a talk at the potential theory seminar in Paris, Séminaire Brelot–Choquet–Deny named after the three leading French mathematicians in the subject. In the 1960s, Brelot and his students developed an abstract potential theory called harmonic spaces. A harmonic space is a locally compact topological space, where a vector space of continuous real valued functions is specified for each open subset, modeling the harmonic functions in open sets of a Euclidean space. Brelot established a few fundamental axioms that these abstract spaces had to satisfy in order that many of the key results of potential theory could be deduced. The significance of the abstract theory was that one could obtain these key results for solution spaces to PDEs related to the Laplace operator by simply checking the axioms.

In harmonic spaces superharmonic functions can be introduced, but as in the classical theory, these are not always continuous, but only lower semicontinuous. In 1940, Henri Cartan got the idea to replace the classical Euclidean topology by a topology having more open sets, so that superharmonic functions become continuous. Cartan called this new topology the fine topology. The fine topology also appears in the theory of Brownian motion and similar stochastic processes, because a finely open set can be characterized as follows: If the process is in the finely open set at a specific time t_0, then it remains there almost surely for a positive period of time. When Fuglede succeeded proving that the fine topology is locally connected, he was inspired to construct what he called finely harmonic functions in finely open sets, and this was the start of a new era in potential theory and complex analysis. He published a comprehensive presentation of the theory in a Springer Lecture Notes, *Finely Harmonic Functions*, 1972, but already at the ICM in Nice, 1970, he gave an invited lecture on it.

The topic became hot and was further developed in the following years by Fuglede himself and with contributions from many others. He experienced the great honor that Heinz Bauer's plenary lecture on potential theory at the ICM in Vancouver, 1974, was largely about Fuglede's finely harmonic theory.

Just as harmonic functions in the plane are closely related to holomorphic functions, he succeeded in developing a rich theory for finely holomorphic functions in finely open sets of the complex plane. It also turned out that Fuglede's theory was a natural continuation of Borel's theory for monogenic functions, which was published as a monograph as early as in 1917.

Fuglede corresponded with a large number of mathematicians from home and abroad, and he was always generous with advice and improvements to manuscript drafts. He had a rare ability to penetrate to the heart of a mathematical problem, and through this he was often able to provide a surprising solution to the problem.

In 1974, Fuglede wrote a paper in the *Journal of Functional Analysis*: "Commuting Self-Adjoint Partial Differential Operators and a Group Theoretic Problem". It has around 300 citations. The work, which was inspired by a question from Irving Segal, associates a geometric property of an n-dimensional domain G with a property of the Hilbert space $L^2(G)$. The result is later known as Fuglede's conjecture. The Danish mathematician Steen Pedersen from Ohio published a number of papers on the subject, and later Terence Tao published two papers: "Fuglede's Conjecture Holds for Convex Planar Domains" (2001) and "Fuglede's Conjecture Is False in 5 and Higher Dimensions" (2003).

Fuglede decided to retire in the spring of 1992. He felt his health a little failing after a year's stay at the Institute for Advanced Study in Princeton, where he was accompanied by his wife. Fortunately, his health turned out to be fine and he achieved an emeritus period of approximately 30 years, during which he was very active. Out of the 114 of his works mentioned in Math. Sci. Net, roughly 40 are written after he became emeritus. During this period, he made significant contributions to many different areas, e.g. isoperimetric inequalities, Riemannian manifolds, and moment problems. It should also be mentioned that he published a research monograph together with James Eells: *Harmonic Maps Between Riemannian Polyhedra*, Cambridge Tracts in Mathematics Vol. 142, 2001. Together with Natalia Zorii from Kyiv, he has since 2016 written eight papers dealing with energy problems with respect to Riesz kernels, an impressive achievement for a person over 90 years of age.

Fuglede was a member of the Royal Danish Academy of Sciences and Letters, the Finnish Academy of Sciences, and the Bavarian Academy of Sciences. For a number of years, he was a member of the editorial boards of the journals *Expositiones Mathematicae* and *Potential Analysis*.

With his great and broad knowledge in many areas of mathematics, he was an obvious member of assessment committees at home and abroad.

Bent Fuglede passed away on December 7, 2023.

Christian Berg
Copenhagen, Denmark
June 2024

Notations and Terminology

$\mathbb{N}$ is the set of natural integers $1, 2, ...,$ and $\mathbb{Z}_+$ is the set of the integers 0, 1, 2,...

The space of real valued continuous functions, resp. functions of class $\mathcal{C}^\infty$, with compact support on an open subset $\Omega \subset \mathbb{R}^N$ is denoted by $\mathcal{C}_0(\Omega)$, resp. $\mathcal{C}_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$.

When speaking of a limit or lim inf or lim sup, say of a function $f(x)$ (with values in $\overline{\mathbb{R}} = [-\infty, +\infty]$) as $x \to x_0$, we follow for example Bourbaki and include $f(x_0)$ in the definition. (The alternative definition, where $f(x_0)$ is not included, is used among others by J. L. Doob, for example in his book [Do4].) A function f on an open subset Ω of a topological space X with values in $\overline{\mathbb{R}}$ is said to be lower semicontinuous (l.s.c.) at x_0, respectively upper semicontinuous (u.s.c.) at $x_0 \in \Omega$ if $\liminf_{x\to x_0} f(x) \geq f(x_0)$, respectively if $-f$ is l.s.c. at x_0, that is, $\limsup_{x\to x_0} f(x) \leq f(x_0)$. The function f is said to be l.s.c., respectively u.s.c. on Ω if f is l.s.c., respectively u.s.c. at every point $x \in \Omega$.

By a *signed measure* on a locally compact space X we mean a Radon measure on X. A positive Radon measure on X is simply called a measure on X. The σ-algebra of Borel measurable subsets of topological space X be denoted by $\mathcal{B}(X)$. A σ-finite positive measure on $(X, \mathcal{B}(X))$ is called a Borel measure on X. Let $A \subset X$, the *trace* of $\mathcal{B}(X)$ on A, that is, the set of the Borel subsets of A of the form $C \cap A$, $C \in \mathcal{B}(X)$, is denoted by $\mathcal{B}_A(X)$ and its elements be the Borel subsets of the topological space A equipped with the induced topology by X. If $A \in \mathcal{B}(X)$ then $\mathcal{B}_A(X) = \{C \in \mathcal{B}(X) : C \subset A\} \subset \mathcal{B}(X)$. The restriction of a Borel measure μ on X to a μ-measurable set $A \subset X$, also called *the part of* μ *on* A is the measure $\mu_A = 1_A\mu$ on X, also given by $\mu_A(C) = \mu(C \cap A)$ for $C \in \mathcal{B}(X)$. It is generally identified with the Borel measure μ' on the topological space A defined as the restriction of μ to $\mathcal{B}_A(X) = \mathcal{B}(A)$. Likewise, a Borel measure μ' on A is identified with its extension μ to X given by $\mu(C) = \mu'(A \cap C)$ for $C \in \mathcal{B}(X)$ (this identification is compatible with the above because $\mu = \mu_A$). A measure μ is said to be *carried* by a set $A \subset X$ if A is μ-measurable and $\mu(\complement A) = 0$.

