Applied and Numerical Harmonic Analysis

$$\hat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma} dx$$

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Editors

The Mathematical Heritage of Guido Weiss

Birkhäuser

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis* (ANHA) book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art ANHA series.

Our vision of modern harmonic analysis includes a broad array of mathematical areas, e.g., wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, deep learning, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods.

The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of ANHA. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

*Analytic Number theory * Antenna Theory * Artificial Intelligence * Biomedical Signal Processing * Classical Fourier Analysis * Coding Theory * Communications Theory * Compressed Sensing * Crystallography and Quasi-Crystals * Data Mining * Data Science * Deep Learning * Digital Signal Processing * Dimension Reduction and Classification * Fast Algorithms * Frame Theory and Applications * Gabor Theory and Applications * Geophysics * Image Processing * Machine Learning * Manifold Learning * Numerical Partial Differential Equations * Neural Networks * Phaseless Reconstruction * Prediction Theory * Quantum Information Theory * Radar Applications * Sampling Theory (Uniform and Non-uniform) and Applications * Spectral Estimation * Speech Processing * Statistical Signal Processing * Super-resolution * Time Series * Time-Frequency and Time-Scale Analysis * Tomography * Turbulence * Uncertainty Principles *Waveform design * Wavelet Theory and Applications

The above point of view for the ANHA book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but is a fundamental tool for analyzing the ideal structures of Banach algebras. It also provides the proper notion of spectrum for phenomena such as white light. This latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems. These problems, in turn, deal naturally with Hardy spaces in complex analysis, as well as inspiring Wiener to consider communications engineering in terms of feedback and stability, his cybernetics. This latter theory develops concepts to understand complex systems such as learning and cognition and neural networks; and it is arguably a precursor of deep learning and its spectacular interactions with data science and AI.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory.

The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'etre of the ANHA series!

College Park, MD, USA John Benedetto
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Preface

Guido Weiss passed away on December 25, 2021, at the age of 92.

With his intensive mathematical production over a 65 year period he has enormously influenced the development of harmonic analysis.

With his strong and inclusive personality, he has supported generations of students and young researchers, favoring the diffusion of this mathematical field to various parts of the world.

The Editors of this volume have been part of this large group and represent two countries, Italy and Spain, where Guido Weiss's influence has been particularly wide and significant. The invited contributors are collaborators, students, and colleagues of Guido, who had a particularly intense relationship with him.

Guido's Mathematics

Guido's mathematical activity can be divided into three periods, with large chronological overlaps and interactions.

The first period (1956–1970) is marked by his collaboration with E. Stein and his first wife Mary Weiss. The main topics are pioneering applications of complex interpolation on Hardy spaces and, most remarkably, the extension of complex-variable methods to real analysis, in particular with the introduction of generalized Cauchy–Riemann systems in real Euclidean spaces. The latter led to the extension of the classic holomorphic Hardy spaces to spaces in several real variables, which preluded the later developments of C. Fefferman and Stein. The collaboration with Stein culminated in the book "Introduction to Harmonic Analysis on Euclidean Spaces," Princeton University Press (1971), which is among the most cited books in this area. It is still used as basic reference in Fourier Analysis.

The second period (1969–1994) is the most articulated for varieties of themes and is marked by his collaboration with R. Coifman. Though many other co-authors were involved, M. Taibleson and R. Rochberg in the first place, the leading force was the strong synergy between him and R. Coifman when both were at “Wash U.” The main themes of this period have been

- Atomic characterization of Hardy spaces and their extension to spaces of homogeneous type, with applications to multiplier theory on Lie groups, and other function spaces (Besov, Triebel-Lizorkin, bloch spaces, etc.).
- The transference method, a generalization to group representations of methods already present in the works of Calderón, Herz, and others.
- Analytic families of Banach spaces and complex interpolation theorems.

The third is the wavelet period. Guido was interested in wavelets since its discovery, mainly through the works of Yves Meyer, who visited Washington University often in the 1980s. He gave Graduate courses at Wash U at the beginning or the 1990s, that lead to the book “A First Course on Wavelets” (CRC Press, 1996).

Guido found his personal way to focus in the most theoretical aspects of wavelets, without forgetting his applications. Among his many collaborators in this period, we must single out Ed Wilson, who was his main research partner in this area. Some of the themes of this period are:

- Foundations of the mathematical theory of wavelets in dimension one
- Extensions to several dimensions, introducing the generalized translation invariant systems, the composite dilation wavelets, and the shearlets
- Frames and Riesz systems generated by unitary representations of locally compact abelian groups

Guido’s Mathematical Community

Guido had an exuberant personality and very high energy. He influenced the life, professional and personal, of many people. He had many graduate students, all of whom he followed well beyond their graduation and remained in strict contact with. Guido mentored many young students, helping them find the best next step in their professional careers. Perhaps most remarkably, Guido had a very large group of collaborators, young and his senior, and was involved in the activities of many of his colleagues at Washington University.

After graduating from the University of Chicago, he moved to Washington University in St. Louis where he remained for all of his life. The Department of Mathematics at Wash U became a gravitational center for harmonic analysis in the world panorama. With his charisma, he attracted mathematicians from all over the world. With his drive and dedication, Guido invited young students and collaborators from Argentina, Spain, Italy, China, Poland, France, and Croatia, just to name the most numerous groups.

