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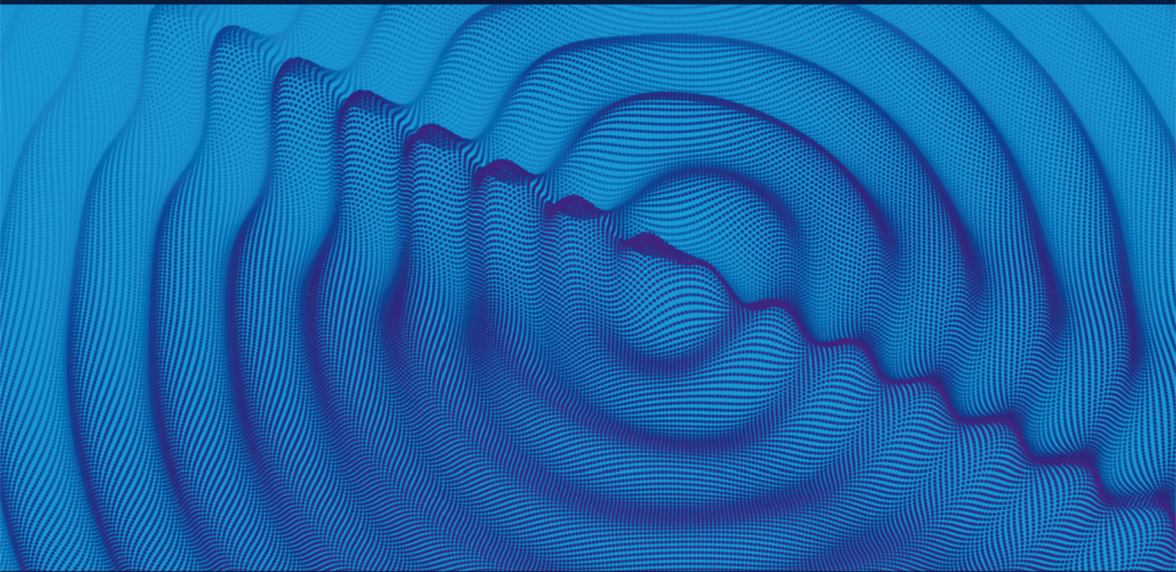
WAVES AND SCATTERING SET



Volume 3

Advanced Studies in the Mathematical Theory of Scattering

Jean-Michel L. Bernard

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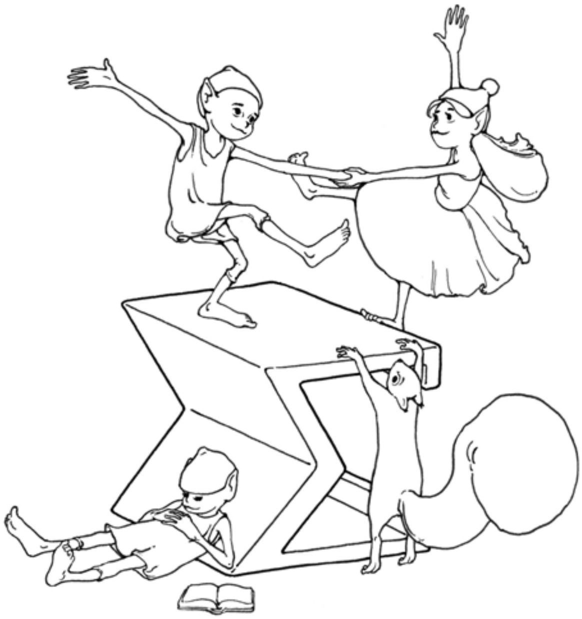
Advanced Studies in the Mathematical

Theory of Scattering

Dedicated to my wife



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coordinated by
Jean-Michel L. Bernard

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First published 2024 in Great Britain and the United States by ISTE Ltd and John Wiley & Sons, Inc.

ISTE Ltd
27-37 St George's Road
London SW19 4EU
UK

www.iste.co.uk

John Wiley & Sons, Inc.
111 River Street
Hoboken, NJ 07030
USA

www.wiley.com

Library of Congress Control Number: 2024940000

British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
ISBN 978-1-78630-857-3

Contents

Introduction

This book is a collection of independent mathematical studies, describing the analytical reduction of complex generic problems in the theory of scattering and propagation of electromagnetic waves in the presence of imperfectly conducting objects. Their subjects are as follows:

– a global method for the scattering by a multimode plane;

– diffraction by an impedance curved wedge;

– scattering by impedance polygons;

– advanced properties of spectral functions in frequency and time domains;

– bianisotropic media and related coupling expressions;

– exact and asymptotic reductions of surface radiation integrals.

Each of our approaches can be qualified as analytical, when it leads to exact explicit expressions, or, as semi-analytical, when it drastically reduces the mathematical complexity of studied problems. Therefore, they can be used in mathematical physics and engineering, to analyze and model, as well as in applied mathematics, to calculate for a low computational cost, the scattered fields in electromagnetism.

All of these works derive from original methods initiated in our publications that we here detail, develop and extend.

The first chapter is devoted to original exact expressions of the diffraction by a multilayered plane that can be partly composed of metamaterials. In whole generality, we then determine the fields as depending on potentials attached to arbitrary passive or active modes whose combination will give the passivity of the complete system. Our expressions directly take account of primary sources composed of electric and magnetic dipoles with arbitrary orientations, and profit of a novel exact development of incomplete Bessel function as an exact series of error functions. This latter characteristic permits a complete uniform analysis for arbitrary complex parameters, contrary to previous known results with error functions that were only approximations. Exact and complete asymptotics (at any order) are described, allowing us to particularly analyze the contribution of guided waves (forward and backward) at any distance.