The Euclidean norm on $\mathbb{R}^N$ $(N \geq 1)$ is denoted by $|\cdot|$ or $\|\cdot\|$. For $x \in \mathbb{R}^N$ and $r > 0$ we denote by $B(x, r)$, resp. $\overline{B}(x, r)$, the open, resp. the closed, ball of radius r centered at x. In the absence of other indication, Ω denotes a *Greenian domain* (that is, a connected Greenian subset of $\mathbb{R}^N, N \geq 2$), mostly serving as an ambient (locally compact) space; see Sect. 1.1.4. We denote by ℓ_N Lebesgue measure on $\mathbb{R}^N$. The restriction of ℓ_N to a ℓ_N-measurable set $X \subset \mathbb{R}^N$ is also denoted by ℓ_N and $L^p(X)$ denotes the space $L^p(X, \ell_N)$ $(1 \leq p \leq \infty)$.

A map into $\overline{\mathbb{R}} = [-\infty, +\infty]$ is called a *numerical* function. For any function $f : \Omega \to \overline{\mathbb{R}}$ write $f_+ = f \vee 0 (= \max\{f, 0\}), f_- = (-f) \vee 0$. Following Bourbaki's terminology, we say that a function is *positive* resp. *negative* if $f_+ = f$, resp. $f_- = -f$.

For the *upper* and *lower* integrals of a numerical function f with respect to a measure μ we follow [Bou6, Chap. 4, § 4, Exercises 5 and 6]. Thus the upper integral $\int^* f\, d\mu$ is defined as the infimum of the integrals $\int g\, d\mu$ where g ranges over all numerical μ-integrable lower l.s.c. functions $\geqslant f$. (If there is no such function g we define $\int^* f\, d\mu$ to be $+\infty$.) The lower integral $\int_* f\, d\mu$ is defined analogously, or equivalently as $-\int^*(-f)\, d\mu$. If $f \geqslant 0$ then $\int^* f\, d\mu$ agrees with the usual upper integral as defined in [Bou6, Chap. 4, § 1, No. 3].

For any increasing sequence of numerical functions f_n such that $\int^* f_n\, d\mu > -\infty$ for some n we have $\int^*(\sup_n f_n)\, d\mu = \sup_n \int^* f_n\, d\mu$.

If $\int^* f\, d\mu = \int_* f\, d\mu$ we may simply write $\int f\, d\mu$ in their place. This occurs in particular if f is μ-*measurable* and if in addition $\int^* f^+\, d\mu < +\infty$ or $\int^* f^-\, d\mu < +\infty$, thus for example if $f \geqslant 0$ is μ-measurable. In order that f be μ-*integrable*, it is necessary and sufficient that $\int^* f\, d\mu$ and $\int_* f\, d\mu$ be equal and finite.

Contents

Chapter 1
Background in Potential Theory

By classical potential theory, we mean the potential theory of the Laplace equation in an open set of Euclidean space $\mathbb{R}^N$ ($N \geq 2$). This chapter is devoted to some background in classical potential theory used in all of the book.

The first section deals with fundamental tools and basic results (without proofs) of this theory. We recall there some fundamental properties of [hyper/super]harmonic functions and potentials. We define Greenian sets, their Greenian kernels and Green potentials, polar sets (sets of singularities of superharmonic functions), and Green capacity and state the fundamental convergence theorem. The Dirichlet problem on balls and on general Greenian open sets, harmonic measures, and regular and irregular points are discussed.

Section 1.2 is devoted to the notion of thin sets, introduced by Brelot in studying the Dirichlet problem, and the Cartan-Brelot fine topology, which is a powerful instrument in potential theory. Deep properties of this topology such as regularity, Baire property, quasi-Lindelöf property, local connectedness (which will be proved in Chap. 2), and Borel measurability of finely continuous functions are also considered. The section is closed by a discussion of thinness at infinity.

Reduction and sweeping of functions, which play a central role in classical potential theory and also in the development of fine potential theory, are thoroughly studied in Sect. 1.3. Polarity and thinness are characterized by means of sweeping of positive functions. Sweeping of measures, studied in Sect. 1.4, allows to study the base of a set or a function and the base operation. It also allows to define the harmonic measure on finely open sets. It is by means of (fine) harmonic measure that the finely [super]harmonic functions on finely open sets are defined in Chap. 2.

In Sect. 1.5, we study "quasi-topological" concepts (quasi-open and quasi-closed sets, quasi-continuous functions) relative to a suitable kind of capacity, not just the Green capacity. Capacity is a suitable countably subadditive set function defined on subsets of a Greenian domain Ω. Inspired by Doob [Do1], we also consider

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capacity as a suitable sublinear functional on the lattice cone $\mathcal{F}^+$ of all functions $f : \Omega \to [0, +\infty]$. As an application, we obtain the remarkable and very useful Brelot property (see Theorem 1.5.29).

1.1 Basics of Classical Potential Theory

For proofs and further information concerning this section, we refer to the monograph of Armitage and Gardiner [AG] and the monograph of Doob [Do4].

1.1.1 Denote by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_N^2}$$

the Laplacian on $\mathbb{R}^N$, $N \geqslant 2$. Let Ω denote an open subset of $\mathbb{R}^N$. A function $u : \Omega \to \mathbb{R}$ is said to be *harmonic* if $u \in \mathcal{C}^2(\Omega)$ and u satisfies the Laplace equation $\Delta u = 0$ identically on Ω. The harmonic functions on Ω form a linear subspace of the vector space of all functions $\Omega \to \mathbb{R}$, closed under locally uniform convergence.