Guido was able to establish a personal relationship with essentially everybody. He organized social events, parties at the department and his own home, naturistic excursions, sports events. With his seemingly endless energy, Guido knitted the connection among the various people he had been interacting with.

To summarize, Guido created a large mathematical community. The significance of this community was very clear to many already, and was exemplified at Guido's Fest, held at Wash U in October 2022, where so many people rediscovered that they had been long-time friends *because of Guido*.

Guido's Personality

We have already mentioned Guido's exuberant personality. Such kind of personality was a necessary requirement in order to create the vast community at Wash U. Guido *loved* to be at the center of the community he created, and gave back a lot in return. Guido spent a lot of his time practically helping his visitors, were they incoming graduate students, short-time visitors, and long-standing collaborators. Guido helped everybody, in one way or another. We could write a book just telling stories of Guido caring for someone.

Guido was very generous, not just in teaching, advising, and sharing his mathematical views, but also in helping *his* community to deal with life events, whether were these daily or serious matters.

Guido's Humanity

Guido has been a very good friend to many people. His desire to help everyone around him led him also to offer his friendship. Guido opened up his home to everyone in his community: dinners, parties, Italian lessons, swimming at the pool, and we could go on. He had a witty sense of humor, and we can safely say that his jokes are legendary. Guido's humanity was warm-hearted, and caring.

Conclusions

We make no pretense of sketching his biography, not to illustrate here all of Guido's achievements. The reader who is interested about Guido's biography is referred to the article of S. Kelly and R. Torres, "Guido Weiss: From Immigrant Boy to International Renowned Research Mathematician" (*The Journal of Geometric Analysis*, 2021, 31: 9146–9179).

Let us mention, however, that he was the recipient of the Chauvenet price in 1971 for his expository writings on Harmonic Analysis. Also, he played an essential role in the academic growth of the Department of Mathematics of Washington University, and held positions in the American Mathematical Society, the National

Research Council, and the National Science Foundation, as well as in the Athletic Department of Wash U.

We are dedicating this volume to Guido's vision of a mathematical community, and ultimately its creation and success. We believe this a very remarkable achievement.

Thank you, Guido!

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May, 2024	

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Chapter 1
Guido Weiss: A Few Memories of a Friend and an Influential Mathematician

Pascal Auscher and Aline Bonami

Abstract This contribution starts with an exchange between us on the way we met Guido and he influenced our mathematical lives. Then it is mainly a survey paper that illustrates this influence by describing different topics and their subsequent evolution after his seminal papers and courses. Our main thread is the notion of a space of homogeneous type. In the second section we describe how it became central in pluricomplex analysis and consider particularly the existence of weak factorization for spaces of holomorphic functions. In the last section, one revisits the construction of a basis of wavelets in a space of homogeneous type and the way it allows a Littlewood-Paley analysis.

1.1 Introduction

The authors of this chapter belong to two different generations of harmonic analysts. We both met Guido at a very early stage in our careers. His influence and friendship were important for both of us. We found it interesting to cross our memories of Guido. In a first section, we will tell how we met Guido and try to describe what has been his influence on our mathematical choices. This will be done through a dialog between us.

The rest of the paper will be more classical and look like a survey paper on particular points for which the influence of Guido was like a starting point for us. We will not be exhaustive on the choice of topics and will certainly not cite all the appropriate literature in the fields that we have chosen to describe in more detail. We

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apologize in advance for these choices, which are linked with our personal views and memories and may diverge from other choices and memories.

The main thread of this paper is the notion of space of homogeneous type that was introduced by Raphy Coifman and Guido in the early seventies. In the second section, mainly written by Aline, we will speak of its influence in pluricomplex analysis. In the third part, mainly written by Pascal, we will see how it may lead to develop all necessary tools for Littlewood-Paley theory in a very general context.

1.2 The Past, the Past…

Pascal. You start, of course!

Aline. For me, it begins in the academic year 1970–1971, when Guido Weiss and Raphy Coifman gave a course in Orsay, now University Paris-Saclay. Their collaboration had started a few years earlier, and at the time they were interested in developing multiplier theorems related to actions of non-commutative groups. I was in the audience, which was not so numerous: even if harmonic analysis was still one of the main topics in the maths department, there was not such a feverish atmosphere as three or four years before. Moreover, Jean-Pierre Kahane and Yves Meyer were mainly abroad. It was perhaps a chance for us. Guido and Raphy asked for volunteers to help them in writing notes, which I did in collaboration with Jean-Louis Clerc and Bernard Mischler. So I met Guido and Raphy every week. I can still see them at the blackboard, in their very different styles, Guido writing calmly with his magnificent writing long formulas for spherical harmonics, while Raphy tried to convince us of the simplicity of notions. The notes that followed, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes* [17] went much further than the initial program and, in a sense, were the first steps into a new paradigm, the world of *spaces of homogeneous type.*

Pascal. Which influence had this course on you?

