# WAVES SERIES



## Volume 3 Advanced Studies in the Mathematical Theory of Scattering

Jean-Michel L. Bernard





Advanced Studies in the Mathematical Theory of Scattering

Dedicated to my wife



Waves and Scattering Set

coordinated by Jean-Michel L. Bernard

Volume 3

### Advanced Studies in the Mathematical Theory of Scattering

Jean-Michel L. Bernard





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### Contents

Introduction	xv
Chapter 1. A Global Method for the Scattering by a Multimode Plane with Arbitrary Primary Sources and Complete Series with Error Functions	1
1.1. Introduction	1
boundary conditions       1.2.1. Fields and potentials         1.2.2. Multimode boundary conditions for a multilayer backed by an	2 2
impedance plane	4 5
1.2.4. Properties of $R_{e,e}$ and $R_{h,h}$ , and consequences on the $g_j^{e(,h)}$	7
1.3. Incident potentials $(\mathcal{E}_s, \mathcal{H}_s)$ for arbitrary bounded primary sources	8 9
1.4.1. A global expression of $(\mathcal{E}_s, \mathcal{H}_s)$ for a multimode plane 1.4.2. Reciprocity principle from our expressions	9 11
1.5. When primary sources J and M are arbitrarily oriented dipoles 1.5.1. Tensorial expressions of fields	12 12
1.5.2. Exact reduction of derivative orders after sums over $\epsilon'$ indices 1.6. Integral reduction of $\mathcal{J}_g$ and exact remarkable expansion with	15
error functions	18
as $ \operatorname{Re}(\theta_1)  \leq \pi/2$ 1.6.2. Exact remarkable expansion of $\mathcal{M}(a, b)$ with error functions	18 19
1.6.3. Exact expansion of $\mathcal{J}_g( ho,-z)$ for arbitrary parameters $\ldots$	22

1.7. Asymptotic expansion of $\mathcal{J}_q$ with complementary error functions	23
1.7.1. Asymptotics of $\mathcal{M}(a, b)$	23
1.7.2. Asymptotics of $\mathcal{J}_q$ for large $ k_0 \rho cos \theta_1 $	
or $ k_0 R(1+\sin(\theta_1\mp\varphi)) $	27
1.8. Forward and backward guided waves	29
1.8.1. Guided waves from asymptotics (1.92) and (1.94)	29
1.8.2. Guided-like waves from the exact expansion $(1.75)-(1.76)$	30
1.9. References	30
1.A. Appendix A: complex spectral properties	33
1.A.1. Wave spectrum and reflection coefficients	33
1 A 2. Complex Poynting vector and reflection coefficients	34
1 B Appendix B: some equalities derived from (1.4) and (1.11)	35
1 C Appendix C: other types of bounded sources	35
1 C 1 Bounded volume sources	35
1 C 2 Bounded surface sources	36
1 D Annendix D: some miscellaneous results on	50
T.D. Appendix D. some miscentateous results on $\mathcal{T}(a_{-} - \alpha)$ and $W_{-}(a)$	36
$\mathcal{J}_g(p,-z)$ and $\mathcal{W}_m(s)$	50
Chapter 2 Diffraction by an Impedance Curved Wedge with	
Arbitrary Angle and Uniform Higher Order Asymptotics	30
	57
2.1. Introduction	39
2.2. Higher order asymptotics in the region without	
creeping waves terms	40
2.2.1. Spectral function and asymptotic boundary conditions on	
both faces	40
2.2.2. Detailed calculation of first term influenced by the curvatures	44
2.3. Several orders asymptotic expressions in a region with creeping	
waves terms	48
2.3.1. Plane wave illumination: observation at infinity	48
2.3.2. Plane wave illumination: observation at finite distance	49
2.3.3. Source and observation points at finite distance	51
2.4. Some developed expressions concerning $f = \sum_{0 \le n \le m} f_n / k^n$ for	
arbitrary wedge angle	53
2.4.1. A developed expression of $\chi$ for $f_1(\alpha) = \Psi(\alpha)\chi(\alpha)$	53
2.4.2. Perfectly conducting case: curved half plane and	
discontinuity of curvature	56
2.4.3. Discontinuity of curvature in an impedance surface	57
2.5. Rewriting the field with fringe waves of second order	58
2.5.1. Definition of fringe waves $u_{fr}$ when $ \varphi  < \Phi$	58
2.5.2. The field term $u_{\pi_{n-1}}^{\pm}$ and the expression of $u_{\pi_{n-1}}$	20
at large distance	60
2.5.3 The expressions of $\mu_c$ for $ \rho_c  < \Phi$ and $ \rho  < \Phi$	61
2.5.5. The expressions of $u_{jr}$ for $ \varphi_0  < \varphi$ and $ \varphi  < \varphi$	62
$2.5.1$ . The term of futuration $a_{op}$ derived from $a_{OGe}$	02

2.5.5. Continuation of $u_{fr}^{\pm}(\rho',\varphi',\rho,\varphi)$ in shadow zone	
when $\pm \varphi' > \Phi$	65
2.6. References	66
2.A. Appendix A: on an integral expression with asymptotic kernel for the	
field near tangence of faces at $\varphi = \pm \Phi$ , permitting an explicit transition at	
arbitrary order	67
2.A.1. Expression when the far field function $F$ has no singularity	67
2.A.2. Expression when $F$ has a singularity $\dots \dots \dots$	72
2.A.3. Why we need to consider $\sin \theta = \sin \theta^{\perp}$ for the transition function	
at $\varphi = \pm \Phi$	73
2.B. Appendix B: first-order asymptotic of transition integral above	74
tangent faces $\varphi = \pm \Phi$	74
2.B.1. when F has no singularity	74
2.B.2. When F has a simple real pole	13 77
2.C. Appendix C: regularity of a sum of residues at coalescent poles	11
2.D. Appendix D. analytical expression of $u_{OGa}$ from behavior at reflection poles of f	79
	70
Chapter 3. Spectral Equations for Scattering by Impedance	
Polygons: Properties and Solutions	81
3.1. Introduction	81
3.2. Generalities	81
3.3. Single-face expression of $f$ for a scatterer enclosed by a surface with	
two semi-infinite polygonal faces	84
3.4. Polygonal surface with impedance boundary conditions	86
semi-infinite planes	86
3.4.2. Functional equations due to boundary conditions	
on finite segments	87
3.4.3. A remarkable relation between $f_{a,p}^{\pm}$ and $f_{b,p}^{\pm}$ deriving from (3.17)	
when (3.10) applies	88
3.5. Formulation of the three-part polygonal problem: spectral functions	
in Sommerfeld–Maliuzhinets representation and functional difference	0.0
equations in complex plane	89
3.5.1. Definition	89
3.5.2. About functional difference equations in complex plane $\dots$	90
5.5.5. Functional difference equations for $f_{br}(\alpha) = f_b(\alpha + \frac{1}{2}),$	00
$f_{ar}(\alpha) = f_a(\alpha - \frac{\pi a}{2}) \dots \dots$	92
5.0. The integral expressions and integral equations for the three-part	02
3.6.1 Elementary integral solutions for difference equations	93
3.0.1. Elementary integral solutions for the form $f_{1}(x)$ and $f_{2}(x)$	93
3.6.2. Coupled integral expressions for $\int_{br}(\alpha)$ and $\int_{ar}(\alpha) \dots \dots \dots$ 3.6.3. Integral equations when $\Phi = \sum_{n=1}^{n} \frac{\pi}{2} \exp[(ib)] < \pi/2$	94
5.6.5. Integral equations when $\Psi_{a,b} > -\frac{1}{2}$ , as $ \arg(i\kappa)  < \pi/2 \dots$	73