For any point $y \in \mathbb{R}^N$ the function U_y defined on $\mathbb{R}^N \setminus \{y\}$ by

$$U_y(x) = \begin{cases} -\log|x-y|, & (N = 2) \\ |x-y|^{2-N}, & (N \geqslant 3) \end{cases}$$

is harmonic on $\mathbb{R}^N \setminus \{y\}$; and U_y is called the *fundamental harmonic function* with pole at y. We extend U_y to y by continuity, so that $U_y(y) = +\infty$.

1.1.2 The *Poisson kernel* of an (open) ball $B = B(x_0, r)$ in $\mathbb{R}^N$ is the function

$$K_B(x, y) = \frac{1}{\sigma_N r} \frac{r^2 - |x - x_0|^2}{|x - y|^N} \quad (y \in \partial B,\ x \in \mathbb{R}^N \setminus \{y\}),$$

where σ_N denotes the surface area of the unit sphere in $\mathbb{R}^N$. Clearly, $K_B(x, y) > 0$ for $x \in B$, $K_B(x, y) < 0$ for $x \in \mathbb{R}^N \setminus \overline{B}$, and $K_B(x, y) = 0$ for $x \in \partial B \setminus \{y\}$. The *Poisson integral* $P_B\mu$ of a signed measure μ on ∂B is well defined by

$$P_B\mu(x) = \int_{\partial B} K_B(x, y)\, d\mu(y) \quad (x \in B)$$

because $K_B(x, \cdot)$ for any $x \in B$ is bounded and continuous on the compact set ∂B carrying μ. Also, $P_B\mu$ is harmonic on B, see, for example, [AG, Theorem 1.3.3], also for (1.1.1) below.

If a function $f : \partial B \to \mathbb{R}$ (or a function defined on some set containing ∂B, for example on $\overline{B}$) is integrable with respect to the surface measure σ on ∂B, we may write $P_B f$ for $P_B(f\sigma)$. For $y \in \partial B$ we have

$$\liminf_{\partial B \ni z \to y} f(z) \leqslant \liminf_{B \ni x \to y} P_B f(x) \leqslant \limsup_{B \ni x \to y} P_B f(x) \leqslant \limsup_{\partial B \ni z \to y} f(z). \tag{1.1.1}$$

If f is continuous (in the extended sense) at $y \in \partial B$, it follows that $P_B f(x) \to f(y)$ for $x \to y$, $x \in B$.

Thus $P_B f$ solves the *Dirichlet problem* for B when f is finite and continuous on ∂B. In other words, $K_B(x, \cdot)\, d\sigma$ is the *harmonic measure* at the point $x \in B$, denoted by μ_x^B, and so

$$P_B f(x) = \int_{\partial B} K_B(x, \cdot) f \, d\sigma = \int f \, d\mu_x^B. \tag{1.1.2}$$

The pointwise supremum of any upper directed family of harmonic functions on a connected open set $\Omega \subset \mathbb{R}^N$ is either harmonic or identically $+\infty$. This is Harnack's *convergence theorem*. It depends on the *Harnack inequalities*. See, for example, [AG, Sections 1.4 and 1.5].

A function $u : \Omega \to \overline{\mathbb{R}} = [-\infty, +\infty]$ (also called a numerical function on Ω) is said to be *hyperharmonic* if (i) u is l.s.c. and $> -\infty$, and (ii)

$$u(x) \geqslant \frac{1}{\sigma_N r^{N-1}} \int^*_{\partial B(x,r)} u \, d\sigma \quad \text{when } \overline{B}(x, r) \subset \Omega, \tag{1.1.3}$$

σ_N denoting the surface area of the unit ball in $\mathbb{R}^N$.

If, in addition to (i) and (ii), we have (iii) $u \not\equiv +\infty$ in each component of Ω, then u is called *superharmonic*. If u satisfies (i) and (iii), one may replace in (ii) the spherical mean-value by the ball mean-value

$$\frac{1}{\ell_N(B(x, r))} \int_{B(x,r)} u \, d\ell_N = (P_{B(x,r)} u)(x),$$

ℓ_N denoting N-dimensional Lebesgue measure. Moreover, one may then replace (ii) by

$$u \geqslant P_B u \quad \text{on } B \text{ for every ball } B \text{ with } \overline{B} \subset \Omega.$$

Beginning after (1.1.3), it suffices to consider balls $B(x, r)$ with sufficiently small radius r depending on x.

It follows from the above that every superharmonic function u on an open set Ω is locally Lebesgue integrable and hence finite Lebesgue almost everywhere, e.g. [AG, Theorem 3.1.3]. In particular, u is finite on a dense subset of Ω. Furthermore, if u_1 and u_2 are superharmonic on Ω and if $u_1 = u_2$ ℓ_N-a.e., then the same holds everywhere, e.g. [AG, Corollary 3.2.7]. Similar assertions hold with ℓ_N replaced by the surface measure σ on a closed ball.

For every $y \in \mathbb{R}^N$ the function U_y is superharmonic on $\mathbb{R}^N$ (and harmonic on $\mathbb{R}^N \setminus \{y\}$).

A function $u : \Omega \to \overline{\mathbb{R}}$ is called *hypoharmonic*, resp. *subharmonic*, if $-u$ is hyperharmonic, resp. superharmonic.

It is known that a numerical function u is harmonic if and only if u is both hyperharmonic and hypoharmonic. Equivalently, u shall be finite and continuous and

$$u(x) = \frac{1}{\sigma_N r^{N-1}} \int_{\partial B(x,r)} u \, d\sigma \quad \text{when } \overline{B}(x,r) \subset \Omega. \tag{1.1.4}$$

Again one may replace the spherical mean-value by the ball mean-value in (1.1.4); and one may replace (1.1.4) by $u = P_B u$ for any ball B with $\overline{B} \subset \Omega$.

A harmonic function on all of $\mathbb{R}^N$ is constant if upper or lower bounded. For this version of Liouville's theorem for holomorphic functions of a complex variable, see, for example, [AG, Theorem 1.2.6].

The hyperharmonic functions on Ω form a convex subcone of $\overline{\mathbb{R}}^\Omega$, stable under pointwise supremum for upper directed families and under pointwise infimum for finite families. The pointwise supremum, resp. infimum, of a family $\mathcal{F}$ of numerical (in particular: of hyperharmonic) functions on Ω is denoted by $\sup \mathcal{F}$, resp. $\inf \mathcal{F}$. We write $u \vee v$ or $\max\{u, v\}$ for $\sup\{u, v\}$ and $u \wedge v$ or $\min\{u, v\}$ for $\inf\{u, v\}$ when u and v are numerical functions, in particular (finite or infinite) constants.

