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# Preface

During my long scientific career, which began in 1955, I worked on a number of processes in the atmosphere, in the ocean, and in astrophysics, and in geophysics on some of them, I would like to think I left a mark. About 350 articles have been written, of which over 60 were written by me only and six monographs. I taught a lot at the Department of Atmospheric Physics of the Faculty of Physics of Moscow M. B. Lomonosov State University.in 1975—2010 and from 1973 to 2019 at the Department of Ocean Thermohydrodynamics of the Moscow Institute of Physics and Technology. Teaching served as a stimulus for writing monographs.

My main achievements until 2011 were published in the book "Statistics and Dynamics of Natural Processes and Phenomena", M.: Krasand, 2012, 398 pages. During my scientific activity, I devoted a lot of time to scientific, public, and scientific-organizational topics not only in the country but also abroad, participating in various councils, committees, commissions, editorial boards, etc. From 1988 to 2002, I was elected three times as a member of the Presidium of the Academy. From 01.01.1990 to 31.12.2008, I was the director of the Institute of Atmospheric Physics of the Russian Academy of Sciences, i.e., day after day for 19 years. In the years of the Presidium, my employment was there, and in the turbulent 1990s, I led the Institute voluntarily, drawing no money, albeit meager, from the IFA salary fund. During this period, our IFA received various foreign grants. But in these years, I did significant work, for example, explaining the energy spectrum of cosmic rays, Gutenberg—Richter law of frequency—magnitude of earthquakes, I found patterns of hurricane development—necessary, but not sufficient, conditions for their development and others. The basis for the construction of these theories was the theory of similarity and dimension, as in previous work.

However, I remembered the late 1950s—early 1960s, when many scientists said that "the theory of similarity is a similarity of a theory" and even in the late 1990s, after my lecture at the Moscow Lebedev Institute of Physics on the spectrum of cosmic rays, they demanded the physical model, i.e. kinetic equations. Only in 2017, I did suddenly realize that such a model could be the probabilistic laws of Andrey Nikolaevich Kolmogorov and his school, developed by his students A. M. Obukhov, then A. M. Yaglom, A. S. Monin, then G. I. Barenblatt, who developed the general principles of their teacher into practical methods and applied these methods to specific phenomena and processes, primarily to turbulence (see Chap. 2). Now the theory of similarity and dimension is an exact theory (see Barenblatt, 2003: Scaling (Vol. 34). Cambridge University Press.)

Therefore, the time has come to present a description of many processes precisely from the point of view of Andrei Nikolayevich's main two-page work in 1934, which contains the principles of a description of not only turbulence, as I understood by mid-2017, but also sea wind waves, tropical and polar hurricanes, mini-hurricanes on the sea surface in the form of spiral eddies discovered by satellites only at the end of the twentieth century, the statistical structure of inhomogeneities of the gravitational field and the surface topography of celestial bodies, so called the Kaula's rule, etc. The main understanding to apply is that the mean square of velocity is the process energy per unit mass.

These are all the latest discoveries, deeply understood only in the context of the ideas of A. N. Kolmogorov in 1934, brought by his students listed above to practical methods for describing specific phenomena and processes. I would hope, as the scientific grandson of a great mathematician and physicist and as a student and colleague of his students mentioned above, that the examples presented here describing specific processes and phenomena, many of which remained mysteries for decades, will serve as examples for understanding future discoveries. Probabilistic laws and the theory of similarity and dimensions are the ways to understand the world around us. I have two important epigraphs from the classics of science. Initially, I wanted to have a third epigraph. Here it is:

Upon this gifted age, in its dark hour,
Rains from the sky a meteoric shower
Of facts … they lie unquestioned, uncombined
Wisdom enough to leech us of our ill
Is daily spun, but there exists no loom
To weave it into fabric.
Edna St. Vincent Millay (1892–1950).

Edna is a wonderful American poet, known not only for her magnificent poems but also for her stormy romances. These lines clearly reflect the dramatic situation in the science of the surrounding world of the twentieth century. With too many poets, inexplicable by them the existing science. The wisdom to cure us is the daily bustle.

This book is something like an attempt to develop a unified view of the macroworld. Each of the paragraphs was written in such a way that after Chap. 1 it could be understood independently.

I would like to thank Vera Grigorievna Kochina for her invaluable assistance in the work, for the repeated typing and re-typing of the chapters of the book, which were discussed many times with Evgeniy Borisovich Gledzer and Otto Guramovich Chkhetiani, long-term collaborators.

The following is a list of the most frequently occurring links in the text, which are marked with abbreviations:

AHK34. Kolmogorov A. N. Zufällige Bewegungen // Ann. Math. 1934. V. 35. P. 116–117.
G18. Golitsyn G. S. The laws of random motions by A.N. Kolmogorov. Meteorology and Hydrology 2018. №3, 5–15.
MY71, 75. Monin A. S. Yaglom A. M. Statistical Hydromechanics V.1, V2 MIT Press. 1971. 1975.
GLG10. Gledzer E. B. and Golitsyn G. S. Scaling and finite ensembles of particles in motion with energy influx. Dokl. (2010). 433, (3), 466-470.
BPW. Bridgman P. W. (1932) Dimensional Analysis—Yale Univ. Press. 2nd Ed.
B09. Barenblatt G. I. Scaling—CUP, 2003—171 p.).
G12. Golitsyn G. S. Statistics and Dynamics of processes and phenomena in Nature. Moscow: Krasand Poll, 2012 (in Russian). 198 p.