3.6.4. Modification of integration path, and extension	
including $k$ real	97
3.7. Existence and uniqueness for the integral equations on $C_{\epsilon}$	102
3.7.1. Uniqueness for the integral equations as $ \arg(ik\Delta)  \le \pi/2$	102
3.7.2. Existence and uniqueness of solution	104
3.8. Some particular features of the system of integral equations for the	
three-part impedance polygon and their consequences	105
3.8.1. Decoupling in the case of unsymmetric three-part	
impedance planes	105
3.8.2. Decoupling for symmetric three-narts polygons	
$(\Phi_{\rm L} = \Phi_{\rm a} \text{ and } \sin \theta_{\rm L} = \sin \theta_{\rm c})$	106
3.8.3 Partial inversion and new kernels for small $k\Delta$	107
3.8.4 Asymptotics for large $k\Delta$	108
3.9 Exact first order expressions for a small complex cavity in a step	100
when $\Phi_{\mu} = -\Phi_{\mu}$ and $\sin \theta_{\mu} = \sin \theta_{\mu}$	111
3.9.1 Impedance boundary conditions and elementary equations for	111
spectral functions	111
3.9.2 Reduction of far field function when $\sin \theta_{\perp} = \sin \theta$	111
and $\Phi_{} \Phi_{}$	114
and $\Psi_a = -\Psi_b$	117
far field function	115
3.9.4 Exact first-order expression for the diffracted field	115
3.10 References	117
3. A Appendix A: About $\Psi_{\alpha}$ ( $\alpha$ , $\Phi$ ) and the solution for	11/
an impedance wedge	110
an impedance wedge $\ldots$	119
3.A.1. $\Psi_{+-}(\alpha)$ in passive case	119
(passive or active)	121
$(passive of active) \dots \dots$	121
or active impedances	122
2 D. Appendix D. principle of somi inversion for our system	122
5. B. Appendix B. principle of semi-inversion for our system	100
2 C. Annen die Courieren auf falle after immeden as heur demo	122
3.C. Appendix C: uniqueness of fields when impedance boundary	107
conditions on piecewise regular geometry	127
3.C.1. First Green theorem for uniqueness: nullity of field outside $S'$	
when $ \arg(ik)  < \pi/2$ , and of field and its normal derivative on regular	105
parts of S' when $ \arg(ik)  = \pi/2$	127
3.C.2. Uniqueness for $ \arg(ik)  \le \pi/2$ comes from nullity of field and	1.00
its normal derivative on a regular part of boundaries	129
3.C.3. Some extension in 3D electromagnetism	130
3.D. Appendix D: asymptotics for $\mathcal{R}_a(\alpha)$ and $\mathcal{R}_b(\alpha)$ taking account of	
complex poles	131

3.E. Appendix E: the scattering diagram from the solutions of	
the integral equations	134
3.F. Appendix F: approximated second-order ameliorations for	
small cavity	135
3 F1 Amelioration of equivalent cavity impedance $\sin \theta'_{1}$	135
$3 \text{ E}_2$ A change of $M$	136
$5.1.2$ . A change of $w_u$	150
Chapter 4 Advanced Properties of Spectral Functions in Frequency	
and Time Domains for Diffraction by a Wedge-shaped Region	137
4.1. Introduction	137
4.2. Basic properties of spectral function in	
Sommerfeld–Maliuzhinets representation	138
4.2.1. Sommerfeld–Maliuzhinets representation for scattering by a	
wedge-shaped sector	138
4.2.2. Basic properties of total field $u$ and spectral function $f$	141
4.2.3. Higher order expressions with	
Sommerfeld–Maliuzhinets integrals	143
4.3. Spectral function attached to radiation of a single	
face and properties	145
4.3.1. Basic integral expression of radiation of one face	145
4.3.2. Spectral function associated with $H_0^{(2)}(kR)$	146
4.3.3. Expression of spectral function associated with $u_+$	
and properties	147
4.4. Far-field radiation of one face and single-face expression of spectral	
function $f$	148
4.4.1. The spectral function $f$ derived from fields on a semi-straight line	
and properties	148
4.4.2. The spectral function $f$ derived from fields on a piecewise	
smooth semi-line	150
4.5. Expression of $f$ from the far field function $F$	
and its consequences	152
4.5.1. Integral expression of $f$ relatively to $F$	152
4.5.2. Equations on $f$ from functional equations on $F$	
and consequences	153
4.5.3. Expression of f from F with a shift of origin	154
4.6. The spectral function $f_0(\alpha)$ for the diffraction by a wedge, with	
passive or active impedance faces, and an illumination	
normal to the edge	156
4.6.1. Definition of $f_0(\alpha)$ in general case, with passive	
or active faces	156
4.6.2. General exact derivation of $\Psi(\alpha) = \Psi^+(\alpha)\Psi^-(\alpha)$	157
4.6.3. Complete asymptotic expressions of $\frac{\Psi^{\pm \prime}}{\Psi^{\pm}}$ and $\Psi^{\pm}$	162
$\Psi^{\pm}$	

	4.6.4. Links between $\Psi^{\pm}$ for passive and active impedances, and	
	positions of poles	165
	4.6.5. Determination of $P_n(\sin(\mu\alpha))$ for passive or active faces	166
	4.7. Analysis for a wedge at skew incidence $(2D1/2)$ and associated special	
	functions	167
	4.7.1. Our solution given in [9]	168
	4.7.2. The relation with the solution given by Lyalinov and Zhu	171
	4.7.3. $\Psi_{arr}$ in terms of $\Psi_{\Phi}$ functions and expressions of $\Psi_{\Phi}$	173
	4.7.4. Expressions for the Bobrovnikov–Fisanov function $v_{\Phi}$	177
	4.8. Explicitly causal expression in time domain for a dispersive	1,1
	wedge-shaped region	179
	4.8.1. Basic elements on causality and integral expressions	179
	4.8.2. Properties of spectral function $f$ and trajectories of poles	181
	4.8.3. Elementary transform $\mathcal{F}_{\pm}^{\pm}$ of f and its analytical	
	continuation $\mathcal{F}^{\pm}$	182
	4.8.4. Fourier transform of $f$ and vanishing property	
	due to causality	186
	4.8.5. Explicitly causal expression of fields in time domain	
	with $\mathcal{F}_{a} \xrightarrow{\Lambda \to \infty} \cdots $	187
	4.9. References	189
	4.A. Appendix A: analysis of the solution of	
	$s(\alpha \pm \Phi) - \varepsilon s(-\alpha \pm \Phi) = S^{\pm}(\alpha) \dots \dots$	191
	4.B. Appendix B: analysis of the reflection coefficient attached to a	
	multilayered face of a wedge	193
	4.C. Appendix C: some miscellaneous properties of spectral functions for	
	an impedance wedge with passive impedance faces	194
	4.C.1. The spectral function $f_0(\alpha)$ for a wedge with straight	
	passive faces	194
	4.C.2. Some miscellaneous results as $\Phi = \pi/2$ and $\operatorname{Re}(\sin \theta^{\pm}) \geq 0$	195
	4.D. Appendix D: functional equations of high order and behavior of fields	
	at the edge	196
	4.D.1. Combinations of derivatives at the edge for the $c_n^{\pm}$	197
	4.D.2. Combinations of derivatives at the edge isolating $\frac{\partial^m}{\partial (ik_0)^m} \frac{\partial u}{\partial k_0 \partial u}$ .	199
	4.E. Appendix E: on the poles of $(\arctan \frac{n}{2})'$ for the evaluation of $v_{an}$ .	200
Ch	apter 5. General Integral Identities for Bianisotropic Media and	
Re	lated Equations, Properties and Coupling Expressions	201
	5.1 Introduction	201
	5.2. General integral identities for piecewise continuous media with	201
	finite losses	202
		-02