The second chapter concerns the diffraction of an impedance wedge with curved faces of arbitrary angle that supports distinct surface boundary conditions of impedance type. We then distinguish the domains above and below the tangent planes at the edge. Our method permits an asymptotic evaluation at arbitrary order of curvatures of both faces for arbitrary passive impedance parameters. The uniformity at the crossing of the tangent plane is a characteristic remaining at arbitrary order, permitting us to analyze reflected, guided waves, edge-diffracted waves, but also waves originating from the edge that creep along the faces (creeping waves when faces are convex).

The diffraction in free space of a imperfectly conducting polygons (finite or with semi-infinite faces) is a particularly delicate problem that we study in third chapter, using Sommerfeld–Maliuzhinets integral representation of fields in a novel manner to rigorously consider several discontinuities, without any approximations. Indeed, contrary to common methods which consider large facets to admit asymptotic coupling between edges, we consider a rigorous development valid for arbitrary dimensions of facets, by establishing novel spectral equations that we can solve exactly or asymptotically, from a novel analysis of properties of spectral functions. This is particularly permitted by using their single-face representations, which is perfectly adapted to directly consider boundary conditions on faces with piecewise smooth geometries, as in polygonal cases.

Chapter 4 explores spectral functions in Sommerfeld–Maliuzhinets integral representation, their properties in the complex plane and new developments of them and special functions, attached to the resolution of multiple problems concerning the diffraction by a wedge with impedance boundaries conditions (passive or active). By beginning the study in frequency domain, we also analyze the representation of fields in time domain, in particular for an efficient explicit expression of causality in the case of dispersive (not constant relatively to frequency) multimode faces.

The fifth chapter analyzes the coupling influence between two imperfectly conducting objects in presence of a third one, all constituted by bianisotropic media. After beginning with developments of integral equalities, in particular, a generalized reciprocity one, we derive different properties of fields that will permit a complete analysis of coupling influences. We then give an example of application for an efficient and simple numerical post-process suppression of the influence of one object, that is, its direct but also its coupling contributions, on a second object.

We conclude this book with the determination of explicit contour integral expressions for an efficient evaluation of surface radiation integrals at arbitrary distance. This reduction concerns radiation of plane or curved plates of arbitrary contours, when the fields, highly oscillatory or not, are analytically defined on them, which is particularly the case for physical optics radiation surface integrals, when the surface fields are defined in closed form from geometrical optics.

1

A Global Method for the Scattering by a Multimode Plane with Arbitrary Primary Sources and Complete Series with Error Functions

1.1. Introduction

In [1], we considered the field scattered by an arbitrary impedance plane in electromagnetism, and we here exploit this formalism to analyze the scattering by a structure composed of several homogeneous planar layers, with isotropy or uniaxial anisotropy, illuminated by arbitrary bounded sources. In this study, the plane is supposed to be either grounded, that is, a multilayer backed by an impedance plane, or not grounded, that is, a multilayer slab in free space; this will lead us to generalize our previous approach for a multilayer given in [2].

The field scattered of such structures is usually given by its plane wave expansion (Fourier representation) [3]-[6], which presents the particularity to have reflection coefficients that are meromorphic functions. Each one can be then modeled as a rational function with a set of N simple poles $\{-g_j\}_{j=1,..,N}$, which permits us to assume a multimode boundary condition of order N [2].

The Fourier expansion is well adapted in far field or for plane wave illuminations, but is not suitable for an analysis at any distance or for complex incident waves. Even when double Fourier integrals are reduced to single Fourier–Bessel integrals, calculation is lengthy and delicate because of functions in the integral that remain highly oscillating and, most often in literature [3]–[9], analytic expansions are not strictly convergent but asymptotic. Besides, an additional difficulty comes from that, and in multimode case, we have to take into account that the constants g_j can have real parts of any sign, which signifies that passive but also active modes are present, even if the complete system is strictly passive.

In this frame, after expanding potentials into a combination of Fourier–Bessel integrals depending on each g_j, we are led to transform them to derive a more efficient integral representation, which is able to take account of active modes. Among other specificities, the definition of a parameter ϵ, attached to each pole, is then particularly important to permit complete exact and asymptotic series with error functions. These series allow us to exhibit guiding waves terms near and far from the sources above the multilayer, generalizing [1] and refining [2].

Otherwise, our approach, as in [1], uses a new representation of potentials for the incident field, which possesses the originality to directly consider arbitrarily oriented electric and magnetic primary currents sources. Thus, we have no more to solve separately the problem for vertical or horizontal dipolar source as commonly done in the literature for passive impedance planes [7]–[14], isotropic or uniaxial slabs [15]–[17] or multilayers [3]–[6], [18]–[22]. In practice, the analytic method so developed can be applied in whole generality to various problems, in particular for the determination of coupling between antennas above an imperfectly reflective plane, or for the calculus of Green's functions for planar lines printed on a multilayer.