A hyperharmonic function $u \geqslant 0$ on a connected open set $\Omega \subset \mathbb{R}^N$ is either everywhere > 0 or $\equiv 0$. In fact, $\sup_{n\in\mathbb{N}}(nu)$ is hyperharmonic, in particular l.s.c. and (if $\not\equiv +\infty$) locally ℓ_N-integrable, so $\{u > 0\}$ is an open ℓ_N-null set, hence void.

1.1.3 Denote by $\partial^\infty\Omega$ the boundary of an open set $\Omega \subset \mathbb{R}^N$ relative to the one point compactification $\mathbb{R}^N \cup \{\infty\}$ of $\mathbb{R}^N$. (Thus $\partial^\infty\Omega = \partial\Omega$ if Ω is bounded, whereas $\partial^\infty\Omega = (\partial\Omega) \cup \{\infty\}$ if Ω is unbounded.)

Let u be hyperharmonic on an open set $\Omega \subset \mathbb{R}^N$. If

$$\liminf_{x\to y} u(x) \geqslant 0 \text{ for every } y \in \partial^\infty\Omega,$$

then $u \geqslant 0$. This is the classical *boundary minimum principle.*

1.1.4 An open subset Ω of $\mathbb{R}^N$ is called a *Greenian set* if for some, and hence for any, $y \in \Omega$ the fundamental harmonic function U_y has a *greatest harmonic minorant* h_y on Ω. A connected Greenian set is called a Greenian domain. Any open subset of a Greenian set is Greenian.

Suppose that $\Omega \subset \mathbb{R}^N$ is a Greenian set. The function $G_\Omega : \Omega \times \Omega \to\ ]0, +\infty]$ defined by

$$G_\Omega(x, y) = U_y(x) - h_y(x)$$

is called the *Green kernel* for Ω. For each $y \in \Omega$, $G_\Omega(\cdot, y)$ is called the *Green function* on Ω with pole at y. The Green kernel is symmetric $G_\Omega(x, y) = G_\Omega(y, x)$, l.s.c., and continuous in the extended sense on $\Omega \times \Omega$; see, for example, [AG, Theorem 4.1.9] or [Do4, 1.VII.4].

For $N \geqslant 3$ (only), the entire space $\mathbb{R}^N$ is a Greenian set, and U_y is then itself the Green function on $\mathbb{R}^N$ with pole at y.

For $N = 2$, an open set $\Omega \subset \mathbb{R}^2$ is a Greenian set if for example $\mathbb{R}^2 \setminus \partial\Omega$ is disconnected, thus in particular if Ω is disconnected or bounded (see further Sect. 1.1.12).

Let $B = B(x_0, r)$ in $\mathbb{R}^N$ $(N \geq 2)$ and

$$\phi(x, y) = \frac{(r^2 - |x - x_0|^2)(r^2 - |y - x_0|^2)}{r^2|x - y|^2} \quad (x, y \in B; x \neq y).$$

Then the Green kernel for the ball B in $\mathbb{R}^N$ is given for $x, y \in B$ with $x \neq y$ by

$$G_B(x, y) = \begin{cases} 2^{-1}\log(1 + \phi(x, y)) & (x \neq y) \\ +\infty & (x = y) \end{cases}$$

if $N = 2$, and

$$G_B(x, y) = \begin{cases} (1 - (1 + \phi(x, y))^{1-\frac{N}{2}})|x - y|^{2-N} & (x \neq y) \\ +\infty & (x = y) \end{cases}$$

if $N \geqslant 3$ (see [AG, p. 91]).[1]

1.1.5 For any (positive) measure μ on a Greenian set $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$, the function $G_\Omega\mu$ on Ω defined by

$$G_\Omega\mu(x) = \int_\Omega G_\Omega(x, y)\, d\mu(y)$$

is hyperharmonic. By Fubini's theorem, we have the *reciprocity theorem*: $\int G_\Omega\mu\, d\nu = \int G_\Omega\nu\, d\mu$ for $\mu, \nu \geqslant 0$. If superharmonic, $G_\Omega\mu$ is called the (Green) *potential* of μ. If $f \geqslant 0$ on Ω is Lebesgue integrable, we may write $G_\Omega f$ for $G_\Omega(f\ell_N)$. A measure μ is uniquely determined by its potential $G_\Omega\mu$ (assumed $\not\equiv +\infty$ on each component of Ω); this is contained in Sect. 1.1.6 below.

1.1.6 Let u denote a superharmonic function on a Greenian set $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$, and suppose that u has a subharmonic minorant on Ω. Then there exists a unique positive measure μ_u on Ω, called the *Riesz measure* associated with u (relative to Ω), such that u has the decomposition

$$u = G_\Omega\mu_u + h,$$

where h is the greatest harmonic minorant, or equivalently: the greatest subharmonic minorant, of u on Ω. This decomposition is called the *Riesz decomposition* of u, e.g. [Do4, 1.IV.7–8].

[1] For the explicit expression for the Green kernel for a ball in $\mathbb{R}^N$ $(N \geqslant 3)$ and a half space in $\mathbb{R}^N$ $(N \geqslant 2)$, see, for example, [AG, p. 92].

On a Greenian set Ω, a superharmonic function $u \geqslant 0$ with associated Riesz measure μ_u is a potential (namely of μ_u) if and only if every harmonic (or equivalently: every subharmonic) minorant of u is $\leqslant 0$ (equivalently: every positive harmonic [or positive subharmonic] minorant of u is the function 0). Clearly, every positive superharmonic minorant of a potential is a potential.

Any finite sum of potentials is a potential. For example, if p_1 and p_2 are potentials and if h is a harmonic minorant of $p_1 + p_2$ then $h - p_1$ is a subharmonic minorant of the potential p_2 and therefore $h - p_1 \leqslant 0$, that is, $h \leqslant p_1$, hence $h \leqslant 0$ since p_1 is a potential. This shows that $p_1 + p_2$ indeed is a potential.

The pointwise sum p of a family $(p_i)_{i\in I}$ of potentials is a potential if p is superharmonic. To see this, let h be a positive harmonic minorant of p. For any finite subset J of I, $h - \sum_{i\in I\setminus J} p_i$ is a hypoharmonic minorant of the potential $\sum_{i\in J} p_i$ and is therefore $\leqslant 0$. It follows that h equals zero on the dense set $\{p < +\infty\}$ (see Sect. 1.1.2) and hence $h \equiv 0$, by continuity.