nite losses	202
5.2.1. Bianisotropic media and adjoint characteristics	202
5.2.2. Basic general properties of fields	203

5.2.3. Initial surface and volume integral identities for piecewise	
continuous media	204
5.3. Generalized reciprocity theorem and associated fields properties	205
5.3.1. Property 1: principle of equivalence of surface sources enclosing	
the domain of a volume source in open space	206
5.3.2. Property 2: generalized reciprocity theorem in open space and	
generalized reciprocity principle	207
5.3.3. Uniqueness properties	207
5.3.4. Property 5: vanishing surface integral equation	209
5.4. Generalized reciprocity principle and dyadic tensors fields	211
5.4.1. Dirac sources in open space and reciprocity principle	211
5.4.2. Generalized reciprocity for co-dipolar dyadic tensors	212
5.4.3. Generalized reciprocity for cross-dipolar dyadic tensors	212
5.4.4. Dyadic tensorial integral expressions of fields and reciprocity	213
5.5. Fields influenced by a perturbation $A$ of a bianisotropic object	213
5.5.1. Definitions of fields	213
5.5.2. Definition of primary sources	214
5.5.3. Integral equalities for $(E_{1S}, H_{1S})$ and $(E_{1S}^0, H_{1S}^0)$ on $S$	214
5.5.4. Integral expressions for $(E_{1S}, H_{1S})$ from fields known on $S$	215
5.5.5. Definition of influence tensors from $E_{1S} - E_0$ and $H_{1S} - H_0$ .	219
5.6. Fields influenced by coupling between two scatterers A and $A'$	219
5.6.1. Definitions of first-order coupling terms $E_{S,S'}$ and $H_{S,S'}$	220
5.6.2. Expressions of $(E_{S,S'}, H_{S,S'})$ for	
$M'_1 = M_1 = 0$ and for $J'_1 = J_1 = 0$	221
5.6.3. First-order coupling tensors $[E_{S,S'}^{c(,m)}(r_1,r_1')]$	
and $[H^{c(,m)}_{S,S'}(r_1,r_1')]$	225
5.6.4. Reciprocity relations for coupling tensors	226
5.7. Application to the numerical removal of couplings	228
5.7.1. Expression of the total field	228
5.7.2. A particular surface radiation integral, its properties and its relation	
with coupling	228
5.7.3. Efficient determination of the radiation of A without $A'$	229
5.8. Numerical results	230
5.9. References	233
5.A. Appendix A: on some identities for conjugate fields	233
5.B. Appendix B: on some identities for surface integrals	234
5.B.1. On the FP of a surface integral expression	234
5.B.2. On the principal value $(pv)$ of a surface integral	236
5.B.3. On properties of the surface divergence operator Div	238
5.C. Appendix C: note on distributions and some integrals expressions	239
5.C.1. Elementary equalities for distributions	239

5.C.2. Some integrals concerning the indicator function	
$1_{\Omega'}(r)$ up to $\partial \Omega'$	239
5.D. Appendix D: integral expressions and equations when	
$S$ is piecewise smooth $\ldots$	242
5.D.1. General integral expressions for piecewise smooth $S$ 5.D.2. Generalities when the domain $\Omega \equiv R^3 \backslash \Omega_S$ holds the isotropic	242
free space	243
5.D.3. Integrals expressions in free space and discontinuities	245
5.D.4. Integral equations for impedance boundary conditions on $S$	246
5.D.5. Integral expressions when $S$ is a perfectly conducting surface in	
free space	248
Chapter 6. Exact and Asymptotic Reductions of Surface Radiation	
Integrals with Complex Exponential Arguments to	
Efficient Contour Integrals	251
6.1 Introduction	251
6.2 Formulation	251
6.2 A combination of two fundamental ways for an afficient reduction to	232
a contour integral	254
6 3 1 First way	255
6.3.2 Second way	255
6.3.3. Why these distinct ways are complementary:	201
a practical example	264
6.4 Reduction to contour integrals in case (a) for a pure dipolar	201
illumination of a large perfectly conducting flat plate	267
6.4.1. Basic expansion of kernels in general radiation surface integral	207
expressions	267
6.4.2. Exact reduction to contour integrals when S is a flat plate $S'$	
perfectly conducting	269
6.5. Reduction to contour integrals in case (a) for a weighted dipolar	
illumination of a large perfectly conducting flat plate	273
6.5.1. Integral expressions for a weighted dipolar illumination	273
6.5.2. Reduction to contour integrals for even/odd	
weight functions $f_{e,h}$	274
6.6. Definition of complex scaling in case (b) and elementary exact surface	
integral reduction	278
6.6.1. Transformation of the polynomial argument $\mathcal{R}^2$	279
6.6.2. Reduction of the surface integral to a contour integral	282
6.7. Complex scaling and extended cases of applications for	• • • •
exact reductions	289
6./.1. Exact reduction when $g \equiv 1$	289
6.7.2. Exact reduction when $g(\mathbf{r'})$ is a polynomial of coordinates	290