This chapter is organized as follows. In section 1.2, we give a discussion on the representation of the field with potentials, on the boundary conditions and on the positions of g_j in the complex plane when metamaterials can be present. Next, we give a global expression of potentials attached to the fields radiated by arbitrary bounded sources in free space in section 1.3, and above the multilayer in section 1.4, which we develop and expand for arbitrarily oriented dipoles in section 1.5. In sections 1.6 and 1.7, we then detail a compact expression of the special function involved in the potentials attached to each mode, intimately depending on a parameter ϵ that is necessary to correctly take account of active modes. The definition of ϵ will be useful for the development of exact (section 1.6) and asymptotic (section 1.7) expansions with error functions for arbitrary cases, allowing in particular a general analysis of guided waves in section 1.8, including backward waves, near and far from the sources.

1.2. Potentials, reflection coefficients and multimode boundary conditions

1.2.1. *Fields and potentials*

We consider the scattering by an imperfectly reflective plane when it is illuminated by the field radiated by a bounded primary source, which is composed of arbitrary electrical and magnetic currents J and M (see Figure 1.1). In the space of points r with Cartesian coordinates (x, y, z), this plane is defined by $z = 0$. A harmonic time dependence $e^{i\omega t}$, from now on assumed, is suppressed throughout. The constants ε_0 and μ_0 are, respectively, the permittivity and the permeability of the free space above the plane, and $k_0 = \omega(\mu_0\varepsilon_0)^{1/2}$ is its wavenumber. Each component of the scattered field is assumed to be regular in the domain $z > 0$, and $O(e^{-\gamma|r|})$ with $\gamma > 0$ as $|r| \to \infty$ when $|\arg(ik_0)| < \pi/2$ (note: no loss is a limit case).

The electric field E and the magnetic field H above the multilayer, following Harrington [23, p. 131] (see also Jones [3, p. 19]), can be written with two scalar potentials $\mathcal{E}$ and $\mathcal{H}$, as follows:

$$E = -ik_0\text{curl}(\mathcal{H}\,\widehat{z}) + (\text{grad}(\text{div}(.)) + k_0^2)(\mathcal{E}\widehat{z}),$$
$$Z_0 H = ik_0\text{curl}(\mathcal{E}\,\widehat{z}) + (\text{grad}(\text{div}(.)) + k_0^2)(\mathcal{H}\,\widehat{z}), \tag{1.1}$$

where the Helmhotz equations $(\Delta + k_0^2)\mathcal{E} = 0$ and $(\Delta + k_0^2)\mathcal{H} = 0$ are verified outside the sources, $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$. Thereafter, we denote $(\mathcal{E}_i, \mathcal{H}_i)$ and $(\mathcal{E}_s,\mathcal{H}_s)$ the potentials corresponding, respectively, to the incident field (incoming wave) (E^i, H^i) and the scattered field (outgoing wave) (E^s, H^s), and we write (1.1) in the compact form:

$$(E{,}Z_0H) = (\mathcal{L}_1(\widehat{z}\mathcal{E}, \widehat{z}\mathcal{H}), \mathcal{L}_2(\widehat{z}\mathcal{E}, \widehat{z}\mathcal{H})) = \mathcal{L}(\widehat{z}\mathcal{E}, \widehat{z}\mathcal{H}),$$
$$\mathcal{L}_1(u{,}v) = ((\text{grad}(\text{div}(.)) + k_0^2)(u) - ik_0\text{curl}(v)),$$
$$\mathcal{L}_2(u{,}v) = (ik_0\text{curl}(u) + (\text{grad}(\text{div}(.)) + k_0^2)(v)). \tag{1.2}$$

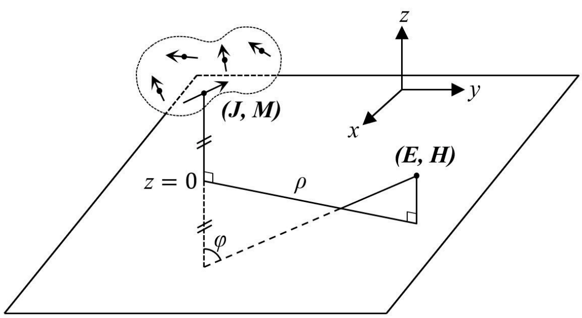

Figure 1.1. *Geometry: sources (J, M) and observation point above the plane $z = 0$*

1.2.2. *Multimode boundary conditions for a multilayer backed by an impedance plane*

Let us consider a multilayer plane composed of uniform isotropic (or z-axial anisotropic) layers. Any plane wave (E^i, H^i), incident at angle β with the normal $\widehat{z}$, is then scattered as a reflected plane wave (E^s, H^s) that satisfies $E_z^s|_{z=0^+} = R_{e,e}E_z^i|_{z=0^-}$ (i.e. $H_\perp^s|_{z=0^+} = R_{e,e}H_\perp^i|_{z=0^-}$ in TM polarization), and $H_z^s|_{z=0^+} = R_{h,h}H_z^i|_{z=0^-}$ (i.e. $E_\perp^s|_{z=0^+} = R_{h,h}E_\perp^i|_{z=0^-}$ in TE polarization) [4], [5], [21] (see details in appendix A). If the multilayer is backed by a constant impedance plane, the reflection coefficients $R_{e,e}$ and $R_{h,h}$ are meromorphic functions of the variable $\cos\beta$, which we can model as rational functions [2] with simple poles, following:

$$R_{e,e}(\beta) = \prod_{j=1}^{N_e} \frac{\cos\beta - g_j^e}{\cos\beta + g_j^e},\ R_{h,h}(\beta) = \prod_{j=1}^{N_h} \frac{\cos\beta - g_j^h}{\cos\beta + g_j^h}, \tag{1.3}$$

for which we have the basic equalities (without superscripts e and h):

$$\prod_{j=1}^{N} \frac{\cos\beta - g_j}{\cos\beta + g_j} = (\pm 1)^N + \sum_{j=1}^{N} a_j \frac{\left(\frac{-\cos\beta}{g_j}\right)^{\frac{1\mp 1}{2}}}{\cos\beta + g_j},$$

$$\frac{a_j}{2g_j} = -\prod_{i\neq j}^{N} \frac{g_j + g_i}{g_j - g_i}, (-1)^N - 1 = \sum_{j=1}^{N} \frac{a_j}{g_j}, \tag{1.4}$$

where $N \geq 1$, and $\prod_{i\neq j}^{1} \frac{g_j+g_i}{g_j-g_i} \equiv 1$ for $N = 1$. The constants $(\pm 1)^N$ refer to limit values when $|\cos\beta|^{\pm 1} \to \infty$.

The $g_j^{e(,h)}$ are constants attached to complex modes with Im$(g_j^{e(,h)}) \neq 0$, passive when Re$(g_j^{e(,h)}) \geq 0$ or active when Re$(g_j^{e(,h)}) < 0$, and ordered such that $|g_{j+1}^{e(,h)}| \geq |g_j^{e(,h)}|$, while, as considered in [26]–[28], we assume that:

when (1.3) applies, N_e and N_h and thus N are positive odd numbers, (1.5)

(note: this restriction on $N_{e(,h)}$ will be removed in the more general case of extended boundary conditions). Considering plane waves representation of fields (see appendix A), we can then write a multimode boundary conditions at $z = 0^+$ [2]:

$$\prod_{j=1}^{N_e}\left(\frac{\partial}{\partial z} - ik_0 g_j^e\right) E_z^s(z)|_{0^+} = \prod_{j=1}^{N_e}\left(\frac{\partial}{\partial z} + ik_0 g_j^e\right) E_z^i(-z)|_{0^+},$$

$$\prod_{j=1}^{N_h}\left(\frac{\partial}{\partial z} - ik_0 g_j^h\right) H_z^s(z)|_{0^+} = \prod_{j=1}^{N_h}\left(\frac{\partial}{\partial z} + ik_0 g_j^h\right) H_z^i(-z)|_{0^+}. \tag{1.6}$$

From the symmetry at normal incidence, the condition $R_{h,h}(0) = -R_{e,e}(0)$ must apply, which leads us to write:

$$\prod_{j=1}^{N_h} \frac{\pm 1 - g_j^h}{\pm 1 + g_j^h} = -\prod_{j=1}^{N_e} \frac{\pm 1 - g_j^e}{\pm 1 + g_j^e}, \tag{1.7}$$

and implies that $R_{h,h}(\pi) = -R_{e,e}(\pi)$. The condition (1.7) has crucial importance to avoid non-physical behaviors of fields derived from potentials, as examined further in this paper. Besides, the reader will notice that (1.7) implies $g_1^e = 1/g_1^h$ when $N_{e(,h)} = 1$, as well known for monomode (impedance) boundaries conditions [1]. The numbers $N_{e(,h)}$ correspond to truncated infinite products, where the less significant $g^{e(,h)}$ have been neglected, while some $g^{e(,h)}$ have to be modified so that $R_{h,h}(0) = -R_{e,e}(0)$ remains.

Considering (1.1), we can use:

$$E_z = \frac{\partial^2 \mathcal{E}}{\partial z^2} + k_0^2 \mathcal{E},\ Z_0 H_z = \frac{\partial^2 \mathcal{H}}{\partial z^2} + k_0^2 \mathcal{H}, \tag{1.8}$$

in (1.6), and we are led to search scattered potentials $\mathcal{E}_s$ and $\mathcal{H}_s$, satisfying the Helmholtz equation as $z > 0$, regular and exponentially vanishing as $z \to \infty$ when $|\arg(ik_0)| < \pi/2$, that verify as $z > 0$:

$$\prod_{i=1}^{N_e} \Big(\frac{\partial}{\partial z} - ik_0 g_j^e\Big) \mathcal{E}_s(z) = \prod_{j=1}^{N_e} \Big(\frac{\partial}{\partial z} + ik_0 g_j^e\Big) \mathcal{E}_i(-z),$$

$$\prod_{j=1}^{N_h} \Big(\frac{\partial}{\partial z} - ik_0 g_j^h\Big) \mathcal{H}_s(z) = \prod_{j=1}^{N_h} \Big(\frac{\partial}{\partial z} + ik_0 g_j^h\Big) \mathcal{H}_i(-z), \tag{1.9}$$

where $\mathcal{E}_i$ and $\mathcal{H}_i$ potentials are attached to radiation of arbitrary primary sources.