Aline. The scientific life in Orsay was so intense at that time that it is difficult to sort out influences. It was my third course on real analysis given by foreigners. Eli Stein had given one in 1967–1968, just when I started research, the course that led to his book *Singular integrals.…* I had adored this course and had thought at the time that it was really what I wanted to do. But how? It seemed so difficult and the school of Chicago seemed so much in advance! At the same time I followed a course of Yves Meyer, who started to ask me questions on multipliers that led to my thesis on what is now called hypercontractivity. Working with Yves was a huge luck. At the time of the venue in Orsay of Guido and Raphy, I defended my thesis and even worked during a few months in probability theory. In the mean time, I had also attended a course of N. Riviere, who died prematurely. He talked on singular integrals related to parabolic equations. After the course of Guido and

Raphy, one saw precisely how the two courses I had followed before dealt with two examples of their *Espaces de nature homogène*. One had a very clear description of the geometric objects involved. In fact their course opened the door to a lot of problems in the theory of singular integrals. Afterwards, it was obvious for me that this was the kind of mathematics I preferred.

It would be unfair to limit the influence of Guido and Raphy at Orsay that year to mathematics. From the beginning, and particularly at that time when relations between mathematicians were still somewhat formal in France, we were all stuck by their generosity and kindness, which they expressed directly to everybody.

Pascal. And what influence had this course on the harmonic analysis team in Orsay at the time?

Aline. It had a direct influence on Jean-Louis Clerc, Noël Lohoue and me, both for the content of their course itself and for the way to do mathematics, to interact between us. As I said before, relationships at work in France were then not so natural. There was still the idea one should publish alone. We started to work together, the three of us, in line with the course, trying to extend to other contexts known properties in Fourier Analysis. Inspired by their course, we read together the book of Helgason *Differential geometry and symmetric manifolds.* And with Jean-Louis I studied Cesàro means for expansions in spherical harmonics. I went two months to Washington University (WU) during the spring 1972. Of course Guido and Raphy had a large influence on the harmonic analysts in Orsay, but a little later. In fact, Yves Meyer became a frequent visitor at WU and started to work with Raphy a little later, in 1974. Afterwards the interactions between WU and French harmonic analysts increased.

Again, friendship mixes with mathematics and it is impossible not to speak of this also. I visited regularly WU after 1972, even if much more rapidly. We met also frequently in Europe. To come back to the far past, I particularly remember having been at WU in 1981, at the same time as Jean-Lin Journé. We were both invited in Guido's and Barbara's summer house for a kind of country party. Conversations, canoeing on the small lake, Guido removing splinters in my hand with the same meticulousness he used to show in his classes…, there was such a charm, as in a film. Guido's jokes made a lot to create a friendly atmosphere around him. Nevertheless Guido's life had its share of mystery and suffering, starting with his childhood in Italy and the intellectual heritage of his parents. He was also a man of conviction, deeply attached to the values of human solidarity. He was a model for many of us.

Nearly twenty years later, Pascal spends two and a half academic years at WU in the context of his military service that all French men had to accomplish at that time. He is not the first student of Yves Meyer to do this: Jean-Lin Journé was there in 1981–82. Its influence on Jean-Lin may be measured when looking at the notes of the course that he gave there and which were published under the title Calderón-Zygmund Operators, Pseudo-Differential Operators and the Cauchy Integral of Calderón.

Aline. It is time for me to ask you a question. Did you know Guido before going to WU ? How was your encounter?

Pascal. No, I did not know him, nor, I must say, had read his books or articles. I was working on wavelet theory at the time, during the years 1986–87 and 1987–88. The mathematical theory of wavelets was just starting and I focused on that. For my military service, one possibility was to do it abroad in a teaching or research institution. Yves suggested that I go to WU. He contacted Guido and I recall reading his enthusiastic reply (a mailed letter at the time) dated December 31, 1987. I was just stuck by the date. Then started some exchanges with Guido (by the new email tool!) in the spring to set up things for the next fall (and for arrangements with French military office). I also read his book with Eli Stein [72] before leaving. The encounter was made easy in all means by Guido's welcoming help. With common interests in tennis and bird-watching, it became a strong friendship.

Aline. How was the scientific life in WU at the time? Were there seminars, for instance? How did you interact with others?

Pascal. First, the mathematics department was a highly renowned place for harmonic analysis. Raphy was long gone, but Al Baernstein, Björn Dalhberg, Steven Krantz, Richard Rochberg, Mitch Taibleson were very active members. This department also attracted many doctoral students from abroad (Spain, Italy, Argentina, China..) and post-docs, some became good friends (Estela Gavosto and Rodolfo Torres, Carlo Morpurgo, Marco Peloso, María-Jesús Carro. . .) and became influential mathematicians back home after having benefited from the formidable and stimulating scientific atmosphere. I was the only French person (the French system did not promote much foreign post-docs in those days). Guido wanted me to learn new maths so he proposed to Maria-Jesús, who had done her thesis on interpolation theory, and me to work on some problems on multipliers around the Riesz means, transference and other things. This led to two articles. The activity in the department was intense: regular analysis seminar, colloquiums, lots of master courses, and meetings in Guido's office. I still have the notes of the course of Guido on transference, based on his work with Raphy. All details were concisely given on the board: "too many details" was I thinking, but I realize now that this allowed the topic to profoundly print in my mind. He was always encouraging people so as to make them available for more (integrating the f+ according to the words of A. Zygmund). I recall he brought me to visit Raphy at Yale, and this visit had tremendous impact for me, as an example of how curiosity drives mathematics. After my thesis, I was trying to understand (solve, why not?) the Kato conjecture in any dimension using wavelet methods as Ph. Tchamitchian had just reproved nicely the strongly linked T(b) Theorem of David-Journé-Semmes via regular adapted wavelets [76]. When wandering in the Yale math department, I found a table with help yourself free printed (the pdf's did not yet exist and some references could be hard to find) articles that people were giving away. I discovered the article of McIntosh on bounded holomorphic calculus from the Proceedings of the Center for Mathematical Analysis of the Australian National University and also the article of Jacques-Louis Lions pursuing the interpolation theory of Kato for maximal accretive operators, which inspired me for later research.