6.8. Complex scaling and applications to asymptotic reductions	292
6.8.1. The case with $f(\mathcal{R}^2) = e^{\mathcal{R}^2}$ and semi-asymptotic reduction	292
6.8.2. The case with $f(\mathcal{R}^2) = \frac{e^{2\sqrt{\mathcal{R}^2}}}{\sqrt{\mathcal{R}^2}}$ and asymptotic reduction for	
curved plate	292
6.9. Reduction in case (c) for surface radiation integrals	
on curved plates	295
6.9.1. Reduction to a contour integral for curved plates when $f_h \equiv 0$	296
6.9.2. Reduction to a contour integral for curved plates when $f_e \equiv 0$	298
6.10. Miscellaneous results	299
6.11. Numerical results	302
6.11.1. Exact contour reduction of a surface integral whose integrand has	
an exponential term with a quadratic argument	302
6.11.2. Asymptotic contour reduction of near-field radiation surface	
integral in physical optics for a large plate illuminated by a	
point source with spherical pattern	304
6.12. References	317
6.A. Appendix A: primitives for theorems 6.5 and 6.6, and properties	319
6.A.1. Primitives for the theorem 6.5	319
6.A.2. Primitives for the theorem 6.6 $\ldots$	320
6.A.3. Expressions and properties of successive primitives of $\frac{e^{-vR}}{(\widetilde{R})^n}$	320
6.B. Appendix B: reduction of physical optics surface integrals to contour	
integrals in bistatic case and quadratic approximation	
of exponential arguments	322
6.B.1. Physical optics surface integrals and asymptotics	322
6.B.2. An asymptotic development of $\stackrel{\sim}{R}$	324
6 B 3 Quadratic approximation of exponential arguments and reduction	021
of surface integrals to contour ones	324
6 C Appendix C: reduction of physical optics radiation integrals in	521
monostatic case for an imperfectly conducting flat plate	325
6 C 1 Physical ontics fields and reduction for	525
an electric dinolar source	326
6 C 2 Physical ontics fields and reduction for	520
a magnetic dinolar source	327
6 D Appendix D: particular uses of asymptotics with	521
theorems 6.1 and 6.2	320
6 D 1 A particular use of lemma 6.2 with theorem 6.1	320
6.D.2. A particular use of lemma 6.2 with theorem 6.2	330
6 F Appendix E: complements on the use of proposition 6.2 for	551
$c(\pi') = c(\pi')f(\mathcal{D}^2)$ when we have $f(\mathcal{D}^2) = c\mathcal{R}^2$ and $c(\pi')$ is a	
$s(\mathbf{r}') = g(\mathbf{r}') f(\mathbf{k}^{-})$ , when we have $f(\mathbf{k}^{-}) = e^{-\mathbf{r}}$ and $g(\mathbf{r}')$ is a smooth analytical function on $C'$	222
smooth analytical function on $S^2$	332

6.F. Appendix F: complements on theorem 6.19 for some	
complicated case	334
6.G. Appendix G: some equalities in relation with proposition 6.11	335
Index	337

### Introduction

This book is a collection of independent mathematical studies, describing the analytical reduction of complex generic problems in the theory of scattering and propagation of electromagnetic waves in the presence of imperfectly conducting objects. Their subjects are as follows:

- a global method for the scattering by a multimode plane;
- diffraction by an impedance curved wedge;
- scattering by impedance polygons;
- advanced properties of spectral functions in frequency and time domains;
- bianisotropic media and related coupling expressions;
- exact and asymptotic reductions of surface radiation integrals.

Each of our approaches can be qualified as analytical, when it leads to exact explicit expressions, or, as semi-analytical, when it drastically reduces the mathematical complexity of studied problems. Therefore, they can be used in mathematical physics and engineering, to analyze and model, as well as in applied mathematics, to calculate for a low computational cost, the scattered fields in electromagnetism.

All of these works derive from original methods initiated in our publications that we here detail, develop and extend.

The first chapter is devoted to original exact expressions of the diffraction by a multilayered plane that can be partly composed of metamaterials. In whole generality, we then determine the fields as depending on potentials attached to arbitrary passive or active modes whose combination will give the passivity of the complete system. Our expressions directly take account of primary sources composed of electric and magnetic dipoles with arbitrary orientations, and profit of a novel exact development of incomplete Bessel function as an exact series of error functions. This latter characteristic permits a complete uniform analysis for arbitrary complex parameters, contrary to previous known results with error functions that were only approximations. Exact and complete asymptotics (at any order) are described, allowing us to particularly analyze the contribution of guided waves (forward and backward) at any distance.

The second chapter concerns the diffraction of an impedance wedge with curved faces of arbitrary angle that supports distinct surface boundary conditions of impedance type. We then distinguish the domains above and below the tangent planes at the edge. Our method permits an asymptotic evaluation at arbitrary order of curvatures of both faces for arbitrary passive impedance parameters. The uniformity at the crossing of the tangent plane is a characteristic remaining at arbitrary order, permitting us to analyze reflected, guided waves, edge-diffracted waves, but also waves originating from the edge that creep along the faces (creeping waves when faces are convex).

The diffraction in free space of a imperfectly conducting polygons (finite or with semi-infinite faces) is a particularly delicate problem that we study in third chapter, using Sommerfeld–Maliuzhinets integral representation of fields in a novel manner to rigorously consider several discontinuities, without any approximations. Indeed, contrary to common methods which consider large facets to admit asymptotic coupling between edges, we consider a rigorous development valid for arbitrary dimensions of facets, by establishing novel spectral equations that we can solve exactly or asymptotically, from a novel analysis of properties of spectral functions. This is particularly permitted by using their single-face representations, which is perfectly adapted to directly consider boundary conditions on faces with piecewise smooth geometries, as in polygonal cases.

Chapter 4 explores spectral functions in Sommerfeld–Maliuzhinets integral representation, their properties in the complex plane and new developments of them and special functions, attached to the resolution of multiple problems concerning the diffraction by a wedge with impedance boundaries conditions (passive or active). By beginning the study in frequency domain, we also analyze the representation of fields in time domain, in particular for an efficient explicit expression of causality in the case of dispersive (not constant relatively to frequency) multimode faces.

The fifth chapter analyzes the coupling influence between two imperfectly conducting objects in presence of a third one, all constituted by bianisotropic media. After beginning with developments of integral equalities, in particular, a generalized reciprocity one, we derive different properties of fields that will permit a complete analysis of coupling influences. We then give an example of application for an efficient and simple numerical post-process suppression of the influence of one object, that is, its direct but also its coupling contributions, on a second object.

We conclude this book with the determination of explicit contour integral expressions for an efficient evaluation of surface radiation integrals at arbitrary distance. This reduction concerns radiation of plane or curved plates of arbitrary contours, when the fields, highly oscillatory or not, are analytically defined on them, which is particularly the case for physical optics radiation surface integrals, when the surface fields are defined in closed form from geometrical optics.

1

### A Global Method for the Scattering by a Multimode Plane with Arbitrary Primary Sources and Complete Series with Error Functions

### 1.1. Introduction

In [1], we considered the field scattered by an arbitrary impedance plane in electromagnetism, and we here exploit this formalism to analyze the scattering by a structure composed of several homogeneous planar layers, with isotropy or uniaxial anisotropy, illuminated by arbitrary bounded sources. In this study, the plane is supposed to be either grounded, that is, a multilayer backed by an impedance plane, or not grounded, that is, a multilayer slab in free space; this will lead us to generalize our previous approach for a multilayer given in [2].

The field scattered of such structures is usually given by its plane wave expansion (Fourier representation) [3]-[6], which presents the particularity to have reflection coefficients that are meromorphic functions. Each one can be then modeled as a rational function with a set of N simple poles  $\{-g_j\}_{j=1,..,N}$ , which permits us to assume a multimode boundary condition of order N [2].