1.1.7 Let u be hyperharmonic on an open set Ω, and v be hyperharmonic on an open set $\omega \subset \Omega$. If

$$\liminf_{\omega\ni x\to y} v(x) \geqslant u(y) \quad \text{for every } y \in \Omega \cap \partial\omega$$

then the function w defined on Ω by

$$w(x) = \begin{cases} u(x) \wedge v(x) & \text{for } x \in \omega \\ u(x) & \text{for } x \in \Omega \setminus \omega \end{cases}$$

is hyperharmonic on Ω. In fact, w is clearly hyperharmonic on ω and on $\Omega \setminus \overline{\omega}$. And for any point $x \in \Omega \cap \partial\omega$ and any ball $B(x, r)$ small enough so that $\overline{B}(x, r) \subset \Omega$, the hyperharmonic mean-value property holds:

$$w(x) = u(x) \geqslant \frac{1}{\sigma_N r^{N-1}} \int^*_{\partial B(x,r)} u \, d\sigma \geqslant \frac{1}{\sigma_N r^{N-1}} \int^*_{\partial B(x,r)} w \, d\sigma.$$

From this, we shall obtain the following *relative boundary minimum principle*: Let u be hyperharmonic on an open subset ω of a Greenian set Ω, and suppose that

$$\liminf_{\omega\ni x\to y} u(x) \geqslant 0 \quad \text{for every } y \in \Omega \cap \partial\omega.$$

Suppose in addition that there exists a Green potential p on Ω such that $u \geqslant -p$ on ω. Then $u \geqslant 0$ on ω. In fact, by the above result with u replaced by 0 and v by u, the extension w of $0 \wedge u$ by 0 on $\Omega \setminus \omega$ is hyperharmonic and $\geqslant -p$ on Ω, whence actually $w \geqslant 0$, and in particular $0 \wedge u \geqslant 0$ on ω, that is, $u \geqslant 0$ on ω.

1.1.8 Even if an open set Ω is not Greenian, there is associated with any superharmonic function u on Ω a unique measure μ_u (again called the Riesz measure associated with u) whose restriction to any relatively compact open subset ω of Ω is the Riesz measure associated with the restriction of u to ω; see, for example, [Do4, 1.IV.7]. We have *Poisson's formula*

$$\Delta u = -a_N \mu_u$$

in the distributional sense, where a_N denotes the Poisson constant: $a_N = (N-2)\sigma_N$ for $N > 2$, $a_2 = 2\pi$; see, for example, [AG, Section 4.3].

1.1.9 A superharmonic function u on an open set Ω is *harmonic* on a given open set $\omega \subset \Omega$ if and only if $\mu_u(\omega) = 0$. In particular u is *harmonic* off the (closed) support Supp μ_u of μ_u (relative to Ω); see Sect. 1.1.8. At a point $x \in$ Supp μ_u, u is *continuous* provided that the restriction of u to Supp μ_u is continuous at x. This latter assertion is the *continuity principle* of Evans and Vasilescu; see, for example, [AG, Theorem 4.5.1] or [Do4, 1.V.8].

1.1.10 Every hyperharmonic function $\geqslant 0$ on a Greenian set Ω is the pointwise limit of an increasing sequence of finite continuous (even smooth) potentials of measures with compact supports in Ω. See, for example, [Do4, 1.IV.10].

1.1.11 A set E in $\mathbb{R}^N$ is called *polar* if there is a superharmonic function u on some open set $\Omega \subset \mathbb{R}^N$ containing E such that $u = +\infty$ on E. Since u is locally ℓ_N-integrable we have $\ell_N(E) = 0$, similarly with ℓ_N replaced by the surface measure σ on a closed ball in Ω.

An assertion involving a variable point of a given set $A \subset \mathbb{R}^N$ is said to hold *quasi-everywhere* (q.e.) on A if it holds at every point of A off some polar subset. Thus a hyperharmonic function u on an open set $\omega \subset \mathbb{R}^N$ is superharmonic if and only if $u < +\infty$ q.e. on ω. A countable union of polar subsets of $\mathbb{R}^N$ is polar; see, for example, [AG, Corollary 5.1.4]. A single point of $\mathbb{R}^N$ forms a polar set (because $N > 1$). It follows that every countable set is polar.

Let E be a polar subset of a Greenian set Ω and let $x \in \Omega \setminus E$ be given. There exists then a measure μ on Ω with superharmonic potential $G_\Omega\mu$ such that $G_\Omega\mu = +\infty$ on E, $G_\Omega\mu(x) < +\infty$, e.g. [AG, Theorem 5.1.3]. It follows that if E is a relatively closed polar subset of a connected open set $\Omega \subset \mathbb{R}^N$ then $\Omega \setminus E$ is connected, e.g. [AG, Corollary 5.1]. Furthermore, the Riesz measure associated with a *locally bounded* superharmonic function does not charge the polar sets, e.g. [AG, Theorem 5.1.9].

1.1.12 Polar sets are removable singularity sets for [super]harmonic functions: If u is (super)harmonic on $\Omega \setminus E$, where E is a polar and relatively closed subset of Ω, and if u is bounded (from below) near points of E, then u has a unique extension to a (super)harmonic function on Ω; see [AG, p. 128]. For a version of this in which E need not be relatively closed, see [Do4, 1.V.5].

An open set $\Omega \subset \mathbb{R}^2$ is Greenian if and only if $\mathbb{R}^2 \setminus \Omega$ is nonpolar (Myrberg's theorem, e.g. [Do4, 1.V.6], [AG, Theorem 5.3.8]).

1.1.13 For any open set $\Omega \subset \mathbb{R}^N$ the *l.s.c. regularization* $\widehat{f}$ of a numerical function f on Ω is defined as the greatest l.s.c. minorant $\widehat{f}$ of f. Thus

$$\widehat{f}(y) = \liminf_{x \to y} f(x). \tag{1.1.5}$$

Let $\mathcal{F}$ be a family of hyperharmonic functions on an open set Ω in $\mathbb{R}^N$, and let $u = \inf \mathcal{F}$ (pointwise infimum). If $\mathcal{F}$ is locally uniformly bounded from below, that is, if u is locally bounded from below, then $\widehat{u}$ is hyperharmonic,

$$\widehat{u}(y) = \liminf_{x \to y,\, x \neq y} u(x),$$

and $\widehat{u} = u$ q.e. This result is known as the *fundamental convergence theorem*, due to Brelot and H. Cartan. See, for example, [AG, Theorem 5.7.1 and Notes to it]. Here a topological lemma due to Choquet applies to $\mathcal{F}$. This lemma states that any family $\mathcal{F}$ of numerical functions on a Hausdorff space with a countable basis of open sets has a countable subfamily $\mathcal{F}^*$ such that $\inf \mathcal{F}^*$ and $\inf \mathcal{F}$ have the same l.s.c. regularization. If $\mathcal{F}$ is lower directed, we may arrange that $\mathcal{F}^*$ is formed by the elements of a decreasing sequence. See, for example, [Do4, 1.VI.1 and 1.III.3], [AG, Lemma 3.7.4].