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The 2021 Nobel Prize in Physics has been awarded to Klaus Hasselmann, Syukuro Manabe, and Giorgio Parisi for their Study of Climate Change, and Complex Systems. In particular, the Nobel Committee for the first two said: "for the physical modelling of Earth's climate, quantifying variability and reliably predicting global warming" and for the third one "for the discovery of the interplay of disorder and fluctuations in physical systems from atomic to planetary scales".

This award was in recognition of the contribution of Complex Systems science to the Climate Change phenomenon.

This book illustrates not only the current knowledge about the stochasticity of nature but also underlines the most basic unanswered relevant questions of environmental dynamics. The second part of the book is devoted to the climate system considered as a complex and complicated system. It mainly focuses on the major contributions of my research group to the topics mentioned in the first part of the book which is excellently written by Prof. Georgy Golitsyn. I must confess that writing this book with him was an unforgettable venture in which I gained an excellent academic experience.

Prof. Costas Varotsos
Athens, Greece

# About this book

The first part of the book introduces the equation of the random motions by A. N. Kolmogorov proposed in 1934. In 1959 A. M. Obukhov found the second moments of probability distribution for it as $\langle u_i^2(t)\rangle = \varepsilon t$, $\langle u_i x_i(t)\rangle = \varepsilon t^2$, $\langle x_i^2(t)\rangle = \varepsilon t^3$, where $\varepsilon$ is the diffusion coefficient in the velocity space, the rate of the energy (or any intensity) generation per unit mass. From these moments A. M. Obukhov has obtained the laws of inertial turbulence of 1941 including the Richardson—Obukhov one. From the energy (intensity) moment is producing histograms and differential probability distributions for the events flux such as earthquakes and this makes the Gutenberg—Richter empirical law as the theory of probability law, TPL. A number of other empirical laws become TPL. Pareto's rule in social sciences and the Zipf law can be understood also in this way. The third moment describes the random areas, such as histograms for lithospheric plates size distributions, mass of spiral galaxies, some features of clouds. Cosmic rays spectra, Kaula's rule for relief spectra of celestial bodies etc. also became TPLs. The empirics and those moments may be used for $\varepsilon$.

The book's second part highlights significant findings in the study of nonlinear dynamics within climate system components. It begins by introducing a tool that separates dynamical and chemical variability in ozone, challenging the previous belief that chemical processes were the primary influence on the ozone layer. Then, a key discovery of the second author of this book from 2002 revealed that major stratospheric sudden warmings could occur in the Southern Hemisphere, leading to the splitting of the Antarctic ozone hole. This emphasized the importance of accounting for long-range correlations in predictive models of global geophysical variables. This understanding is crucial for grasping the complexities of geophysical dynamics over extended time scales.

In the following, it examines the linearity assumption in reducing solar and volcanic forcings to radiative equivalents. It finds that variability in models is weak at centennial scales, and that solar and volcanic forcings combine in a nonlinear manner over longer periods, affecting model sensitivity.

Finally, the introduction of the concept of “natural time” is presented as a valuable tool for predicting extreme geophysical events. This approach has led to the development of a nowcasting tool that successfully analyzes the evolution of complex systems, with examples including El Niño, solar radiation, air pollution, cyclones, and heatwaves, all of which have significant societal impacts.

# Contents

# Chapter 1
# Necessary Notions from the Theory of Stochastic Processes

## 1.1 Correlation and Structure Functions, Energy Spectra

To facilitate the reader's understanding, we will now present the fundamental formulas of the theory of stochastic processes that are essential for the subsequent discussions. While some of these formulas are well-known, others are only found in obscure publications, and a few are being cited for the first time. Although these formulas primarily focus on the initial two moments of probability distributions, they possess a straightforward structure and content, yet their level of familiarity is not widespread. Certain formulas can be derived by considering similarity and dimensionality, thus the material presented here may offer further validation for their accuracy and applicability limits, as well as for the analysis of empirical data. The exposition will specifically address temporal processes, which pertain to processes in a one-dimensional space. The statistical theory of random vector fields was originally developed by A. M. Obukhov in the 1940s, and a comprehensive explanation can be found in volume II of the book authored by A. S. Monin and A. M. Yaglom, referred to as MY75.