The Fourier expansion is well adapted in far field or for plane wave illuminations, but is not suitable for an analysis at any distance or for complex incident waves. Even when double Fourier integrals are reduced to single Fourier–Bessel integrals, calculation is lengthy and delicate because of functions in the integral that remain highly oscillating and, most often in literature [3]–[9], analytic expansions are not strictly convergent but asymptotic. Besides, an additional difficulty comes from that, and in multimode case, we have to take into account that the constants  $g_i$  can have

real parts of any sign, which signifies that passive but also active modes are present, even if the complete system is strictly passive.

In this frame, after expanding potentials into a combination of Fourier–Bessel integrals depending on each  $g_j$ , we are led to transform them to derive a more efficient integral representation, which is able to take account of active modes. Among other specificities, the definition of a parameter  $\epsilon$ , attached to each pole, is then particularly important to permit complete exact and asymptotic series with error functions. These series allow us to exhibit guiding waves terms near and far from the sources above the multilayer, generalizing [1] and refining [2].

Otherwise, our approach, as in [1], uses a new representation of potentials for the incident field, which possesses the originality to directly consider arbitrarily oriented electric and magnetic primary currents sources. Thus, we have no more to solve separately the problem for vertical or horizontal dipolar source as commonly done in the literature for passive impedance planes [7]–[14], isotropic or uniaxial slabs [15]–[17] or multilayers [3]–[6], [18]–[22]. In practice, the analytic method so developed can be applied in whole generality to various problems, in particular for the determination of coupling between antennas above an imperfectly reflective plane, or for the calculus of Green's functions for planar lines printed on a multilayer.

This chapter is organized as follows. In section 1.2, we give a discussion on the representation of the field with potentials, on the boundary conditions and on the positions of  $g_j$  in the complex plane when metamaterials can be present. Next, we give a global expression of potentials attached to the fields radiated by arbitrary bounded sources in free space in section 1.3, and above the multilayer in section 1.4, which we develop and expand for arbitrarily oriented dipoles in section 1.5. In sections 1.6 and 1.7, we then detail a compact expression of the special function involved in the potentials attached to each mode, intimately depending on a parameter  $\epsilon$  that is necessary to correctly take account of active modes. The definition of  $\epsilon$  will be useful for the development of exact (section 1.6) and asymptotic (section 1.7) expansions with error functions for arbitrary cases, allowing in particular a general analysis of guided waves in section 1.8, including backward waves, near and far from the sources.

### 1.2. Potentials, reflection coefficients and multimode boundary conditions

#### 1.2.1. Fields and potentials

We consider the scattering by an imperfectly reflective plane when it is illuminated by the field radiated by a bounded primary source, which is composed of arbitrary electrical and magnetic currents J and M (see Figure 1.1). In the space of points rwith Cartesian coordinates (x, y, z), this plane is defined by z = 0. A harmonic time dependence  $e^{i\omega t}$ , from now on assumed, is suppressed throughout. The constants  $\varepsilon_0$ and  $\mu_0$  are, respectively, the permittivity and the permeability of the free space above the plane, and  $k_0 = \omega(\mu_0 \varepsilon_0)^{1/2}$  is its wavenumber. Each component of the scattered field is assumed to be regular in the domain z > 0, and  $O(e^{-\gamma |r|})$  with  $\gamma > 0$  as  $|r| \to \infty$  when  $|\arg(ik_0)| < \pi/2$  (note: no loss is a limit case).

The electric field E and the magnetic field H above the multilayer, following Harrington [23, p. 131] (see also Jones [3, p. 19]), can be written with two scalar potentials  $\mathcal{E}$  and  $\mathcal{H}$ , as follows:

$$E = -ik_0 \operatorname{curl}(\mathcal{H}\,\hat{z}) + (\operatorname{grad}(\operatorname{div}(.)) + k_0^2)(\mathcal{E}\,\hat{z}),$$
  

$$Z_0 H = ik_0 \operatorname{curl}(\mathcal{E}\,\hat{z}) + (\operatorname{grad}(\operatorname{div}(.)) + k_0^2)(\mathcal{H}\,\hat{z}),$$
(1.1)

where the Helmhotz equations  $(\Delta + k_0^2)\mathcal{E} = 0$  and  $(\Delta + k_0^2)\mathcal{H} = 0$  are verified outside the sources,  $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$ . Thereafter, we denote  $(\mathcal{E}_i, \mathcal{H}_i)$  and  $(\mathcal{E}_s, \mathcal{H}_s)$  the potentials corresponding, respectively, to the incident field (incoming wave)  $(E^i, H^i)$ and the scattered field (outgoing wave)  $(E^s, H^s)$ , and we write (1.1) in the compact form:

$$(E, Z_0 H) = (\mathcal{L}_1(\hat{z}\mathcal{E}, \hat{z}\mathcal{H}), \mathcal{L}_2(\hat{z}\mathcal{E}, \hat{z}\mathcal{H})) = \mathcal{L}(\hat{z}\mathcal{E}, \hat{z}\mathcal{H}),$$
  

$$\mathcal{L}_1(u, v) = ((\operatorname{grad}(\operatorname{div}(.)) + k_0^2)(u) - ik_0\operatorname{curl}(v)),$$
  

$$\mathcal{L}_2(u, v) = (ik_0\operatorname{curl}(u) + (\operatorname{grad}(\operatorname{div}(.)) + k_0^2)(v)).$$
(1.2)



**Figure 1.1.** Geometry: sources (J, M) and observation point above the plane z = 0

### **1.2.2.** Multimode boundary conditions for a multilayer backed by an impedance plane

Let us consider a multilayer plane composed of uniform isotropic (or z-axial anisotropic) layers. Any plane wave  $(E^i, H^i)$ , incident at angle  $\beta$  with the normal  $\hat{z}$ , is then scattered as a reflected plane wave  $(E^s, H^s)$  that satisfies  $E_z^s|_{z=0^+} = R_{e,e}E_z^i|_{z=0^-}$  (i.e.  $H_{\perp}^s|_{z=0^+} = R_{e,e}H_{\perp}^i|_{z=0^-}$  in TM polarization), and  $H_z^s|_{z=0^+} = R_{h,h}H_z^i|_{z=0^-}$  (i.e.  $E_{\perp}^s|_{z=0^+} = R_{h,h}E_{\perp}^i|_{z=0^-}$  in TE polarization) [4], [5], [21] (see details in appendix A). If the multilayer is backed by a constant impedance plane, the reflection coefficients  $R_{e,e}$  and  $R_{h,h}$  are meromorphic functions of the variable  $\cos \beta$ , which we can model as rational functions [2] with simple poles, following:

$$R_{e,e}(\beta) = \prod_{j=1}^{N_e} \frac{\cos \beta - g_j^e}{\cos \beta + g_j^e}, \ R_{h,h}(\beta) = \prod_{j=1}^{N_h} \frac{\cos \beta - g_j^h}{\cos \beta + g_j^h},$$
(1.3)

for which we have the basic equalities (without superscripts e and h):

$$\prod_{j=1}^{N} \frac{\cos \beta - g_j}{\cos \beta + g_j} = (\pm 1)^N + \sum_{j=1}^{N} a_j \frac{\left(\frac{-\cos \beta}{g_j}\right)^{\frac{1 \pm 1}{2}}}{\cos \beta + g_j},$$
$$\frac{a_j}{2g_j} = -\prod_{i \neq j}^{N} \frac{g_j + g_i}{g_j - g_i}, (-1)^N - 1 = \sum_{j=1}^{N} \frac{a_j}{g_j},$$
(1.4)

where  $N \ge 1$ , and  $\prod_{i \ne j}^{1} \frac{g_j + g_i}{g_j - g_i} \equiv 1$  for N = 1. The constants  $(\pm 1)^N$  refer to limit values when  $|\cos \beta|^{\pm 1} \to \infty$ .