1.1.14 The Green *capacity* $C(K)$ of a compact subset K of a Greenian set Ω is defined by

$$C(K) = \sup\left\{\int d\mu : \mu \in \mathcal{M}_+(K),\ G_\Omega \mu \leqslant 1 \text{ on } \Omega\right\},$$

where $\mathcal{M}_+(K)$ denotes the convex cone of all (positive) measures on Ω supported by K. The *inner capacity* $C_*(A)$ and the *outer capacity* $C^*(A)$ of any subset A of Ω are defined by

$$C_*(A) = \sup\{C(K) : K \text{ compact },\ K \subset A\},$$
$$C^*(A) = \inf\{C_*(O) : O \text{ open in } \Omega,\ O \supset A\}.$$

A set A is called *capacitable* if $C^*(A) = C_*(A)$, and we then define the *capacity* $C(A)$ of A by $C(A) = C^*(A) = C_*(A)$. Clearly, every open set is capacitable. It is well known that so is any compact set K, and the supremum in the above definition of $C(K) = C_*(K)$ is finite and attained uniquely by the so-called *equilibrium measure* on K.

The set function C^* is countably subadditive, and $C^*(A_n) \to C^*(A)$ for any increasing sequence of sets A_n with the union A. As shown by Choquet [Ch1], the latter property implies that every analytic set (in particular every Borel set) is capacitable. As shown by Cartan [Ca1], a set $E \subset \Omega$ is polar if and only if $C^*(E) = 0$; see, for example, [AG, p. 141].

We shall not make much use of the concept of Green capacity until Sect. 1.4, but rather use some analogous set functions defined in terms of *reduction* and *sweeping* of functions; see Sects. 1.3 and 1.5. For the Green capacity, we therefore merely refer to [Do4, 1.XIII], [AG, Sections 5.5–5.9]. For $\Omega = \mathbb{R}^N$ with $N \geqslant 3$, Green capacity is also termed *Newtonian capacity*.

1.1.15 Let u be superharmonic on a Greenian set $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$), and let B be a ball with $\overline{B} \subset \Omega$. The function u_B, defined to equal $P_B u$ on B and to equal u elsewhere on Ω, is then superharmonic $\leqslant u$ on Ω and harmonic on B; cf. Sect. 1.1.2. This function u_B is called the *Poisson modification* of u relative to B.

A non-empty lower directed family Γ of superharmonic functions on Ω, each of which has a subharmonic minorant, is called a *Perron family* if every Poisson modification of any $u \in \Gamma$ belongs to Γ. For any Perron family Γ it was shown by Perron that, on each component of Ω, the pointwise infimum $\inf \Gamma$ of any Perron family Γ is either harmonic or identically $-\infty$; cf., for example, [AG, Theorem 3.6.2]. The former possibility occurs of course if the functions in Γ are positive. See, for example, [AG, Theorem 3.6.2].

For any superharmonic function u on Ω which has a subharmonic minorant, and for any cover $\mathcal{W}$ of Ω by balls B with $\overline{B} \subset \Omega$, the family of functions of the form $P_{B_n} P_{B_{n-1}} \dots P_{B_1} u$, where $n \in \mathbb{N}$ and $(B_1, \dots, B_n)$ is a finite sequence in $\mathcal{W}$, is termed the *Perron family generated by u and $\mathcal{W}$*. In the absence of other indication, $\mathcal{W}$ is taken as the cover of Ω by all balls B with $\overline{B} \subset \Omega$, and we simply speak of the Perron family generated by u.

For a *positive* superharmonic function u on Ω, the following are equivalent (e.g. [CC3, Proposition 2.2.1]):

(a) u is a potential,

(b) the infimum of some, and hence of any, Perron family containing u is identically 0,

(c) every subharmonic minorant of u is $\leqslant 0$.

It is enough to verify the equivalence of (a), (b), and (c) when Ω is connected.

(a)⇒(b). If u is a potential then so is any positive superharmonic minorant of u, in particular any Poisson modification of u and therefore any Perron family containing u, and likewise the greatest l.s.c. minorant of such a family according to the fundamental convergence theorem (Sect. 1.1.13). As mentioned above, such a minorant is a harmonic potential, hence $\equiv 0$.

(b)⇒(c). Let v be a subharmonic minorant of u. Let Γ be a Perron family generated by u and suppose that $\inf \Gamma \equiv 0$. Then $u - v$ is a positive superharmonic function and $P_B v \geqslant v$ for any ball B such that $\overline{B} \subset \Omega$. Hence

$$(P_{B_n} P_{B_{n-1}} \dots P_{B_1} u) - v \geqslant P_{B_n} P_{B_{n-1}} \dots P_{B_1} (u - v) \geqslant 0,$$

that is, $u' - v \geqslant 0$ for any $u' \in \Gamma$. This implies (c) as follows for any $x \in \Omega$:

$$-v(x) = -v(x) + \inf_{u' \in \Gamma} u'(x) = \inf_{u' \in \Gamma} (u' - v)(x) \geqslant 0.$$

(c)⇒(a) because any positive harmonic minorant of u is subharmonic and hence $\equiv 0$.

1.1.16 The *Dirichlet problem* for a Greenian set $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is treated by the Perron-Wiener-Brelot (PWB) method. Given an extended real valued function f, defined at least on $\partial^\infty\Omega$, a hyperharmonic and lower bounded function u on Ω is called an *upper function* for f if

$$\liminf_{\Omega\ni x\to y} u(x) \geqslant f(y) \quad \text{for each } y \in \partial^\infty\Omega,$$

and a hypoharmonic upper bounded function u on Ω is called a *lower function* for f if

$$\limsup_{\Omega\ni x\to y} u(x) \leqslant f(y) \quad \text{for each } y \in \partial^\infty\Omega.$$

The class of all upper, resp. lower, functions for f is denoted by $\overline{\Phi}_f^\Omega$, resp. $\underline{\Phi}_f^\Omega$. Clearly $\underline{\Phi}_f^\Omega = -\overline{\Phi}_{-f}^\Omega$.