1.2.3. *Extended multimode boundary conditions*

More generally, we can consider an extended form when we want to include the case of a multilayer slab in free space, which is composed of isotropic [4]-[5] (or z-axial anisotropic [21]) layers. The reflection coefficients $R_{e,e}$ and $R_{h,h}$ remain meromorphic functions of $\cos\beta$, but we now model them in a more general form, following:

$$R_{e,e}(\beta) = R_0^e \frac{\prod_{j=1}^{N'_e} (\cos\beta - g_j^{e\prime})}{\prod_{j=1}^{N_e} (\cos\beta + g_j^e)},\ R_{h,h}(\beta) = R_0^h \frac{\prod_{j=1}^{N'_h} (\cos\beta - g_j^{h\prime})}{\prod_{j=1}^{N_h} (\cos\beta + g_j^h)}, \tag{1.10}$$

with simple complex poles $-g_j^{e(,h)}$, $N'_{e(,h)} \le N_{e(,h)}$, constants R_0^e and R_0^h (this time for odd or even $N_{e(h)}$), for which we notice the basic equalities (without supercripts e and h):

$$R_0 \frac{\prod_{j=1}^{N'}(\cos\beta - g'_j)}{\prod_{j=1}^{N}(\cos\beta + g_j)} - a_{0\tau} = \sum_{j=1}^{N} a_j \frac{(\frac{-\cos\beta}{g_j})^{\frac{1-\tau}{2}}}{\cos\beta + g_j} = \sum_{j=1}^{N} \frac{\tau \frac{a_j}{g_j}}{(\frac{\cos\beta}{g_j})^{\tau} + 1},$$

$$a_{j\neq 0} = R_0 \frac{\prod_{i=1}^{N'}(-g'_i - g_j)}{\prod_{i\neq j}^{N}(g_i - g_j)}, \left|{}^{a_{0+}=\left|{}^{=R_0 \text{ if } N'=N}_{=0 \text{ if } N'<N}\right.}_{a_{0-}=R_0(\prod_{i=1}^{N'} -g'_i)/(\prod_{i=1}^{N} g_i)}\right., \tag{1.11}$$

where the constants $a_{0\tau}$ refer to limit values when $|\cos\beta|^{\tau} \to \infty$ with $\tau = +1$ or -1. We have $N \geq N'$, $N \geq 1$, $N' \geq 0$, and we can let $\prod_{i\neq j}^{1}(g_i - g_j) \equiv 1$ for $N = 1$ and $\prod_{j=1}^{0}(\cos\beta - g'_j) \equiv 1$ for $N' = 0$.

As in the previous section, we consider that $R_{h,h}(0) = -R_{e,e}(0)$, and assume additionally that $R_{h,h}(\pi) = -R_{e,e}(\pi)$. This implies, after using (1.11) when $\cos\beta = \pm 1$:

$$\sum_{j=1}^{N_e} \frac{\tau_e \frac{a_j^e}{g_j^e}(g_j^e)^{\tau_e}}{(g_j^e)^{\tau_e} \pm 1} + \sum_{j=1}^{N_h} \frac{\tau_h \frac{a_j^h}{g_j^h}(g_j^h)^{\tau_h}}{(g_j^h)^{\tau_h} \pm 1} = -(a_{0\tau_e} + a_{0\tau_h}), \tag{1.12}$$

with $\tau_{e(,h)} = +1$ or -1. The condition (1.12), as previously noticed for (1.7), has a crucial importance to avoid non-physical behaviors of fields derived from potentials.

Considering (1.10) and plane waves representation of fields (see appendix A), we can write:

$$\prod_{j=1}^{N_e}\Big(\frac{-\partial}{ik_0\partial z} + g_j^e\Big) E_z^s(z)|_{0^+} = R_0^e \prod_{j=1}^{N'_e}\Big(\frac{-\partial}{ik_0\partial z} - g_j^{e'}\Big) E_z^i(-z)|_{0^+},$$

$$\prod_{j=1}^{N_h}\Big(\frac{-\partial}{ik_0\partial z} + g_j^h\Big) H_z^s(z)|_{0^+} = R_0^h \prod_{j=1}^{N'_h}\Big(\frac{-\partial}{ik_0\partial z} - g_j^{h'}\Big) H_z^i(-z)|_{0^+}, \tag{1.13}$$

which applies for any primary sources that illuminates the plane. Using (1.8), we can solve (1.13) with $(\mathcal{E}_s, \mathcal{H}_s)$ that satisfy as $z > 0$:

$$\prod_{i=1}^{N_e}\Big(\frac{-\partial}{ik_0\partial z} + g_j^e\Big)\mathcal{E}_s(z) = R_0^e \prod_{j=1}^{N'_e}\Big(\frac{-\partial}{ik_0\partial z} - g_j^{e'}\Big)\mathcal{E}_i(-z),$$

$$\prod_{j=1}^{N_h}\Big(\frac{-\partial}{ik_0\partial z} + g_j^h\Big)\mathcal{H}_s(z) = R_0^h \prod_{j=1}^{N'_h}\Big(\frac{-\partial}{ik_0\partial z} - g_j^{h'}\Big)\mathcal{H}_i(-z). \tag{1.14}$$

NOTE 1.1.– For z-uniaxial chiral layers [17], non-diagonal terms in (1.100) (see appendix A on plane waves representation of fields) do not vanish, and

$$\begin{pmatrix} \mathcal{P}_{N_e}^{e}(\frac{\partial}{\partial z})\mathcal{E}_s(z) \\ \mathcal{P}_{N_h}^{h}(\frac{\partial.}{\partial z})\mathcal{H}_s(z) \end{pmatrix} = \begin{pmatrix} \mathcal{Q}_{N'}^{e,e}(\frac{\partial.}{\partial z}) & \mathcal{Q}_{N'}^{e,h}(\frac{\partial.}{\partial z}) \\ \mathcal{Q}_{N'}^{h,e}(\frac{\partial.}{\partial z}) & \mathcal{Q}_{N'}^{h,h}(\frac{\partial.}{\partial z}) \end{pmatrix} \begin{pmatrix} \mathcal{E}_i(-z) \\ \mathcal{H}_i(-z) \end{pmatrix}, \tag{1.15}$$

applies with polynomials $\mathcal{P}^{e(,h)}$ and $\mathcal{Q}^{e(,h)}$. By linearity, it can be solved by addition of terms similar to ones derived for the solution of equations (1.14) (note: in this case, adjoint characteristics have to be considered to apply the generalized reciprocity principle).