Aline. You have a paper with Guido on Wilson bases. How was it?

Pascal. During my stay at WU, Guido was curiously NOT interested to work on wavelets, despite my attempts to suggest problems, like the one which I solved later showing that a mild sufficient condition on the wavelet generating an orthonormal basis suffices to conclude it arises from a multiresolution analysis, introducing at the same time a special series whose values are the dimensions of certain vector spaces, that was next baptised by Guido as the "dimension function" and which became a central tool as beautifully explained in Guido's reference book with Eugenio Hernández [43]. That trip to Yale was when a new interest came in Guido's mind. Raphy explained to us the concept of local Fourier bases and Guido wanted to understand the calculations behind. We started to work out the details, trying in Guido's style to make them as conceptually simple and elegant as possible. The observation that these bases and the Meyer orthonormal wavelet basis were built from the same tricks fascinated him. Victor Wickerhauser joined the discussion and the article was ready for publication [4]. This is how he plunged into the mathematical theory of wavelets, trying to exhibit its finest structures and as always in the most simple and accessible terms. This was when I left WU at the end of 1990…

Pascal. You also have papers with Guido in the nineties. Can you say something on them?

Aline. As you said, Guido started to work on the theory of wavelets in the nineties. Just before, he was particularly interested by characterizations coming from Littlewood-Paley theory. I like very much his notes with Frazier and Jawerth in the CBMS Regional Conference Series (1991), [32]. He had all reasons to be fascinated by wavelets and had questions related to constructions. Yes, we wrote two papers, the first one [8] on band-limited wavelets. Our joint work was pursued by Gustavo Garrigós who was then his PhD student. Gustavo came later to Orléans as a post-doc, which was really great for me. But then Gustavo and I worked on Bergman spaces in tube domains, where of course we used a lot his book with Eli Stein and the properties of Hardy spaces in tube domains given there.

Aline. From your point of view, what is the mathematical legacy of Guido?

Pascal. Of course, the discoveries he made all along his career, some that we are going to elaborate on in this article afterwards, which paved the way to create a strong school in harmonic analysis and linked topics: interpolation, singular integrals, Hardy spaces, spaces of homogeneous type…In all of these, there was always the will to leave no black holes in his way of writing, going into every details even the simplest ones, to make the reading accessible. In that respect, he was particularly proud of the Chauvenet prize he obtained in 1967 [78], the highest American award for mathematical expository writing. For example, he told me that for his book with Eli Stein, the first to propose a synthetic exposition of the extension of harmonic analysis from one to several Euclidean dimensions that occurred in the sixties, he thought at length about notation avoiding as much as possible using coordinates, to present the computations in the most conceptual manner (and with almost no typos). This notation is still accurate as of today. The legacy is then clear. He posed the solid bricks on which many new results can be elaborated. Still, despite there are now many references on each of these topics, my first reaction when I look for a reference or a proof is to browse his masterpieces as I am sure to find what I need (and sometimes understand some new points that I left aside on earlier reading). When I write research articles, I always have in mind the words of Guido: try to stay simple and reader friendly.

Pascal. Same question.

Aline. I will only speak of my own heritage. Of course I mention first his books, primarily his book with Eli Stein and the Notes of Orsay, which played different roles in my personal Pantheon. I had the impression to have the Notes in mind, just as they were engraved during the courses of Guido and Raphy, while I came back to the book with Eli Stein to read it in all details each time I gave a course myself. The book with Eugenio reaches the same degree of perfection. May be you find elsewhere a better source for intuition, for example in Yves's books, but if you want to understand all details, the book of Eugenio and Guido makes the job remarkably. These are three books that I keep on hand in my library. Then there are the works of Guido that inspired me for long. I would mention first the paper in Annals with Raphy and Richard Rochberg on commutators and weak factorization. I have the impression to have turned around this paper part of my life. We will come back to this later. I would like also to mention some parts of his work that surprised me when I heard him speak of them and that I regret not to have understood more deeply, not to have come back to them later. You know, this kind of sensation one has that there, there is something I do not understand enough, there is some new music with which I would like to familiarize myself. For example, I recall how Guido was excited with Block spaces, that is, spaces that are defined from an atomic decomposition [75], but with atoms that are not of mean 0, and for which one has almost everywhere convergence results, for instance. Of course, the subject has been deepened afterwards, in particular by Fernando Soria. I have nevertheless the feeling I should have tried to understand better what was around. The same with interpolation. Being exhaustive is impossible. But I will try to show, in the next section, how the work of Guido influenced mine. And of course, as you said, there is a lot more in the legacy of Guido than a list of results. There is a way to do mathematics, to write and to talk, and to do much more than only mathematics, there is a way to be curious of life and others, there is a lot.