The  $g_j^{e(,h)}$  are constants attached to complex modes with  $\operatorname{Im}(g_j^{e(,h)}) \neq 0$ , passive when  $\operatorname{Re}(g_j^{e(,h)}) \geq 0$  or active when  $\operatorname{Re}(g_j^{e(,h)}) < 0$ , and ordered such that  $|g_{j+1}^{e(,h)}| \geq |g_j^{e(,h)}|$ , while, as considered in [26]–[28], we assume that:

when (1.3) applies,  $N_e$  and  $N_h$  and thus N are positive odd numbers, (1.5)

(note: this restriction on  $N_{e(,h)}$  will be removed in the more general case of extended boundary conditions). Considering plane waves representation of fields (see appendix A), we can then write a multimode boundary conditions at  $z = 0^+$  [2]:

$$\prod_{j=1}^{N_e} \left(\frac{\partial}{\partial z} - ik_0 g_j^e\right) E_z^s(z)|_{0^+} = \prod_{j=1}^{N_e} \left(\frac{\partial}{\partial z} + ik_0 g_j^e\right) E_z^i(-z)|_{0^+},$$

$$\prod_{j=1}^{N_h} \left(\frac{\partial}{\partial z} - ik_0 g_j^h\right) H_z^s(z)|_{0^+} = \prod_{j=1}^{N_h} \left(\frac{\partial}{\partial z} + ik_0 g_j^h\right) H_z^i(-z)|_{0^+}.$$
(1.6)

From the symmetry at normal incidence, the condition  $R_{h,h}(0) = -R_{e,e}(0)$  must apply, which leads us to write:

$$\prod_{j=1}^{N_h} \frac{\pm 1 - g_j^h}{\pm 1 + g_j^h} = -\prod_{j=1}^{N_e} \frac{\pm 1 - g_j^e}{\pm 1 + g_j^e},\tag{1.7}$$

and implies that  $R_{h,h}(\pi) = -R_{e,e}(\pi)$ . The condition (1.7) has crucial importance to avoid non-physical behaviors of fields derived from potentials, as examined further in this paper. Besides, the reader will notice that (1.7) implies  $g_1^e = 1/g_1^h$  when  $N_{e(,h)} = 1$ , as well known for monomode (impedance) boundaries conditions [1]. The numbers  $N_{e(,h)}$  correspond to truncated infinite products, where the less significant  $g^{e(,h)}$  have been neglected, while some  $g^{e(,h)}$  have to be modified so that  $R_{h,h}(0) =$  $-R_{e,e}(0)$  remains.

Considering (1.1), we can use:

$$E_z = \frac{\partial^2 \mathcal{E}}{\partial z^2} + k_0^2 \mathcal{E}, \ Z_0 H_z = \frac{\partial^2 \mathcal{H}}{\partial z^2} + k_0^2 \mathcal{H},$$
(1.8)

in (1.6), and we are led to search scattered potentials  $\mathcal{E}_s$  and  $\mathcal{H}_s$ , satisfying the Helmholtz equation as z > 0, regular and exponentially vanishing as  $z \to \infty$  when  $|\arg(ik_0)| < \pi/2$ , that verify as z > 0:

$$\prod_{i=1}^{N_e} \left(\frac{\partial}{\partial z} - ik_0 g_j^e\right) \mathcal{E}_s(z) = \prod_{j=1}^{N_e} \left(\frac{\partial}{\partial z} + ik_0 g_j^e\right) \mathcal{E}_i(-z),$$
  
$$\prod_{j=1}^{N_h} \left(\frac{\partial}{\partial z} - ik_0 g_j^h\right) \mathcal{H}_s(z) = \prod_{j=1}^{N_h} \left(\frac{\partial}{\partial z} + ik_0 g_j^h\right) \mathcal{H}_i(-z),$$
(1.9)

where  $\mathcal{E}_i$  and  $\mathcal{H}_i$  potentials are attached to radiation of arbitrary primary sources.

#### 1.2.3. Extended multimode boundary conditions

More generally, we can consider an extended form when we want to include the case of a multilayer slab in free space, which is composed of isotropic [4]-[5] (or z-axial anisotropic [21]) layers. The reflection coefficients  $R_{e,e}$  and  $R_{h,h}$  remain meromorphic functions of  $\cos \beta$ , but we now model them in a more general form, following:

$$R_{e,e}(\beta) = R_0^e \frac{\prod_{j=1}^{N'_e}(\cos\beta - g_j^{e'})}{\prod_{j=1}^{N_e}(\cos\beta + g_j^{e})}, \ R_{h,h}(\beta) = R_0^h \frac{\prod_{j=1}^{N'_h}(\cos\beta - g_j^{h'})}{\prod_{j=1}^{N_h}(\cos\beta + g_j^{h})},$$
(1.10)

with simple complex poles  $-g_j^{e(,h)}$ ,  $N'_{e(,h)} \leq N_{e(,h)}$ , constants  $R_0^e$  and  $R_0^h$  (this time for odd or even  $N_{e(h)}$ ), for which we notice the basic equalities (without supercripts e and h):

$$R_{0} \frac{\prod_{j=1}^{N'} (\cos \beta - g_{j}')}{\prod_{j=1}^{N} (\cos \beta + g_{j})} - a_{0\tau} = \sum_{j=1}^{N} a_{j} \frac{(\frac{-\cos \beta}{g_{j}})^{\frac{1-\tau}{2}}}{\cos \beta + g_{j}} = \sum_{j=1}^{N} \frac{\tau \frac{a_{j}}{g_{j}}}{(\frac{\cos \beta}{g_{j}})^{\tau} + 1},$$
$$a_{j\neq0} = R_{0} \frac{\prod_{i=1}^{N'} (-g_{i}' - g_{j})}{\prod_{i\neq j}^{N} (g_{i} - g_{j})}, \left|_{a_{0-}=R_{0}}^{a_{0+}=|\frac{-R_{0} \text{ if } N' = N}{\cos N}} \right|_{a_{0-}=R_{0}(\prod_{i=1}^{N'} - g_{i}')/(\prod_{i=1}^{N} g_{i})},$$
(1.11)

where the constants  $a_{0\tau}$  refer to limit values when  $|\cos\beta|^{\tau} \to \infty$  with  $\tau = +1$  or -1. We have  $N \ge N'$ ,  $N \ge 1$ ,  $N' \ge 0$ , and we can let  $\prod_{i \ne j}^{1} (g_i - g_j) \equiv 1$  for N = 1 and  $\prod_{j=1}^{0} (\cos\beta - g'_j) \equiv 1$  for N' = 0.