NOTE 1.2.– Letting $R = (\cos\beta - \mathcal{Z}(\cos\beta))/(\cos\beta + \mathcal{Z}(\cos\beta))$, $\mathcal{Z}$ is a rational function which is even as (1.3) applies [4], [5], [21] (see also Appendix A of [2]).

NOTE 1.3.– Considering (1.12), we can assume that, when a $g_j^{e(,h)} \to \mp 1$, it exists a $g_m^{h(,e)}$ such that $a_j^{e(,h)}(g_j^{e(,h)} \pm 1)^{-1} + a_m^{h(,e)}(g_m^{h(,e)} \pm 1)^{-1}$ is bounded.

1.2.4. *Properties of $R_{e,e}$ and $R_{h,h}$, and consequences on the $g_j^{e(,h)}$*

The multilayer is characterized by permittivity $\varepsilon(z)$ and permeability $\mu(z)$ as z varies from $z = 0$ to $-h_L$, which corresponds to a set of complex ε_j and μ_j specified in each layer $-h_j \leq z \leq -h_{j-1}$ with $1 \leq j \leq L$, $h_0 = 0$. Considering the hypotheses for (1.3) and (1.10), the last numbered layer shall be impenetrable or be bounded by the free space at $z = -h_L$. The plane wave spectrum of fields follows certain rules (see appendix A), and the reflection coefficients, considered for an harmonic time dependence $e^{i\omega t}$, are analytic functions, called thereafter R to simplify, which have some common elementary properties:

(a) R is an analytic function of the complex variables k_0, $\cos\beta$, $i\omega\varepsilon$, $i\omega\mu$ and layers depths. In the domain of passivity $\Omega_{\varepsilon,\mu}$ with $\text{Re}(i\omega\varepsilon) > 0$ and $\text{Re}(i\omega\mu) > 0$, R has no singularity when β varies from 0 to $i\infty + \arg(ik_0)$ as $\text{Re}(ik_0 \sin\beta) = 0$, and at infinity. This regularity applies in the whole domain with $|\arg(i\omega)| \leq \pi/2$ from the causality principle. Its highest modulus is obtained for real ω from the maximum modulus principle, while, from (1.104), $|R| \leq 1$ when k_0 is real and $0 \leq \beta \leq \pi/2$.

(b) When $i\omega\varepsilon$ and $i\omega\mu$ are purely real positive (perfect lossy case), the multilayer is purely resistive, and, in these circumstances, R is real when k_0 and β are real.

(c) From (a) and (b), we can apply the Schwarz reflection principle (or edge-of-the-wedge theorem) [24, sect.5] (see also [25]), and deduce that R satisfies in $\Omega_{\varepsilon,\mu}$:

$$R((k_0)^*, (\cos\beta)^*, (i\omega\varepsilon)^*, (i\omega\mu)^*) = (R(k_0, \cos\beta, i\omega\varepsilon, i\omega\mu))^*. \tag{1.16}$$

(d) Using $(i\varepsilon)^* = i(-\varepsilon^*)$ and $(i\mu)^* = i(-\mu^*)$ in (1.16), we note, as we let change ε and μ for their "anticonjugate" $-\varepsilon^*$ and $-\mu^*$ (possible in practice with metamaterials, or plasmas), that, for real ω and k_0:

$$\text{every pole } \cos\beta = (-g_j^{e(,h)}) \text{ of } R \text{ in } (1.3) \text{ and } (1.10) \text{ for } \varepsilon \text{ and } \mu,$$

$$\text{gives us its conjugate } (-g_j^{e(,h)})^* \text{ as a pole of } R \text{ for } -\varepsilon^* \text{ and } -\mu^*. \tag{1.17}$$

(e) The set $\{g_j^{e(,h)}\}$ has elements with positive and negative real parts [26]-[28], and for the class of problems including metamaterials, the total set to be considered, for real ω and k_0, is $\{g_j^{e(,h)}\} \cup \{(g_j^{e(,h)})^*\}$; this leaves no quarter-plane of the complex plane empty and explains why we have such a wide range of properties when multilayers include metamaterials.

Let us now consider the examples of two lossless systems for real positive ω, where, in the first case, ε and μ are with purely positive real values, and, in the second case, with negative real values. As defined in (d), both cases then belong, respectively, to anticonjugate classes. Pure imaginary $g_j^{e(,h)}$ are generally of finite number. From the analysis in [19] (respectively in [16]), the complex $g_j^{e(,h)}$ with non-null real part are of infinite number, and they are complex numbers with negative (respectively, positive) imaginary parts, associated with improper (respectively, proper) modes. Therefore, the domains of the $g_j^{e(,h)}$ for both cases of these examples are conjugated with each other, which plainly illustrates and confirms the property (d).