The *upper* and *lower PWB solutions* to the Dirichlet problem with function f are defined pointwise on Ω by

$$\overline{H}_f^\Omega = \inf \overline{\Phi}_f^\Omega, \qquad \underline{H}_f^\Omega = \sup \underline{\Phi}_f^\Omega.$$

Clearly $\underline{H}_f^\Omega = -\overline{H}_{-f}^\Omega$ because $\underline{\Phi}_f^\Omega = \{-u : u \in \overline{\Phi}_{-f}^\Omega\}$.

On any component of Ω, each of the functions $\overline{H}_f^\Omega$, $\underline{H}_f^\Omega$ is *either harmonic or identically* $+\infty$ *or* $-\infty$. To see this it suffices to consider the case of $\overline{H}_f^\Omega$. Suppose first that Ω is connected. If $\overline{\Phi}_f^\Omega = \{+\infty\}$ then $\overline{H}_f^\Omega \equiv +\infty$. Otherwise $\overline{H}_f^\Omega$ is the pointwise infimum of the family of all *superharmonic* upper functions for f. This family is clearly a Perron family, and its pointwise infimum is therefore either identically $-\infty$ or else harmonic, as mentioned in the preceding subsection. For disconnected Ω; use, for example, [AG, Lemma 6.2.4].

Furthermore, $\overline{H}_f^\Omega \geqslant \underline{H}_f^\Omega$. As above we may suppose that Ω is connected. It is enough to show that if u is a superharmonic upper solution and v a subharmonic lower solution then $u \geqslant v$ on Ω. This inequality will follow from the classical boundary minimum principle (Sect. 1.1.7) if we show that

$$\liminf_{x\to y} (u - v)(x) \geqslant 0$$

for each $y \in \partial^\infty\Omega$. If $f(y)$ is finite then

$$\liminf_{x\to y} (u - v)(x) \geqslant \liminf_{x\to y} u(x) - \limsup_{x\to y} v(x) \geqslant f(y) - f(y) = 0. \tag{1.1.6}$$

If $f(y) = +\infty$ then $u(x) \to +\infty$ as $x \to y$, while v is upper bounded on Ω and hence $(u - v)(x) \to +\infty$ as $x \to y$. A similar argument yields the same conclusion if $f(y) = -\infty$. Hence (1.1.6) holds at each point $y \in \partial^\infty\Omega$, as required.

A function $f : \partial^\infty\Omega \to \overline{\mathbb{R}}$ is called *resolutive* (relative to Ω) if $\overline{H}_f^\Omega$ and $\underline{H}_f^\Omega$ are identical and finite valued (and hence harmonic) on Ω. In that case we define $H_f^\Omega := \overline{H}_f^\Omega = \underline{H}_f^\Omega$, and we call H_f^Ω the *PWB solution* (or generalized solution) to the Dirichlet problem on Ω with the boundary function f. Every function $f \in \mathcal{C}(\partial^\infty\Omega)$ is resolutive. See, for example, [AG, Chapter 6].

1.1.17 For any point x of a Greenian set Ω, there exists a unique Borel probability measure μ_x^Ω on $\partial^\infty\Omega$ such that

$$H_f^\Omega(x) = \int_{\partial^\infty\Omega} f\, d\mu_x^\Omega \quad \text{for every } f \in \mathcal{C}(\partial^\infty\Omega). \tag{1.1.7}$$

This measure μ_x^Ω is called the *harmonic measure* relative to Ω and x; cf. Sect. 1.1.2. For any function $f : \partial^\infty\Omega \to \overline{\mathbb{R}}$ we have

$$\underline{H}_f^\Omega(x) \leqslant \int_* f\, d\mu_x^\Omega \leqslant \int^* f\, d\mu_x^\Omega \leqslant \overline{H}_f^\Omega(x) \tag{1.1.8}$$

(integrations over $\partial^\infty\Omega$). Therefore, if f is resolutive then f is μ_x^Ω-integrable over $\partial^\infty\Omega$ for every $x \in \Omega$, and equality prevails in each inequality in (1.1.8). On the other hand, if f is μ_x^Ω-measurable and $\int f\, d\mu_x^\Omega$ exists then $\overline{H}_f^\Omega(x) = \underline{H}_f^\Omega(x) = \int f\, d\mu_x^\Omega$. In particular, if f is μ_x^Ω-integrable over $\partial^\infty\Omega$ for some x in each component of Ω then f is resolutive. See, for example, [AG, Theorem 6.4.6 and its corollary]. For any l.s.c. function $f > -\infty$ we have $\overline{H}_f^\Omega = \int_{\partial^\infty\Omega}^* f\, d\mu_x^\Omega$, and f is resolutive if $\overline{H}_f^\Omega < +\infty$ on Ω, [AG, Lemma 6.4.4].

1.1.18 A point z of $\partial^\infty\Omega$ (Ω Greenian) is called *regular* (for Ω) if

$$\lim_{x \to z} H_f^\Omega(x) = f(z) \quad \text{for each } f \in \mathcal{C}(\partial^\infty\Omega),$$

or equivalently if

$$\lim_{\Omega \ni x \to z} G_\Omega(x, y) = 0 \quad \text{for each } y \in \Omega,$$

or just for at least one point y in each component of Ω; see [AG, Theorem 6.8.3]. Otherwise z is called *irregular*. We say that Ω is *regular* if every point of $\partial^\infty\Omega$ is regular. The irregular points of $\partial\Omega$ always form a polar set (Kellogg's theorem); see, for example, [AG, Theorem 6.6.8]. For any Greenian set $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$), a sufficient condition for regularity of a point $y \in \partial\Omega$ is that $\mathbb{R}^N \setminus \Omega$ contains a truncated cone of revolution with vertex at y; this is the Poincaré-Zaremba exterior cone condition. (For $\Omega \subset \mathbb{R}^2$ the truncated cone may be replaced by a line segment with y as an end-point.) In particular, every ball $B \subset \Omega$ is regular, and the harmonic measure μ_x^B ($x \in B$) has the density $K_B(x, \cdot)$ (the Poisson kernel; see Sect. 1.1.2) with respect to the surface measure on ∂B.