1.3 Orsay's Course and Excursion Inside Complex Analysis

1.3.1 The Notes: A New Paradigm, the One of Space of Homogeneous Type

One of the main new notions of the course was the notion of *space of homogeneous type*. For a topological space X, one first calls *pseudo-distance (or quasi-distance depending on the authors)* a function $\rho : X \times X \mapsto [0, \infty)$ such that there exists $K \geq 1$ such that, for all x, y, z in X,

1. $\rho(x, y) = 0 \iff x = y$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, z) \leq K(\rho(x, y) + \rho(y, z))$.

The pseudo-ball (or quasi-ball, or simply ball) $B(x, r)$ with center x and radius r is the set of points $y \in X$ such that $\rho(x, y) < r$.

We say that X is a *space of homogeneous type* if X is endowed with a pseudo-distance ρ and sith a Borel measure μ such that

1. the balls B centered at x constitute a basis of neighborhoods of x
2. the measure μ is doubling, that is, there exists a constant A such that, for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r) \leq A\mu(B(x, r)).$$

These definitions are sufficient to define the maximal function of $f \in L^1(d\mu)$ by

$$Mf(x) := \sup_{r>0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y)| d\mu(y),$$

to prove the maximal theorem in this context as a consequence of a covering lemma of Vitali type, to develop as well Whitney's covering lemmas and to make it possible to write a Calderón-Zygmund decomposition of a function $f \in L^1(d\mu)$. As in the theory of singular integrals in $\mathbb{R}^n$, it is then possible to prove L^p estimates for singular integrals once one assumes that one already knows them for a particular value $p_0 > 1$. To go ahead and be able to have theorems $T(1)$ or $T(b)$ asks for new tools, and this is what we will tempt to describe later.

Calderón-Zygmund theory was developed at the same time by Coifman and de Guzmán in [15]. This last one studied systematically covering lemmas in relation with differentiation [23], which is always a living field of study (see for instance [69]).

It should be emphasized that, in the seventies, one had in mind some generalizations of Calderón-Zygmund theory, but not all. For example, the idea that this could be used for discrete structures appeared much later. Nevertheless, there is already this idea in the Notes that the doubling property of the measure μ can be replaced by the following one, which is weaker and is expressed only as a geometric property (named nowadays as the geometric doubling property):

2′. There exists a constant N such that, for all $x \in X$ and $r > 0$, the ball $B(x, 2r)$ is covered by at most N balls of radius r.

This will be seen as the right point of view for some constructions later on in this text.

There are also emerging notions in the course, which have been central in the further development of the theory. In particular, the definition of an atom of the Hardy space H^1 and the atomic decomposition of H^1 have only been given by Coifman a little later [14]. But the notion of atoms can already be found in the Notes (page 89 for instance). It is considered there for Riemannian symmetric spaces, and only from examples that are built with approximate identities. The property of singular integrals to transport them into molecules is already present. All this program is developed, this time with precise definitions and statements, in the two fundamental texts [18, 74].

In the Notes, one of the first examples of space of homogeneous type which is given is the unit sphere in $\mathbb{C}^n$ endowed with the Euclidean measure $d\sigma$ and the distance $d_b(z, \zeta) := |1 - \langle z\, , \zeta\rangle|$. We note $\mathbb{B}_n$ the unit ball in $\mathbb{C}^n$, $\partial\mathbb{B}^n$ its boundary, that is, the unit sphere and $\langle\cdot, \cdot\rangle$ the Hermitian inner product on $\mathbb{C}^n$. The distance d_b (the index b stands for the boundary) is clearly invariant under the action of the group $SU(n)$ and the distance from the point $\mathbf{1} = (1, 0, \cdots, 0)$ satisfies the property

$$d(\mathbf{1}, z) \simeq |\operatorname{Im} z_1| + |z'|^2, \qquad z = (z_1, z'),$$

meaning that the quotient of the two quantities lies between two uniform constants in a neighborhood of $\mathbf{1}$. Moreover, $\sigma(B(z, r)) \simeq r^n$ and

$$|z - \zeta|^2 \lesssim d_b(z, \zeta) \lesssim |z - \zeta|.$$

This metric is usually called the Korányi's metric. It had in particular already been used by A. Korányi for generalizing Fatou Theorem of non tangential convergence for harmonic functions in the unit disc to harmonic functions in $\mathbb{B}_n$ with respect to the Laplace-Beltrami operator for the Bergman metric [49]. Indeed, in the same way that the first question for spaces of homogeneous nature was about generalizations of the Lebesgue Differentiation theorem, the first question for holomorphic functions was about generalizations of Fatou Theorem.