NOTE 1.4.– Considering appendix A, the conditions (a)–(e) also apply for anisotropic multilayers, when ε and μ are tensors.

NOTE 1.5.– $k_j = \omega(\mu_j\varepsilon_j)^{1/2}$, with $(\mu_j\varepsilon_j)^{1/2} = \sqrt{\mu_j}\sqrt{\varepsilon_j}$, can be $\neq \omega\sqrt{\mu_j\varepsilon_j}$, and thus, a term like $((\mu_j\varepsilon_j)^{1/2}/\varepsilon_j)\tan(\omega(\mu_j\varepsilon_j)^{1/2}d)$ can be $\neq \sqrt{\mu_j/\varepsilon_j}\tan(\omega\sqrt{\mu_j\varepsilon_j}d)$.

1.3. Incident potentials $(\mathcal{E}_i,\mathcal{H}_i)$ for arbitrary bounded primary sources

To solve problems with conditions (1.9) or (1.14), we need a correct explicit expression of $(\mathcal{E}_i, \mathcal{H}_i)$. We begin by considering the incident field (E^i, H^i) at point r of coordinates (x, y, z), radiated by arbitrary electric and magnetic bounded sources J and M in free space [3]:

$$E^i = \text{curl}(G * M) + \frac{i}{\omega\varepsilon_0}(\text{grad}(\text{div}(.)) + k_0^2)(G * J),$$
$$H^i = -\text{curl}(G * J) + \frac{i}{\omega\mu_0}(\text{grad}(\text{div}(.)) + k_0^2)(G * M), \tag{1.18}$$

where $G(r) = -\frac{e^{-ik_0|r(x,y,z)|}}{4\pi|r(x,y,z)|}$ with $|r| = \sqrt{x^2+y^2+z^2}$ verifies $(\Delta + k_0^2)G = \delta$, $*$ is the convolution product, and J and M are generalized functions [29].

Considering arbitrary J and M in the domain $\pm z > 0$, the potentials $(\mathcal{E}_i, \mathcal{H}_i)$ in the representation of fields E^i and H^i with (1.1), which satisfy the Helmholtz equation as $\mp z > 0$ and exponentially vanish as $\mp z \to \infty$ when $|\arg(ik_0)| < \pi/2$ can be written when $\mp z > 0$ [1]:

$$(\mathcal{E}_i, \mathcal{H}_i)(x, y, z) = [\frac{\widehat{z}}{8\pi k_0^2}.(\mathcal{L}(Z_0 J, M) * \mathcal{W})](x, y, z), \tag{1.19}$$

where $(A, B) * C \equiv (A * C, B * C)$, $\mathcal{L}$ is defined as in (1.2):

$$\begin{aligned}
&\mathcal{L}(Z_0 J, M) = (\mathcal{L}_1(Z_0 J, M), \mathcal{L}_2(Z_0 J, M)),\\
&\mathcal{L}_1(u,v) = ((\text{grad}(\text{div}(.)) + k_0^2)(u) - ik_0\text{curl}(v)),\\
&\mathcal{L}_2(u,v) = (ik_0\text{curl}(u) + (\text{grad}(\text{div}(.)) + k_0^2)(v)),
\end{aligned} \tag{1.20}$$

and $\mathcal{W}$ has the remarkable compact form [1]–[2]:

$$\begin{aligned}
\mathcal{W}(r) &= e^{ik_0|z|}E_1(ik_0(|r|+|z|))\\
&+ e^{-ik_0|z|}(E_1(ik_0(|r|-|z|)) + 2\ln\rho),
\end{aligned} \tag{1.21}$$

with $\rho = \sqrt{x^2 + y^2}$, E_1 being the exponential integral function [30], which verifies, as $\mp z > 0$:

$$\Big(\frac{\partial^2}{\partial z^2} + k_0^2\Big)\frac{\mathcal{W}(r)}{8\pi i k_0} = G(r),\ (\Delta + k_0^2)\mathcal{W}(r) = 0. \tag{1.22}$$

NOTE 1.6.– It is worth noticing that we have $\Delta_{xy}\ln(\rho) = 2\pi\delta(x)\delta(y)$, which implies that $\ln\rho$ in $\mathcal{W}$ has no influence on the expression of the field, except by its singularity at $\rho = 0$.

NOTE 1.7.– Let us divide the primary sources J and M into the sources above and below a plane $z = z_1$, $J^{\pm}_{z_1} = U_s(\pm(z - z_1))J$ and $M^{\pm}_{z_1} = U_s(\pm(z - z_1))M$, where U_s is the unit step function, with $U_s(z) = (\text{sign}(z) + 1)/2$, $U_s(z) = 1$ as $z > 0$, $U_s(z) = 1/2$ as $z = 0$ and $U_s(z) = 0$ as $z < 0$. We can then write, for arbitrary observation point at $z = z_1$:

$$(E^i, Z_0 H^i)(x, y, z_1) = \sum_{\pm}(E^{\pm}, Z_0 H^{\pm})(x, y, z_1 \mp 0^{\pm}), \tag{1.23}$$

where $(E^{\pm}, H^{\pm})$ are the field radiated by the sources $J^{\pm}_{z_1}$ and $M^{\pm}_{z_1}$, following $(E^{\pm}, Z_0 H^{\pm}) = \mathcal{L}(\widehat{z}\mathcal{E}^{\pm}, \widehat{z}\mathcal{H}^{\pm})$ with $(\mathcal{E}^{\pm}, \mathcal{H}^{\pm}) = \frac{\widehat{z}}{8\pi k_0^2}.\mathcal{L}(Z_0 J^{\pm}_{z_1}, M^{\pm}_{z_1}) * \mathcal{W}$, from (1.2) and (1.19).