Consider a time-stationary stochastic process $a(t)$ for which there is an average value:

$$\langle a\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T a(t)dt \qquad (1.1)$$

and correlation function

$$B_a(\tau) = \langle a(t+\tau)a(t)\rangle = \sigma_a^2 f(\tau), \quad \sigma_a^2 \equiv \langle a^2\rangle. \qquad (1.2)$$

It corresponds to the Fourier transform, called the spectral energy density of the process in question:

 
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$$E_a(\omega) = \frac{2}{\pi}\int_0^\infty B_a(\tau)\cos\omega\tau d\tau, \tag{1.3}$$

and vice versa

$$B_a(\tau) = \int_0^\infty E(\omega)\cos\omega\tau d\omega. \tag{1.4}$$

The last formula shows that:

$$B_a(\tau = 0) = \langle a^2 \rangle = \int_0^\infty E_a(\omega)d\omega \equiv \sigma_a^2, \tag{1.5}$$

from where it is evident that the spectral density, the function $E_a(\omega)$ accurately reflects its purpose by providing the energy distribution based on the frequency $\omega = 2\pi/\tau$ where $\tau$ is the time period. The correlation function is normalized to its variance, and therefore in (1.2) the function $f(\tau)$ cannot exceed one.

For the convergence of integrals in (1.3)–(1.5), the corresponding behavior of sub-integral functions at zero and infinity is necessary. Since the function $E_a(\omega)$ describes the distribution across the spectrum of the square (energy) of our random process $a(t)$, it must be non-negative everywhere: $E_a(\omega) \geq 0 \, for \, 0 < \omega < \infty$. This imposes limitations on the kind of correlation function $B_a(\tau)$, according to (1.3) and (1.4). For example, the function $B_a(\tau)$ in (1.3) cannot fall linearly to zero at some $\tau$, and then is zero precisely because, as is easy to see, its Fourier transform then would have negative regions on the frequency axis $\omega$.

Here are three examples MY75 of "correct" correlation functions and their corresponding spectral densities normalized to dispersion $\sigma_\alpha^2$:

$$B(\tau) = e^{-a\tau} \tag{1.6}$$

$$E(\omega) = \frac{2a}{\pi\left(a^2 + \omega^2\right)} \tag{1.7}$$

$$B(\tau) = e^{-a\tau^2}, \quad E(\omega) = \frac{e^{-\omega^{2/4a}}}{(a\pi)^{1/2}}; \tag{1.8}$$

$$B(\tau) = (a\tau)^\nu K_\nu(a\tau), \quad E(\omega) = \frac{\Gamma(\nu+1)a^{2\nu}}{\pi\left(a^2+\omega^2\right)^{\nu+1/2}}, \tag{1.9}$$

where $K_\nu$ is the MacDonald function with an index of $\nu$. The spectrum (1.7) is also called the Cauchy–Lorentz distribution.

Consider now the process $u(t)$, such that:

$$\dot{u}(t) = a(t), \text{ or } u(t) = \int a(t)dt. \tag{1.10}$$

If *the process a* ($t$) was stationary in time, then the integral from it will be a random process with stationary increments. The theory of such processes was developed by A. N. Kolmogorov in 1940, MY75, when he began to create a mathematical tool for studying the theory of turbulence, starting at the invitation of O. Y. Schmidt in 1939 to work in the new Institute of Theoretical Geophysics of the Academy USSR of Sciences. For such processes, it is impossible to determine any average value or correlation function. However, it is possible to construct (and measure) the probability distribution for the average values of the differences in the quantities $u$ ($t$) taken at two points in time: $t_1$ and $t_2$. The corresponding second moment was called by A. N. Kolmogorov a structure function. He showed that for processes with stationary increments, this function depends only on the difference in difference $t_2 - t_1 = \tau$:

$$D_u(\tau) = \left\langle [u(t+\tau) - u(t)]^2 \right\rangle. \tag{1.11}$$

The structure function of this formula can also be determined for stationary stochastic processes. Then the connection between the structure and correlation functions of such processes is obvious:

$$D(\tau) = 2[B(0) - B(\tau)]. \tag{1.12}$$

For the structure function of processes with first-order stationary increments, it is possible to introduce a spectral density with a ratio, also MY 75:

$$D(\tau) = 2\int_0^\infty (1 - \cos\omega\tau)E(\omega)d\omega. \tag{1.13}$$

Most important to us in the future is the example of a power structure function.

$$D(\tau) = A\tau^\gamma, \quad 0 < \gamma < 2, \tag{1.14}$$

the spectrum of which is determined by the ratios:

$$E(\omega) = C/\omega^{\gamma+1}, C = A\pi^{-1}\Gamma(\gamma+1)\sin(\pi\gamma/2), \tag{1.15}$$

where $\Gamma(\gamma+1) = \gamma\Gamma(\gamma)$—gamma function.

Power regularities cannot be implemented over the entire infinite interval of times and their inverse frequencies $\omega = 2\pi/t$. There are always internal and external boundaries for the execution of power regularities. The internal scale is usually associated with the dissipative mechanisms inherent in the system under consideration, and the external scale is determined by boundary conditions, for example, the lifetime of the system or its dimensions.

In the development of these concepts, Barenblatt and Zeldovich (1972), see also B09 introduced the concept of intermediate asymptotics, which is performed on the finite interval of the quantity under study. Sometimes, as in the case of the statistical description of the relief (see Chap. 9), one asymptotic may pass into another due to a change like the forces acting in the system, Chap. 8.