At this point, let us recall definitions before going further. For Ω a smooth bounded domain in $\mathbb{C}^n$ and $0 < p < \infty$, let $L^p(\Omega)$ denote the Lebesgue space with respect to the Lebesgue measure $dV(z)$ and $A^p(\Omega)$ be the corresponding Bergman space, that is, the closed subspace of $L^p(\Omega)$ consisting of holomorphic functions. For $0 < p < \infty$, let $L^p(\partial\Omega)$ denote the Lebesgue space on $\partial\Omega$ with respect to the induced surface measure $d\sigma$ and $H^p(\Omega)$ the Hardy space of holomorphic functions on Ω, with norm given by

$$\|f\|_{H^p}^p := \sup_{0<\varepsilon<\varepsilon_0} \int_\Omega |f(w - \varepsilon\nu_w)|^p d\sigma(w),$$

where ν_w is the unit exterior normal vector at the point w, with ν_w seen as a vector in $\mathbb{C}^n$. The question of generalizing non tangential convergence to higher dimension was treated in the small book of E. Stein [71], which has been a major source of inspiration for its adaptation of potential theory to this context and for its list of open questions. For such smooth domains, functions in $H^p(\partial\Omega)$ have admissible limits a.e.: the admissible regions, roughly speaking, are defined in the same way as non tangential regions in the unit disc, except that the ordinary distance on the circle is replaced by the Korányi pseudo-distance on the boundary $\partial\Omega$, that is,

$$d_b(z,\zeta) := |\langle \nu_z, z-\zeta\rangle| + |z-\zeta|^2.$$

It is not symmetric, but $d_b(z,\zeta) \simeq d_b(\zeta,z)$ and can be symmetrized. These admissible regions are optimal for strictly pseudo-convex domains (that is, domains that are locally diffeomorphic to strictly convex domains), but not for weakly pseudo-convex domains in which there may be a larger swelling of admissible regions in complex tangential directions and a need for a definition that takes this into account. See for instance [62]. Because of admissible limits one may consider $H^p(\Omega)$ as a subspace of $L^p(\partial\Omega)$.

The next question concerns Bergman and Szegö projections, which we denote by P_B and P_S and which are, respectively, the orthogonal projection from $L^2(\Omega)$ to $A^2(\Omega)$ and the orthogonal projection from $L^2(\partial\Omega)$ to $H^2(\Omega)$ (identified with a subspace of $L^2(\partial\Omega)$). They are respectively given by the Bergman kernel $B_\Omega(z,\zeta)$ and the Szegö kernel $S_\Omega(z,\zeta)$. For the unit ball they are given by

$$S(z,\zeta) = \frac{c_n}{(1-\langle z\,,\zeta\rangle)^n}, \qquad B(z,\zeta) = \frac{c'_n}{(1-\langle z\,,\zeta\rangle)^{n+1}}.$$

Korányi and Vági, independently of Coifman and Weiss, developed around the same time a theory of singular integrals on homogeneous spaces, which led to the L^p inequalities for the Szegö projection of the unit ball [50]. This one is now seen as an immediate example of singular integral on $\partial\mathbb{B}^n$. Indeed, it is elementary to see that

$$|S(\mathbf{1},z) - S(\mathbf{1},\zeta)| \lesssim \frac{d_b(z,\zeta)^{1/2}}{|1-\mathbf{1}\cdot\zeta|^{n+1}},$$

which, by invariance by the action of $SU(n)$, gives the required inequality for a singular integral on the space of homogeneous type given by $\partial\mathbb{B}^n$, the Korányi distance and the Euclidean measure.

In order to obtain L^p estimates for the Bergman projection, one may also be consider it as a singular integral for the distance that is given by the ordinary distance in the radial direction and by d_b on the boundary $\partial\mathbb{B}_n$. But $L^p(\mathbb{B}_n)$ estimates can be obtained in a much simpler way by using Schur's lemma. These kinds of estimates have been first developed by Forelli and Rudin (see [30]).

This was the beginning of a long story, which we evoke now. From 1970, a considerable work has been done on Hardy and Bergman spaces as well as Szegö and Bergman projections in smooth bounded pseudo-convex domains of $\mathbb{C}^n$. The link between the adequate pseudo-distance at the boundary and the two kernels, which is obvious for the unit ball, asks for long studies in general. Estimates on the Bergman and Szegö kernels have been developed in the context of strictly pseudo-convex domains, then pseudo-convex domains of finite type in $\mathbb{C}^2$, then convex domains of finite type by Fefferman [28], Nagel et al. [63], McNeal and Stein [56] and many others.

The work of Henkin [42] on strictly pseudo-convex domains play a fundamental role for having explicit representation formulas. They lead to the construction of support functions $H \in \mathcal{C}^\infty(\Omega \times U)$ with U a neighborhood of the boundary, such that $H(\cdot, \zeta)$ is holomorphic on Ω for all $\zeta \in U$ and

$$C^{-1} d(z, \zeta) \leq |H(z, \zeta)| \leq C d(z, \zeta).$$

Here d is defined in a neighborhood of the boundary by

$$d(z, \zeta) = \delta(z) + \delta(\zeta) + d_b(\pi(z), \pi(\zeta)) \tag{1.1}$$

where $\delta(z)$ is the distance of z to $\partial\Omega$ and $\pi(z)$ is the closest point of $\partial\Omega$, which is well-defined in U for U small enough and d_b is the adequate distance on the boundary. When Ω is the unit ball, one can take simply $H(z, \zeta) = 1 - \langle z, \zeta \rangle$. It is well adapted to Szegö or Bergman spaces, which have only singularities at the boundary. For convex domains of finite type (with the adequate pseudo-distance on the boundary), such a construction has been given by Diederich and Fornæss [26]. We will use these support functions later on. Unlike what happens in the unit ball, they are not found in connection with the Bergman kernel.