1.4. Scattered potentials $(\mathcal{E}_s, \mathcal{H}_s)$ for arbitrary primary sources

1.4.1. *A global expression of* $(\mathcal{E}_s, \mathcal{H}_s)$ *for a multimode plane*

Considering the potentials $(\mathcal{E}_i, \mathcal{H}_i)$ attached to the radiation of arbitrary sources from (1.19), we can now express the scalar potentials $\mathcal{E}_s$ and $\mathcal{H}_s$, which satisfy the multimode conditions (1.9) or (1.14), as $z \geq 0$. From the method developed in [1]–[2], we have for conditions (1.9) (for odd numbers $N_{e(h)}$):

$$\begin{aligned}
&\mathcal{E}_s(x,y,z) = \pm\mathcal{E}_i(x,y,-z)\\
&+\Big(\Big(\frac{\widehat{z}}{\omega\varepsilon_0}\frac{\mathrm{grad}(\mathrm{div}(J))+k_0^2J}{8\pi k_0}+\frac{\widehat{z}}{k}\frac{(-ik_0\,\mathrm{curl}(M))}{8\pi k_0}\Big)\\
&*\sum_{\epsilon'=-1,1}\Big((\prod_{j=1}^{N_e}\frac{\epsilon'+g_j^e}{\epsilon'-g_j^e}\mp 1)\mathcal{V}_{\epsilon'}+\sum_{j=1}^{N_e}\frac{\epsilon' a_j^e\mathcal{K}_{g_j^e}}{(g_j^e-\epsilon')}\Big)\Big)(x,y,-z),
\end{aligned}\tag{1.24}$$

and

$$\begin{aligned}
&\mathcal{H}_s(x,y,z) = \mp\mathcal{H}_i(x,y,-z)\\
&+\Big(\Big(\frac{\widehat{z}}{\omega\varepsilon_0}\frac{(ik_0\mathrm{curl}(J))}{8\pi k_0}+\frac{\widehat{z}}{k}\frac{(\mathrm{grad}(\mathrm{div}(M))+k_0^2M)}{8\pi k_0}\Big)\\
&*\sum_{\epsilon'=-1,1}\Big((\prod_{j=1}^{N_h}\frac{\epsilon'+g_j^h}{\epsilon'-g_j^h}\pm 1)\mathcal{V}_{\epsilon'}+\sum_{j=1}^{N_h}\frac{\epsilon' a_j^h\mathcal{K}_{g_j^h}}{(g_j^h-\epsilon')}\Big)\Big)(x,y,-z),
\end{aligned}\tag{1.25}$$

while we have, for extended conditions (1.14) (for odd or even numbers $N_{e(h)}$):

$$\begin{aligned}
&\mathcal{E}_s(x,y,z) = a_{0\tau_e}^e\mathcal{E}_i(x,y,-z)\\
&+\Big(\Big(\frac{\widehat{z}}{\omega\varepsilon_0}\frac{\mathrm{grad}(\mathrm{div}(J))+k_0^2J}{8\pi k_0}+\frac{\widehat{z}}{k}\frac{(-ik_0\,\mathrm{curl}(M))}{8\pi k_0}\Big)\\
&*\sum_{\epsilon'=-1,1}\Big(((R_{e,e}|_{\cos\beta=-\epsilon'})-a_{0\tau_e}^e)\mathcal{V}_{\epsilon'}+\sum_{j=1}^{N_e}\frac{\epsilon' a_j^e\mathcal{K}_{g_j^e}}{(g_j^e-\epsilon')}\Big)\Big)(x,y,-z),
\end{aligned}\tag{1.26}$$

and

$$\begin{aligned}
&\mathcal{H}_s(x,y,z) = a_{0\tau_h}^h\,\mathcal{H}_i(x,y,-z)\\
&+\Big(\Big(\frac{\widehat{z}}{\omega\varepsilon_0}\frac{(ik_0\,\mathrm{curl}(J))}{8\pi k_0}+\frac{\widehat{z}}{k}\frac{(\mathrm{grad}(\mathrm{div}(M))+k_0^2M)}{8\pi k_0}\Big)\\
&*\sum_{\epsilon'=-1,1}\Big(((R_{h,h}|_{\cos\beta=-\epsilon'})-a_{0\tau_h}^h)\mathcal{V}_{\epsilon'}+\sum_{j=1}^{N_h}\frac{\epsilon' a_j^h\mathcal{K}_{g_j^h}}{(g_j^h-\epsilon')}\Big)\Big)(x,y,-z),
\end{aligned}\tag{1.27}$$

where $\mathcal{V}_{\epsilon'}$, $\mathcal{K}_g$ are given by,

$$\begin{aligned}
&\mathcal{V}_{\epsilon'}(x,y,-z) = e^{\epsilon' ik_0z}(E_1(ik_0(|r|+\epsilon' z))+(1-\epsilon')\ln\rho),\\
&\mathcal{K}_g(x,y,-z) = e^{ik_0gz}\mathcal{J}_g(\rho,-z),
\end{aligned}\tag{1.28}$$