In the 1950s, Yaglom developed a general theory of stochastic processes with stationary increments of arbitrary order $n$. For example, for the characteristics of sea waves, oceanographers use the frequency spectrum of elevations of the water surface. This spectrum of vertical displacements is the spectrum of the correlation function of velocities arising from $g$ sin$\beta$, $g = 9.8\,\mathrm{m/s}^2$, and $\beta$—the slope of the surface in the wave, which is a random stationary process. At $n = 2$, the growth of the structure function is limited from above by a power index equal to 4, not 2, as in $n = 1$, see (1.14). This leaves the ratio (1.13) with $n = 2$, binding structure function and spectral density. Using known trigonometric relationships, it can also be rewritten as

$$D(\tau) = 4\int_0^\infty \sin^2\left(\frac{\omega\tau}{2}\right)E(\omega)d\omega. \tag{1.16}$$

Given the limited interval of execution of power regularities both from above and below, the relation (1.16) can be reasonably corrected for its convergence at near-zero frequencies and at infinity.

For random processes with second-order stationary increments, it is possible to determine structure functions and spectra similarly (for particle displacement) (1.13) to (1.15) in the following form (Golitsyn and Fortus 2020):

$$D^{(2)}(\tau) = 2^3\int_0^\infty (1 - \cos\omega t)^2 E(\omega)d\omega \tag{1.17}$$

and with the power form of these functions

$$D^{(2)}(\tau) = A_2\tau^\gamma, \quad 2 < \gamma < 4,$$

connection is maintained $E^{(2)}(\omega) = C_2/\omega^{\gamma+1}$, and the relationship between the constants $A_2$ and $C_2$ found in Golitsyn and Fortus (2020):

$$A_2 = \frac{\Gamma(\gamma+1)\sin\left(-\frac{\pi\gamma}{2}\right)C_2}{\pi\left(2^{\gamma+1} - 2^3\right)}. \tag{1.18}$$

Definitions of Fourier-type transformations such as (1.13) or (1.17) exclude divergences at zero. It should be remembered, however, that power dependencies are always intermediate asymptotics (Barenblatt and Zeldovich, 1972).

## 1.2 Delta-Correlated Stochastic Processes

In nature, most of the processes (for example: turbulence, earthquakes, landslides, etc.) are random in time and space. The primary basis of randomness is the forceful effects on the system. The correlation time of these influences is usually much shorter than the reaction time of the system. For times $\tau_0 << \tau$, an internal time scale, which is often also the time of correlation of random forces acting on the system, in a first approximation their correlation function can be approximated by a delta function (in probability theory, such approximations are called Markovian)

$$B_a(\tau) = \varepsilon\delta(\tau), \quad \varepsilon = \sigma_a^2\tau_0 \tag{1.19}$$

It should be remembered that the delta function has a dimension inverse of the dimension of its argument. For the process $u\,(t)$, the integral of $a\,(t)$, a random process with stationary increments, the structure function (1.11) is (see MY75):

$$D_u(\tau) = 2\varepsilon\tau \tag{1.20}$$

This formula was first published in 1944 in the first edition of the book "Mechanics of Continuous Media" by L. D. Landau and E. M. Lifshitz (the history of many of its rediscoveries is set out in MY75). Turbulence researchers have usually obtained it for reasons of similarity and dimension when $u$ is the increment of velocity, $\varepsilon$ is the rate of generation/dissipation of the kinetic energy of turbulence per unit mass of the liquid.

With the delta-shaped correlation function (1.19), according to the formula (1.3), we calculate the spectral density of the stationary process $a\,(t)$, which turns out to be a frequency-independent constant equal to $2\varepsilon/\pi$, that is, white noise. According to (1.13), the structure function (1.20), proportional to time, corresponds to the spectral density:

$$E_u(\omega) = \varepsilon(2/\pi)\omega^{-2} \approx \varepsilon\omega^{-2}. \tag{1.21}$$

Below we omit the multipliers $2/\pi = 0.637$ everywhere, since in all real situations the theoretical multipliers must be compared with the data of specially staged experiments. For the average square of the process,

$$x(t) = \int u(t)dt$$

equal to

$$\left\langle [\Delta x(\tau)]^2 \right\rangle$$

we have a spectrum:

$$E_x(\omega) = \varepsilon\omega^{-4}. \tag{1.22}$$

This expression can be obtained without a numerical coefficient for reasons of dimensionality, or from the formula (1.21) for the spectral density of the process $u(\tau)$ given that the spectrum of the process $x\,(t)$ for which the equation below exists:

$$\dot{x}(t) = u(t).$$

and associated with the spectrum:

$$\omega^2 E_x(\omega) = E_u(\omega), \tag{1.23}$$

where the multiplier $\omega^2$ arises from the quadratic nature of the structure function (1.11).

For the structure function of displacements $x\,(t)$, a random process with second-order random increments, $n = 2$, the formula (1.13) gives

$$D_x(\tau) = \varepsilon\tau^3 = \left\langle(\Delta x)^2\right\rangle \equiv r^2, \tag{1.24}$$

which is also obtained for reasons of dimensionality.