1.3.2 Weak Factorization, Commutators and Hankel Operators

In this subsection we will concentrate on one of the other fundamental papers of Guido, namely the one on commutators [19]. It was written in 1975 with Richard Rochberg and Raphy and, again, had a large influence on problems that have been considered later. Some of them have been solved recently, some are still open. It is among the top cited papers of Guido and has been generalized in many directions during the nearly passed fifty years. One can only find here a personal point of view, which leaves many developments aside.

The first question that is treated in this paper is the characterization of $BMO(\mathbb{R}^n)$ as the class of functions b for which the commutator $[M_b, R_j]$ are bounded in $L^2(\mathbb{R}^n)$. Here M_b stands for the multiplication by b and R_j is the j-th Riesz transform, that is, the convolution by the distribution $c_n p.v. \frac{x_j}{|x|^{n+1}}$. This has been fundamental and, almost twenty years later, revealed itself to be one of the ways to understand the div-curl lemma in relation with compensated compactness [20].

We will consider in more detail the second part of the article, which deals with holomorphic functions and factorization. Let us recall that in dimension 1 every function $f \in H^p(\mathbb{B}_1)$ may be written as the product gh, with $g \in H^{p_1}(\mathbb{B}_1)$, $h \in H^{p_2}(\mathbb{B}_1)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. This is false in higher dimension (see [35]). In this second part one finds for the first time the idea to replace factorization by weak factorization. Namely, they prove that any function $f \in H^1(\mathbb{B}_n)$ can be written as

$$f = \sum_{j=1}^{\infty} g_j h_j, \qquad \sum_j \|g_j\|_{H^p} \|g_j\|_{H^{p'}} \simeq \|f\|_{H^1}.$$

Their proof uses atomic decomposition of Hardy spaces $H^1(\mathbb{B}_n)$. Namely, a holomorphic atom A is the Szegö projection of an atom a, that is, $A = P_S a$ with a a bounded function of mean zero, which is supported in a pseudo-ball $B \subset \mathbb{B}_n$, and such that $\|a\|_\infty \leq |B|^{-1}$. They prove that H^1 has an atomic decomposition, that is, $f \in H^1(\mathbb{B}_n)$ can be written in terms of atoms a_j related to balls $B_j = B(w_j, r_j)$

$$f = c + \sum_{j=1}^{\infty} \lambda_j a_j, \qquad \sum_j |\lambda_j| \simeq \|f\|_{H^1}.$$

It is then sufficient to factorize an atom a_j related to the ball $B(w_j, r_j)$. At this point their proof may be in some way simplified and can be seen as an easy consequence of the existence of a support function that satisfies (1.1), which makes it easier to find generalizations to other domains. One uses an idea that emerged later on: it is possible to ask for the atoms to have more moments that vanish, so that $|P_S a_j|$, which is not supported in B_j, decreases rapidly outside B_j. Then, one just takes $g_j = H(z, \zeta_j)^{-l}$, where the point ζ_j is chosen at distance r_j of $\partial\mathbb{B}_n$ and such that $\pi(\zeta_j) = w_j$. So for $l > n$ the norm of g_j in H^p is easily estimated, as well as the one of $H(z, \zeta_j)^l P a_j$ in $H^{p'}$ when the atoms have sufficiently vanishing moments.

This kind of proof extends easily to other domains, see [9]. It allows one to generalize weak factorization to H^p for $p < 1$ but does not work when there is no atomic decomposition of H^p, that is, for $p > 1$, even in the unit ball $\mathbb{B}^n$. Up to our knowledge, the existence of weak factorization of H^p is still an open problem in dimension $n > 1$:
For $p > 1$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, is there weak factorization for $H^p(\mathbb{B}^n)$ with products of functions in $H^{p_1}(\mathbb{B}^n)$ and $H^{p_2}(\mathbb{B}^n)$ when $n \geq 2$?

Another way to prove weak factorization consists in proving boundedness of the Hankel operator, which is the anti-linear operator on H^p defined by $H_b(f) = P_S(b\overline{f})$. More precisely, the fact that H^1 may be weakly factorized with products of functions in H^p and $H^{p'}$ is equivalent to the fact that H_b is bounded on H^p if and only if b belongs to the dual of H^1. The same considerations prove that the open problem given above is equivalent to the fact that the Hankel operator H_b maps H^{p_1} into $H^{p_2'}$ if and only if b belongs to $H^{p'}$.

In $\mathbb{R}^n$, such characterizations have been proved recently by Hytönen [45] for commutators $[M_b, R_j]$. It is straightforward that these commutators map L^p into L^q for $b \in L^r$, with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. The challenge was to prove the converse, which is quite intricate. Unfortunately, this result, even if proved for the Szegö projection on $\partial\mathbb{B}_n$, is not sufficient to find weak factorization for $H^p(\mathbb{B}_n)$ for $p > 1$.