1.2 Thin Sets and Fine Topology

For the classical Dirichlet problem for a Greenian domain $\Omega \subset \mathbb{R}^3$, the Poincaré-Zaremba cone condition (see Sect. 1.1.18) is sufficient for regularity of a boundary point for Ω. On the other hand, the Lebesgue spine (or thorn) in $\mathbb{R}^3$ exhibits a limit to how "sparse" the complement $\mathbb{R}^N \setminus \Omega$ can be near a regular boundary point. A remarkable necessary and sufficient condition for regularity of a boundary point for Ω was obtained by Wiener in terms of the Newtonian capacity of the parts of $\mathbb{R}^N \setminus \Omega$ in small annuli centered at y; see, for example, [AG, Theorem 7.7.2].

A different and often more manageable characterization of regularity for a boundary point y for a Greenian set $\Omega \subset \mathbb{R}^N$ was given by Brelot [Br2] in terms of his notion of "effilement" (in English "thinness") of $\mathbb{R}^N \setminus \Omega$ at y. Although $\mathbb{R}^N \setminus \Omega$ is a closed set, Brelot allowed an arbitrary set E in its place.

Definition 1.2.1 (Brelot [Br2]) A set $E \subset \mathbb{R}^N$ $(N \geqslant 2)$ is said to be thin (effilé) at a point $y \in \mathbb{R}^N$ if either $y \notin \overline{E}$ or else there exists a hyperharmonic function u on an open neighborhood of y such that

$$\liminf_{E\setminus\{y\}\ni x\to y} u(x) > u(y).$$

One may clearly assume here that u is superharmonic and that $u \geqslant 0$ on that neighborhood. In the case $N \geqslant 3$ it suffices to consider u defined on all of $\mathbb{R}^N$. A set which is not thin at a point x is said to be *unthin* at x.

Theorem 1.2.2 (Brelot [Br2]) *A set $E \subset \mathbb{R}^N$ $(N \geqslant 2)$ is thin at a point $y \in \mathbb{R}^N$ if and only if, for some and hence any Greenian set $\omega \subset \mathbb{R}^N$ containing y, there exists a (Green) potential $u \geqslant 0$ on ω such that $u(y) < +\infty$ and*

$$u(x) \to +\infty \quad \textit{as } x \to y \textit{ through points } x \in E \setminus \{y\}.$$

See, for example, [AG, Theorem 7.2.3] or [Do4, Theorem 1.XI.2]. It follows by the Riesz decomposition theorem (Sect. 1.1.6) that one may equivalently replace the potential $u \geqslant 0$ by a superharmonic function on ω in Theorem 1.2.2. Clearly, every finite union of sets which are thin at y is likewise thin at y.

In correspondence with Brelot in 1939, H. Cartan observed that the notion of thinness may be described in terms of a new topology on $\mathbb{R}^N$ which he termed *the fine topology*.

Definition 1.2.3 The fine topology on $\mathbb{R}^N$ $(N \geqslant 2)$ is the coarsest topology in which every superharmonic function on $\mathbb{R}^N$ is continuous in the extended sense.

In other words, the fine topology is the intersection of all topologies in which every superharmonic function on $\mathbb{R}^N$ is continuous (such topologies exist, for example the discrete topology). It is easily shown that every superharmonic function on an open set $\Omega \subset \mathbb{R}^N$ is finely continuous, e.g. [AG, Lemma 7.1.2].

The fine topology is indeed stronger (= finer) than the Euclidean topology, and hence Hausdorff. In fact, every ball $B(y,r) \subset \mathbb{R}^N$ with $N \geqslant 3$ equals $\{x \in \mathbb{R}^N : U_y(x) > r^{2-N}\}$ (recall that U_y denotes the fundamental harmonic function, see Sect. 1.1.1), and so $B(y,r)$ is finely open, U_y being superharmonic on $\mathbb{R}^N$, similarly for $N = 2$.

Actually, the fine topology is *strictly finer* than the Euclidean one because there exist superharmonic (hence finely continuous) functions which are discontinuous in the Euclidean topology. Denoting by E the polar set formed by a sequence of distinct points of $\mathbb{R}^N$ converging to a point $x \in \mathbb{R}^N$, there exists (as mentioned in Sect. 1.1.11) a superharmonic function u on $\mathbb{R}^N$ such that $u = +\infty$ on E, but $u(x) < +\infty$, so u is discontinuous at x.

Topological properties relative to the fine topology will henceforth be distinguished by the qualifier *fine* or *finely* from properties relative to the Euclidean topology. We write f-lim for limit in the fine topology, and similarly f-lim inf and f-lim sup. The fine boundary of a set $A \subset \mathbb{R}^N$ is denoted by $\partial_f A$.

A subbasis for the fine topology on $\mathbb{R}^N$ is formed by the sets $\{u < \alpha\}$ and the sets $\{u > \alpha\}$ with u superharmonic on $\mathbb{R}^N$ and $\alpha \in \mathbb{R}$. The sets $\{u > \alpha\}$ being (Euclidean) open we may replace them by balls, and we obtain another subbasis for the fine topology formed by the sets

$$\{x \in B : u(x) < \alpha\}, \tag{1.2.1}$$

where B is an open ball, u is superharmonic on B, and $\alpha \in \mathbb{R}$. This latter subbasis is actually a *basis* for the fine topology as we will see in Theorem 1.2.6 below.

Theorem 1.2.4 *(H. Cartan) The fine neighborhoods of a point $x \in \mathbb{R}^N$ are the complements of those sets $E \subset \mathbb{R}^N \setminus \{x\}$ which are thin at x.*

Proof For any fine neighborhood V of x there exist a ball $B(x,r) = B$, a superharmonic function u on B, and a number $\alpha > u(x)$, such that $\{y \in B : u(y) < \alpha\} \subset V$. Furthermore, x does not belong to the fine closure of $\complement V$. Since $u \geqslant \alpha$ on $B \setminus V$ it follows that

$$\liminf_{\complement V \ni y \to x} u(y) \geqslant \alpha > u(x),$$

and hence $E := \complement V$ is thin at x without containing x. Conversely, suppose that $E \subset \complement\{x\}$ is thin at x. If $x \notin \overline{E}$, then $V := \complement\overline{E}$ is a neighborhood, in particular a fine neighborhood, of x. If $x \in \overline{E}$, there is a superharmonic function u on a ball $B(x,r) = B$ and a number $\alpha > u(x)$ such that

$$\liminf_{E \ni y \to x} u(y) > \alpha.$$

It follows that $u \geqslant \alpha$ on $E \cap B(x,\delta)$ for some $0 < \delta \leqslant r$. Consequently, $\complement E$ contains $\{y \in B(x,\delta) : u(y) < \alpha\}$, which is a fine neighborhood of x. □