With formula (1.20) it is instructive to compare Einstein's formula for the average square of displacement of a Brownian particle.

$$\left\langle x^2\right\rangle = 2nD\tau, \tag{1.25}$$

where $n$ is the dimension of space, $D$ is the diffusion coefficient. It is this identity of the formulas (1.20) and (1.25) that allowed Obukhov (1959) in the case when $u(\tau)$ is the velocity field, in the Lagrange description *of* $T^*$, to call the value $\varepsilon/2$, the rate of generation/ dissipation of the kinetic energy of turbulence, the coefficient of diffusion in the space of velocities.

The question arises, what in specific numbers means the condition of the smallness of the times of correlation of the effects $\tau_0$ compared to the reaction time of the system $\tau$. To answer this question, at least in order of magnitude, let's choose the correlation function of the stationary random process $a\,(t)$ in its simplest form:

$$B_a(\tau) = \frac{\varepsilon}{\tau_0}\exp\left(-\frac{\tau}{\tau_0}\right), \tag{1.26}$$

which at $\tau_0 \to 0$ tends to the $\delta$ -function. The correlation function (1.26) corresponds to Langevin's stochastic equation (MY75). This equation differs from the equation $u = a$ by adding linear friction on the right—the term $\lambda u$, where $\lambda = \tau_0^{-1}$. The frequency spectrum of such a process is described by the formula (1.7) with $\alpha = \lambda = \tau_0^{-1}$.

The structure function of the *process* $u\,(t)$ will be:

$$D_u(t) = \varepsilon\tau_0\left[\frac{\tau}{\tau_0} - 1 + \exp\left(-\frac{\tau}{\tau_0}\right)\right]. \tag{1.27}$$

With an accuracy of a better than 1% already at $\tau = 5\tau_0$, you can neglect the exponential here and get:

$$D_u(\tau) \cong \varepsilon(\tau - \tau_0) \tag{1.28}$$

and the linear dependence on the constant $\varepsilon\tau_0$ is not difficult to take into account in further formulas.

The formulas (1.19)–(1.28) are correct in the probabilistic-theoretic sense, i.e., for infinite ensembles of events or times of observations over time $\tau$, when the number of events can be taken as the value $N = T/\tau_0$, where $T$ is the total period of observations. However, in practice, ensembles are always finite, often the number of events is only on the order of a few dozen. In the work of GlG10, analytically and by numerical counting, the validity of these formulas is checked, and it is shown that asymptotically they work satisfactorily already at $N \geq 10$. This concludes the consideration of continuous processes. Note, in order not to return to this further, that if the correlation time of the effects of the order or greater than the reaction time of the system, then the Boltzmann equilibrium of GlG10 is established in it, which is also carried out if the influx of energy is balanced by its dissipation.

## 1.3 Moments of Distribution Functions of A. N. Kolmogorov

All these results are derived from considerations of dimensionality, which, generally speaking, need a more rigorous justification. Such a justification is given by the works of A. N. Kolmogorov of the early 1930s. Their crown is the work of 1934, ANK34, in which, to describe the evolution of the probability density function, PDF, of the system $p(x_i\, u_i,\ t)$ the Fokker–Planck equation is used in the form:

$$\frac{\partial p}{\partial t} + u_i\frac{\partial p}{\partial x_i} = D\frac{\partial^2 p}{\partial u_i^2} \tag{1.29}$$

which in Soviet literature was called the FPK equation, where K corresponds to the surname of Andrei Nikolaevich. Here, $x_i$, $u_i$ are components of a 6-dimensional vector. It is obtained by the decomposition of the PDF into a Taylor series by a random parameter with an accuracy of a second-order term of smallness and is solved with initial and boundary conditions for the semi-infinite space. At the lower boundary of space, the values of $u$ are set over time, that is, the accelerations distributed according to Markov, or $\delta$-correlated in time and space.

The fundamental solution of the equation (1.29) is of the form (see MY75, and Appendix to Chap. 1):

$$p(u_i, x_i, t) = \left(\frac{\sqrt{3}}{2\pi Dt}\right)^3 \exp\left[-\left(\frac{u_i^2}{Dt} - \frac{3u_i x_i}{Dt^2} + \frac{3x_i^2}{Dt^3}\right)\right]. \quad (1.30)$$

A. M. Obukhov was the first to analyze this equation. He showed that the coefficient $D = \varepsilon/2$ (see Lifshitz and Pitaevski 1979) is proportional to the dissipation rate of the kinetic energy of turbulence $\varepsilon$. Solution (1.30) shows that the desired probability distribution is normal. This solution has three scales (angle brackets mean the average for the distribution over the ensemble):

$$\left\langle u_i^2 \right\rangle = \varepsilon t, \quad (1.31)$$

$$\langle u_i x_i \rangle = \varepsilon t^2 \equiv K, \quad (1.32)$$

$$\left\langle x_i^2 \right\rangle = \varepsilon t^3 \equiv r^2, \quad (1.33)$$

where the scale (1.32) in dimension is equal to the coefficient of turbulentmixing. Expressing the time from (1.33) and substituting it in (1.31) and (1.32), we get (1.34) and (1.35):