Weak factorization for the Hardy space H^1 has also been studied in the polydisc, where there is no atomic decomposition. It was announced in a celebrated paper [29] of Ferguson and Lacey but at this moment there is a gap in their proof as proved in [44]. It is quite curious that the weak characterization of H^1, if true, implies, again on the bidisc, a weak factorization of H^p by products of functions in H^{2p} when $p < 3/2$ (unpublished manuscript of Bonami, Pott, Sehba and Wick).

The paper of Coifman, Rochberg and Weiss finishes with the weak factorization of Bergman spaces. This is by now also completely classical and has been generalized in many contexts. The classical method, now, (see for instance [79]) goes through the atomic decomposition that has been developed by Coifman and Rochberg [16] a little later: for $p > 0$ given, every function in $f \in A^p(\mathbb{B}_n)$ may be written as

$$f(z) = \sum_j \lambda_j \frac{d_j(1 - |w_j|)^N}{(1 - \langle z, w_j\rangle)^{n+N+1}}, \qquad \sum |\lambda_j|^p \simeq \|f\|^p_{A^p},$$

where the points w_j do not depend of the function f but constitute what is called an $\eta-$net: they are at distance at most η for the Bergman distance, but balls of radius $\eta/2$ centered at those points are disjoint while balls of radius $2r$ are almost disjoint. The quantities that appear here, $\frac{\omega_j(1-|w_j|)^N}{(1-\langle z, w_j\rangle)^{n+N+1}}$, are the values at (z, w_j) of the Bergman kernel when the Lebesgue measure dV is replaced by the weighted measure $(1 - |z|)^N dV(z)$. The constant d_j is a normalization factor so that its A^p-norm is 1. One has not only the atomic decomposition, but the weak factorization of Bergman spaces for $p \leq 1$. Generalizations are valid in the same context as for Hardy spaces, as well as estimates for related Hankel operators, now defined as $H_b(f) = P_B(b\overline{f})$. But one can go much further in many directions.

First, the fact that such decompositions are valid for $p > 1$ and a tricky use of Rademacher coefficients has allowed Pau and Zhao [68] to get estimates with loss for the Hankel operator, and so to obtain weak factorization of Bergman spaces of $A^p(\mathbb{B}^n)$ for all values of p.

Secondly, the validity of atomic decomposition and weak factorization of Bergman spaces can be proved far beyond the contexts for which one can conclude for Hardy spaces, as outlined in [16]: one can find such decompositions in tube domains over symmetric cones, for instance, but with restrictions that are due to the lack of validity of L^p inequalities for the Bergman projection for all $p > 1$. In particular, if $\Omega = \mathbb{R}^n + i\Gamma$, where Γ is a symmetric cone (think of the forward light cone $y_n > \sqrt{y_1^2 + \cdots + y_n^2}$), results related to the Bergman space may be adapted (see [64]), but with new constraints that are linked to the validity of L^p inequalities for the Bergman projection. The constraints for these L^p inequalities given in [6] are the best possible in the case of the forward light cone as a consequence of the decoupling inequalities of Bourgain and Demeter [11].

We are then far from the geometry of spaces of homogeneous type and singular integrals, but all this finds its origin in the seminal papers written from the years 1970 by Guido and those researchers that gathered around him.

1.4 Littlewood-Paley Decompositions and Wavelets

1.4.1 The Context

The tools based on covering arguments that only need balls allow, in principle, to forget about algebraic contents to focus more on geometry or distributions of points. Still, familiar tool boxes such as convolution and the Fourier transform need to be replaced by something else.

First one needs approximations to the identity in relation to scales replacing usual mollifiers. Second to develop a rich function space theory and the action of operators on them (such as Calderón-Zygmund operators), one needs Littlewood-Paley decompositions. It took some time to develop those tools in the optimal generality offered by the definition of a space of homogeneous type, that is, without extra hypotheses. The aim of this section is to explain the evolution of the ideas and a solution. We use material from Auscher-Hytönen [3] and also the introduction of Han-Li-Ward [38]. The latter reference contains a rich bibliography as well.

Recall that the definition of a space of homogeneous type involves a set X equipped with a quasi-distance d and a positive doubling Borel measure μ (Borel measure is defined in several ways in the literature: either it is a measure on the Borel σ-algebra and one works only with Borel measurable functions or it is an outer measure for which Borel sets are measurable (in the sense of Carathéodory): in the latter case, we add the regularity property that every measurable set is contained in a Borel set with same measure). As said, extra hypotheses had to be imposed in the proofs. However, it seemed that this would not limit the range of applicability thanks to an article of Macías-Segovia that describes the structure of spaces of homogeneous type [53]. This explains why removing these extra conditions remained unexplored for some time.

A popular assumption is that the volume of a ball $B(x,r) = \{y \in X : d(x,y) < r\}$ is comparable to a power r^α of its radius, uniformly in its center (this is called the Ahlfors-David regularity property for the measure). Another one is that the volume of a ball $B(x, Cr)$ exceeds $(1+\varepsilon)$ times the volume of $B(x,r)$ for some constants $1 < C < \infty$, $\varepsilon > 0$ (this is called the reverse doubling property). One or the other are satisfied in many cases. Nevertheless, if one wants one or the other for all $0 < r$ less than a fraction of the diameter of X, this excludes atomic measures, and for example $\mathbb{Z}$ equipped with the induced absolute value and counting measure.