As in the previous section, we consider that  $R_{h,h}(0) = -R_{e,e}(0)$ , and assume additionally that  $R_{h,h}(\pi) = -R_{e,e}(\pi)$ . This implies, after using (1.11) when  $\cos \beta = \pm 1$ :

$$\sum_{j=1}^{N_e} \frac{\tau_e \frac{a_j^e}{g_j^e} (g_j^e)^{\tau_e}}{(g_j^e)^{\tau_e} \pm 1} + \sum_{j=1}^{N_h} \frac{\tau_h \frac{a_j^h}{g_j^h} (g_j^h)^{\tau_h}}{(g_j^h)^{\tau_h} \pm 1} = -(a_{0\tau_e} + a_{0\tau_h}), \quad (1.12)$$

with  $\tau_{e(,h)} = +1$  or -1. The condition (1.12), as previously noticed for (1.7), has a crucial importance to avoid non-physical behaviors of fields derived from potentials.

Considering (1.10) and plane waves representation of fields (see appendix A), we can write:

$$\prod_{j=1}^{N_{e}} \left( \frac{-\partial}{ik_{0}\partial z} + g_{j}^{e} \right) E_{z}^{s}(z)|_{0^{+}} = R_{0}^{e} \prod_{j=1}^{N_{e}'} \left( \frac{-\partial}{ik_{0}\partial z} - g_{j}^{e'} \right) E_{z}^{i}(-z)|_{0^{+}},$$

$$\prod_{j=1}^{N_{h}} \left( \frac{-\partial}{ik_{0}\partial z} + g_{j}^{h} \right) H_{z}^{s}(z)|_{0^{+}} = R_{0}^{h} \prod_{j=1}^{N_{h}'} \left( \frac{-\partial}{ik_{0}\partial z} - g_{j}^{h'} \right) H_{z}^{i}(-z)|_{0^{+}},$$
(1.13)

which applies for any primary sources that illuminates the plane. Using (1.8), we can solve (1.13) with  $(\mathcal{E}_s, \mathcal{H}_s)$  that satisfy as z > 0:

$$\prod_{i=1}^{N_e} \left( \frac{-\partial}{ik_0 \partial z} + g_j^e \right) \mathcal{E}_s(z) = R_0^e \prod_{j=1}^{N'_e} \left( \frac{-\partial}{ik_0 \partial z} - g_j^{e'} \right) \mathcal{E}_i(-z),$$
  
$$\prod_{j=1}^{N_h} \left( \frac{-\partial}{ik_0 \partial z} + g_j^h \right) \mathcal{H}_s(z) = R_0^h \prod_{j=1}^{N'_h} \left( \frac{-\partial}{ik_0 \partial z} - g_j^{h'} \right) \mathcal{H}_i(-z).$$
(1.14)

NOTE 1.1.– For z-uniaxial chiral layers [17], non-diagonal terms in (1.100) (see appendix A on plane waves representation of fields) do not vanish, and

$$\begin{pmatrix} \mathcal{P}_{N_e}^e(\frac{\partial}{\partial z})\mathcal{E}_s(z)\\ \mathcal{P}_{N_h}^h(\frac{\partial}{\partial z})\mathcal{H}_s(z) \end{pmatrix} = \begin{pmatrix} \mathcal{Q}_{N'}^{e,e}(\frac{\partial}{\partial z}) \ \mathcal{Q}_{N'}^{e,h}(\frac{\partial}{\partial z})\\ \mathcal{Q}_{N'}^{h,e}(\frac{\partial}{\partial z}) \ \mathcal{Q}_{N'}^{h,h}(\frac{\partial}{\partial z}) \end{pmatrix} \begin{pmatrix} \mathcal{E}_i(-z)\\ \mathcal{H}_i(-z) \end{pmatrix},$$
(1.15)

applies with polynomials  $\mathcal{P}^{e(,h)}$  and  $\mathcal{Q}^{e(,h)}$ . By linearity, it can be solved by addition of terms similar to ones derived for the solution of equations (1.14) (note: in this case, adjoint characteristics have to be considered to apply the generalized reciprocity principle).

NOTE 1.2.– Letting  $R = (\cos \beta - \mathcal{Z}(\cos \beta))/(\cos \beta + \mathcal{Z}(\cos \beta))$ ,  $\mathcal{Z}$  is a rational function which is even as (1.3) applies [4], [5], [21] (see also Appendix A of [2]).

NOTE 1.3.– Considering (1.12), we can assume that, when a  $g_j^{e(,h)} \to \mp 1$ , it exists a  $g_m^{h(,e)}$  such that  $a_j^{e(,h)}(g_j^{e(,h)} \pm 1)^{-1} + a_m^{h(,e)}(g_m^{h(,e)} \pm 1)^{-1}$  is bounded.

### **1.2.4.** Properties of $R_{e,e}$ and $R_{h,h}$ , and consequences on the $g_i^{e(,h)}$

The multilayer is characterized by permittivity  $\varepsilon(z)$  and permeability  $\mu(z)$  as z varies from z = 0 to  $-h_L$ , which corresponds to a set of complex  $\varepsilon_j$  and  $\mu_j$  specified in each layer  $-h_j \leq z \leq -h_{j-1}$  with  $1 \leq j \leq L$ ,  $h_0 = 0$ . Considering the hypotheses for (1.3) and (1.10), the last numbered layer shall be impenetrable or be bounded by the free space at  $z = -h_L$ . The plane wave spectrum of fields follows certain rules (see appendix A), and the reflection coefficients, considered for an harmonic time dependence  $e^{i\omega t}$ , are analytic functions, called thereafter R to simplify, which have some common elementary properties:

(a) R is an analytic function of the complex variables  $k_0$ ,  $\cos \beta$ ,  $i\omega \varepsilon$ ,  $i\omega \mu$  and layers depths. In the domain of passivity  $\Omega_{\varepsilon,\mu}$  with  $\operatorname{Re}(i\omega\varepsilon) > 0$  and  $\operatorname{Re}(i\omega\mu) > 0$ , Rhas no singularity when  $\beta$  varies from 0 to  $i\infty + \arg(ik_0)$  as  $\operatorname{Re}(ik_0\sin\beta) = 0$ , and at infinity. This regularity applies in the whole domain with  $|\arg(i\omega)| \le \pi/2$  from the causality principle. Its highest modulus is obtained for real  $\omega$  from the maximum modulus principle, while, from (1.104),  $|R| \le 1$  when  $k_0$  is real and  $0 \le \beta \le \pi/2$ .

(b) When  $i\omega\varepsilon$  and  $i\omega\mu$  are purely real positive (perfect lossy case), the multilayer is purely resistive, and, in these circumstances, R is real when  $k_0$  and  $\beta$  are real.