$$\left\langle u_i^2 \right\rangle = (\varepsilon r)^{2/3} \quad (1.34)$$

$$K(r) = \varepsilon^{1/3} r^{4/3}. \quad (1.35)$$

i.e., the Kolmogorov–Obukhov law of 1941 for a structure function with zero (small) initial conditions and the Richardson–Obukhov law for turbulent mixing (vortex diffusion). The scales (1.31) and (1.33) are also manifested in the fact that the substitution of the variables:

$$u_i = \tilde{u}_i (Dt)^{1/2}$$
$$x_i = \tilde{x}_i \left(Dt^3\right)^{1/2}$$

(where ~ is the dimensionless symbol), excludes from (1.29) the diffusion coefficient $D$, i.e., the description becomes completely self-similar, GlG10.

Time dependencies (1.31)–(1.33) were checked numerically for ensembles of $N$ randomly moving particles in accordance with the formulation of the problem by A. N. Kolmogorov ANK34, GlG10. Figure 1.1 shows that even at $N = 10$ dependencies (1.31) and (1.33) begin to be fulfilled, and at $N = 100$ numerical dependencies are practically non-existent and are different from the theoretical ones. When there is no motion in the system, the Eq. (1.28) takes on a simplified Fokker-Plank form for the

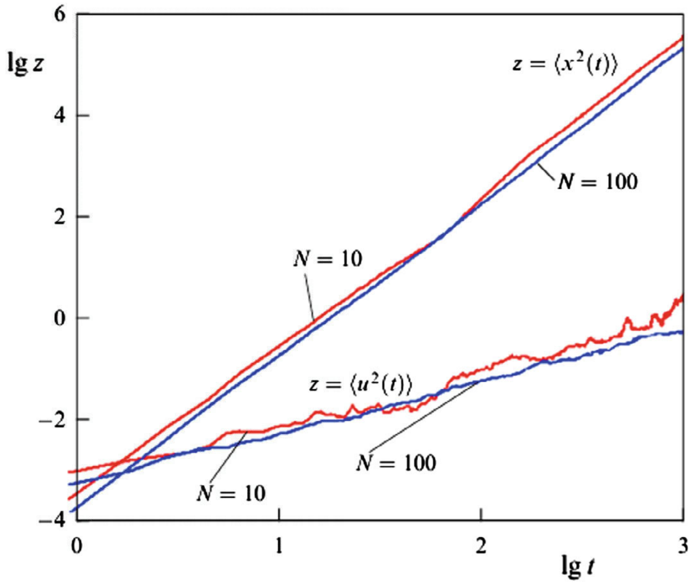


**Fig. 1.1** Second moments of the probability density function for velocities $< u^2(t) >$ and distances between particles $< x^2(t) > \sim t^3$, from GlC10

probability distribution $p(t, x_i)$:

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x_i^2}, \tag{1.36}$$

the solution of which

$$\langle x^2 \rangle = 2nDt, \tag{1.37}$$

where $n$ is the dimension of space (Phillips 1958), which is the average area of the spot occupied by random movements.

This section with formulas (1.19)–(1.37) in theoretical physics is called diffusion approximation (Lifshitz and Pitaevski 1979), but in fact, is a general approach to the consideration of random influences on the system under consideration correlated in one way or another (MY75).

## 1.4 Stochastic Events Flow

A statistical description of the flow of events requires a significant part of what has already been said since events require continuous preparation for their implementation. For example, earthquakes occur when the stress reaches a certain critical value in the crust, depending on the properties of the rock, first of all, and the stress arises as a result of random influences. Events must be characterized by a certain "size", intensity, and energy $E$, which must be a positively defined quadratic value.

Usually, in practice, histograms of the number of events $N(E)$ in the energy range from $E$ to $E + dE$ for some period of observations $T_0$ are constructed.

Then the value $N(E)dE$, normalized by the full number of events during time $T_0$ will be an empirical estimate of the probability of an event with energy $E$. Here, probability $N(E)$ is associated with the frequency of occurrence of an event with energy $E$. In reality, usually the sample size is not large enough, and the number of events in the range from $E$ to $E + dE$ is small and can fluctuate strongly in adjacent intervals. Therefore, in practice, a cumulative histogram is used:

$$N(\geq E) = \int_{E}^{E_{\max}} N(E)dE, \tag{1.38}$$

i.e., the number of events with an energy greater than or equal to $E$, a value with the dimension of the reverse time, i.e., frequency. Due to the operation of integration, in practice—summation, the cumulative distribution is much more stable, that is, with a smaller spread than the differential distribution $N(E)$. The function $N(E)$ obviously has a dimension inverse of the dimensions of time and its argument and is an empirical irregular estimate of the probability density for the flow of events characterized by parameter $E$, and the cumulative distribution has a frequency dimension, i.e., inverse time. Usually, if the integral in (1.29) converges well, then with an accuracy of a multiplier O (1)

$$N(E) = E^{-1}N(\geq E). \tag{1.39}$$

With random influences of the "white noise", or Markovian type, as discussed above, the increment of energy in the system in question occurs linearly with time according to (1.20).

Until we normalize the formula type (1.20)

$$E = \varepsilon\tau \tag{1.40}$$

for the full period of observations $T_0$, i.e., we do not proceed to estimates of empirical probability:

$$p_e(\geq E) = \tau(\geq E)/T_0 = [N(\geq E)T_0]^{-1}, \tag{1.41}$$

that time $\tau(\geq \text{E})$, according to §13.9 of Feller's book (1957), can be considered the average waiting time of an event with an energy $\geq E$. After normalization by $T_0$ this value will be an estimate of the dimensionless probability $p(\geq E)$. As a result, in (1.40) time $\tau \equiv \tau(\geq E)$, and the inverse of it will be an estimate of the cumulative frequency

$$N(\geq E) = [\tau(\geq E)]^{-1}.$$

Without an excursion into the concept of the average waiting time of an event, characterized by the parameter $\geq E$, which is rarely mentioned in many standard books on probability theory, we would not be able to write down the ratio (1.41).

Now (1.40) with (1.39) can be rewritten as

$$N(\geq E) = \varepsilon/E, \quad N(E) = \varepsilon/E^2 \tag{1.42}$$

with precision to the multipliers to be found from observations. The value $\varepsilon$ in (1.40) can be interpreted as the rate of generation of the quantity $E$ both in dimension and in meaning, considering the right side (1.40) as the first term of the decomposition into the Taylor series by the time of increment of the quadratic parameter $E$.

An empirical estimate of the value $\varepsilon$ is given by the formulas (1.42):

$$\varepsilon = EN(\geq E) = E^2N(E), \tag{1.43}$$

down to a multiplier of the order of one. As we shall see later, many empirical frequency-size distributions have a shape close to those of (1.42). This formula and (1.43) were first published by Golitsyn (2004).

The mean velocity square (1.31) has the dimension of energy per unit mass and we denote it as $E$. If we deal with the flux of events with frequency of dimension $t^{-1}$, then from (1.41) we may write (1.42) and for events with random arias $S$, described by (1.33) denoted by $S$:

$$N(> S) = \left(\frac{\varepsilon}{S}\right)^{1/3}, \tag{1.44}$$

$$N(S) = \frac{1}{S}\left(\frac{\varepsilon}{S}\right)^{1/3}. \tag{1.45}$$

In practice the distributions of random values $N(\leq A)$ are called cumulative ones or histograms and $N(A)$ differential ones. In Sect. 11.3 we shall find the histogram for lithospheric plates $N(\leq S) \sim S^{-n}$, $n = -0.33$. The similar form has integral distributions for spiral galaxies, Fig. 11.4.

When the text of this book was in printing stage the first author nearing 90 found the direct way of obtaining the CR spectrum from (1.42) taking into account the unit area as $s^{-1} = \left(\frac{w}{E}\right)^{2/3}$. The final form is the product of $\frac{G}{E}$ which is a unit time and $(w/E)^{2/3}$ is a unit area:

$$I(\geq E) = \frac{G}{E}\left(\frac{w}{E}\right)^{2/3} \sim E^{-5/3},$$

$$I(E) = \frac{dI(E)}{dE} \sim E^{-8/3}.$$

The multiyear measurements by PAMELA (Golitsyn 2004) presented $I(E) \sim E^{-n}$, $n = 2.67 \pm 0.02$. Unfortunately they have not presented data for $I(\geq E)$, there uncertainties in $n$ would be at least an order of magnitude smaller.

## 1.5 Special Spectral Exponents and Their Sense

The difference between the spectrum of excitations and "white noise" should lead to a difference in indicators from one for the structure functions of continuous processes and for histograms, that is, frequency-intensity distributions. Here we will briefly and formally address this issue.

Let the frequency spectrum of the effects be in the form where *the* $\beta$ is related to the index $v$ in (1.9)

$$E_a(\omega) \sim \omega^{-\beta} \tag{1.46}$$

where the cases $\beta > 0$ and $\beta < 0$ will be discussed separately. The time correlation function corresponding to this spectrum exists only for $\beta > 0$, i.e., the spectrum growing towards low frequencies. It has the form, MY75:

$$B_a(\tau) = \tau^{-\beta}\Gamma(1+\beta)\sin\frac{\pi(1-\beta)}{2} \tag{1.47}$$

and at $\beta \to 0$ tends to δ -function. If the spectrum of the process $a(t)$ is (1.46), then the spectrum of the process $u(\tau) = \int \alpha(t)\mathrm{d}t$, $\alpha$ calculation will be of the form:

$$E_u(\omega) \sim \omega^{-2-\beta}, \tag{1.48}$$

with a corresponding structure function:

$$D_u(\tau) = \left\langle \Delta u^2(t) \right\rangle \sim \tau^{1+\beta}, \tag{1.49}$$

from where acting by analogy with the derivation of formulas (1.42), we obtain:

$$N(\geq E) = [\tau(\geq E)]^{-1} \sim E^{-n}, \quad n = (1+\beta)^{-1} \tag{1.50}$$

Thus, when $\beta > 0$, the number of large events increases compared to the case of "white noise" $\beta = 0$. When $\beta > 0$, the energy of low-frequency fluctuations increases with a decrease in frequency, and the correlation increases, which causes an increase in the number of large events that manage to form more often than with a uniform distribution energy impact across the entire spectrum. By analogy with the theory of electromagnetic radiation in physics of the late XIX century, this situation can be called an “infrared catastrophe”. In any case, the spectrum at low frequencies must fall due to natural causes, since infinite energy cannot accumulate there, that is, i.e., $\omega < \omega_0$ and how the wave collapses in wind waves (see Chap. 6).

The case *of* $\beta < 0$, i.e. an increase in the spectrum of influences towards high frequencies, leads to an increase in the indicator $n$ in (1.50) compared to one, i.e. to a decrease in the number of large events compared to the case of “white noise” $\beta = 0$ and to a predominant increase in the number of small events, which is expressed in an increase in the exponent in the distribution (1.50).

From the point of view of similarity and dimensionality, the presence in the spectrum of influences (1.46) of an indicator of the degree of $\beta \neq 0$ means that in the process under consideration, there is a dimensionless parameter depending on the frequency and other dimensional quantities, which does not disappear from consideration, no matter how large or small it may be. This is an example of self-similarity of the second kind according to the terminology of G. I. Barenblatt B02). Of the possible others, for more common reasons, the difference between the indicator $n$ and the unit in the empirical cumulative distributions of the frequency-intensity of events type, it is necessary to mention the insufficient sample size (therefore, confidence intervals for the value of the indicator $n$ should always be estimated), the presence of geometric factors, as for the statistics of earthquakes in thin plates near the mid-ocean ridges in contrast to their statistics in faults, that is, the boundaries of much thicker ones. San Andreas Fault Type Plates in Southern California (see Chap. 3). Another parameter of the difference is the distribution, where the number of events is measured not only per unit of time but also per unit area, which itself may depend on the parameter by which the distribution is sought, as will be seen in the case of cosmic ray statistics (Chap. 4).

Here we have considered various, practically useful questions of statistics of temporal random processes. However, in reality, many tasks require knowledge of the statistics of processes in their spatial manifestation, for example, turbulence, although, as a rule, we have only a time record of the signal at the point. In the latter case, the simplest connection between temporal and spatial characteristics for spatially homogeneous processes is given by the hypothesis of GI. Taylor on so-called “frozen” turbulence (see MY75). Formally, this hypothesis uses the relation:

$$r = U\tau, \tag{1.51}$$

or in wave representation:

$$\omega = kU, k = 2\pi/r, \omega = 2\pi/\tau, \tag{1.52}$$

where $U$ is the average flow velocity, e.g., wind in the atmosphere. It is assumed that the turbulence carried by the wind past the measurement point remains virtually unchanged during the measurement $\tau$.

In general, it is necessary to use a dispersion ratio $\omega = \omega(\tau)$, which does not have to be linear as (1.52). For example, the ratio (1.20) for the inertial turbulence interval can be compared (again, with an accuracy of a numerical coefficient) to the dispersion equation:

$$\omega = \varepsilon^{1/3} k^{2/3} \tag{1.53}$$

Ratios of the dispersion type, as, for example, for waves on the surface of the sea, in the transition from frequency characteristics to spatial ones, can be considered as an operation of replacing variables in the distribution of probability or its moments. Due to the conservation of probability, equality must be fulfilled.

$$P(x)dx = P(y)dy, \; y = y(x). \tag{1.54}$$

Similar relationships can be written for distribution moments, for example, for spectral densities. In the latter case, a physical interpretation of such an equality is also possible: due to (1.5), the integral of the spectrum is equal to the variance of the quantity under study.

Therefore, under the assumption of ergodicity, the magnitude of the variance should not depend on whether we estimate it using temporal or spatial characteristics, that is, whether the amount of variance is estimated using temporal or spatial characteristics,

$$E(\omega)d\omega = E(k)dk. \tag{1.55}$$

When studying the spatial structure of random fields, the question of the diffusion of particles in such fields is practically important. As is known, the motion of the Brownian particle in a random velocity field with a spectrum of "white noise" is carried out with a constant diffusion coefficient - see (1.25). Formally, the diffusion coefficient can be defined as (see Chap. 7 for details):

$$K = \frac{1}{2}\frac{d}{dt}\left\langle (\Delta x)^2 \right\rangle = \frac{1}{2}\frac{dr^2}{dt} = r\frac{dr}{dt} = ru, \tag{1.56}$$

or a time derivative of the area. In this spirit, the old, Taylor (1915), definition works.

Consider the general case of the impact spectrum (1.44): $\mathrm{E}_\alpha(\omega) \sim \omega^{-\beta}$, $\beta > 0$. It corresponds to the spectrum of spatial displacements:

$$E_k(\omega) \sim \omega^{-4-\beta} \tag{1.57}$$

for which the generalized structure function with second-order stationary increments $(0 < \beta < 1)$ will be:

$$D_x(\tau) \sim \tau^{3+\beta} \equiv r^2. \tag{1.58}$$