(c) From (a) and (b), we can apply the Schwarz reflection principle (or edge-of-the-wedge theorem) [24, sect.5] (see also [25]), and deduce that R satisfies in  $\Omega_{\varepsilon,\mu}$ :

$$R((k_0)^*, (\cos\beta)^*, (i\omega\varepsilon)^*, (i\omega\mu)^*) = (R(k_0, \cos\beta, i\omega\varepsilon, i\omega\mu))^*.$$
(1.16)

(d) Using  $(i\varepsilon)^* = i(-\varepsilon^*)$  and  $(i\mu)^* = i(-\mu^*)$  in (1.16), we note, as we let change  $\varepsilon$  and  $\mu$  for their "anticonjugate"  $-\varepsilon^*$  and  $-\mu^*$  (possible in practice with metamaterials, or plasmas), that, for real  $\omega$  and  $k_0$ :

every pole  $\cos \beta = (-g_j^{e(,h)})$  of R in (1.3) and (1.10) for  $\varepsilon$  and  $\mu$ , gives us its conjugate  $(-g_j^{e(,h)})^*$  as a pole of R for  $-\varepsilon^*$  and  $-\mu^*$ . (1.17) (e) The set  $\{g_j^{e(,h)}\}\$  has elements with positive and negative real parts [26]-[28], and for the class of problems including metamaterials, the total set to be considered, for real  $\omega$  and  $k_0$ , is  $\{g_j^{e(,h)}\} \cup \{(g_j^{e(,h)})^*\}\$ ; this leaves no quarter-plane of the complex plane empty and explains why we have such a wide range of properties when multilayers include metamaterials.

Let us now consider the examples of two lossless systems for real positive  $\omega$ , where, in the first case,  $\varepsilon$  and  $\mu$  are with purely positive real values, and, in the second case, with negative real values. As defined in (d), both cases then belong, respectively, to anticonjugate classes. Pure imaginary  $g_j^{e(,h)}$  are generally of finite number. From the analysis in [19] (respectively in [16]), the complex  $g_j^{e(,h)}$  with non-null real part are of infinite number, and they are complex numbers with negative (respectively, positive) imaginary parts, associated with improper (respectively, proper) modes. Therefore, the domains of the  $g_j^{e(,h)}$  for both cases of these examples are conjugated with each other, which plainly illustrates and confirms the property (d).

NOTE 1.4.– Considering appendix A, the conditions (a)–(e) also apply for anisotropic multilayers, when  $\varepsilon$  and  $\mu$  are tensors.

NOTE 1.5.-  $k_j = \omega(\mu_j \varepsilon_j)^{1/2}$ , with  $(\mu_j \varepsilon_j)^{1/2} = \sqrt{\mu_j} \sqrt{\varepsilon_j}$ , can be  $\neq \omega \sqrt{\mu_j \varepsilon_j}$ , and thus, a term like  $((\mu_j \varepsilon_j)^{1/2} / \varepsilon_j) \tan(\omega(\mu_j \varepsilon_j)^{1/2} d)$  can be  $\neq \sqrt{\mu_j / \varepsilon_j} \tan(\omega \sqrt{\mu_j \varepsilon_j} d)$ .

#### **1.3.** Incident potentials $(\mathcal{E}_i, \mathcal{H}_i)$ for arbitrary bounded primary sources

To solve problems with conditions (1.9) or (1.14), we need a correct explicit expression of  $(\mathcal{E}_i, \mathcal{H}_i)$ . We begin by considering the incident field  $(E^i, H^i)$  at point r of coordinates (x, y, z), radiated by arbitrary electric and magnetic bounded sources J and M in free space [3]:

$$\begin{split} E^{i} &= \operatorname{curl}(G \ast M) + \frac{i}{\omega \varepsilon_{0}} (\operatorname{grad}(\operatorname{div}(.)) + k_{0}^{2})(G \ast J), \\ H^{i} &= -\operatorname{curl}(G \ast J) + \frac{i}{\omega \mu_{0}} (\operatorname{grad}(\operatorname{div}(.)) + k_{0}^{2})(G \ast M), \end{split}$$
(1.18)

where  $G(r) = -\frac{e^{-ik_0|r(x,y,z)|}}{4\pi |r(x,y,z)|}$  with  $|r| = \sqrt{x^2 + y^2 + z^2}$  verifies  $(\Delta + k_0^2)G = \delta$ , \* is the convolution product, and J and M are generalized functions [29].

Considering arbitrary J and M in the domain  $\pm z > 0$ , the potentials  $(\mathcal{E}_i, \mathcal{H}_i)$ in the representation of fields  $E^i$  and  $H^i$  with (1.1), which satisfy the Helmholtz equation as  $\mp z > 0$  and exponentially vanish as  $\mp z \to \infty$  when  $|\arg(ik_0)| < \pi/2$ can be written when  $\mp z > 0$  [1]:

$$(\mathcal{E}_i, \mathcal{H}_i)(x, y, z) = \left[\frac{\widehat{z}}{8\pi k_0^2} \cdot (\mathcal{L}(Z_0 J, M) * \mathcal{W})\right](x, y, z), \tag{1.19}$$

where  $(A, B) * C \equiv (A * C, B * C)$ ,  $\mathcal{L}$  is defined as in (1.2):

$$\mathcal{L}(Z_0 J, M) = (\mathcal{L}_1(Z_0 J, M), \mathcal{L}_2(Z_0 J, M)),$$
  

$$\mathcal{L}_1(u, v) = ((\operatorname{grad}(\operatorname{div}(.)) + k_0^2)(u) - ik_0 \operatorname{curl}(v)),$$
  

$$\mathcal{L}_2(u, v) = (ik_0 \operatorname{curl}(u) + (\operatorname{grad}(\operatorname{div}(.)) + k_0^2)(v)),$$
(1.20)

and W has the remarkable compact form [1]–[2]:

$$\mathcal{W}(r) = e^{ik_0|z|} E_1(ik_0(|r|+|z|)) + e^{-ik_0|z|} (E_1(ik_0(|r|-|z|)) + 2\ln\rho),$$
(1.21)

with  $\rho = \sqrt{x^2 + y^2}$ ,  $E_1$  being the exponential integral function [30], which verifies, as  $\mp z > 0$ :

$$\left(\frac{\partial^2}{\partial z^2} + k_0^2\right)\frac{\mathcal{W}(r)}{8\pi i k_0} = G(r), \ (\Delta + k_0^2)\mathcal{W}(r) = 0.$$
(1.22)

NOTE 1.6.— It is worth noticing that we have  $\Delta_{xy} \ln(\rho) = 2\pi \delta(x) \delta(y)$ , which implies that  $\ln \rho$  in W has no influence on the expression of the field, except by its singularity at  $\rho = 0$ .

NOTE 1.7.– Let us divide the primary sources J and M into the sources above and below a plane  $z = z_1$ ,  $J_{z_1}^{\pm} = U_s(\pm(z - z_1))J$  and  $M_{z_1}^{\pm} = U_s(\pm(z - z_1))M$ , where  $U_s$  is the unit step function, with  $U_s(z) = (\text{sign}(z) + 1)/2$ ,  $U_s(z) = 1$  as z > 0,  $U_s(z) = 1/2$  as z = 0 and  $U_s(z) = 0$  as z < 0. We can then write, for arbitrary observation point at  $z = z_1$ :

$$(E^{i}, Z_{0}H^{i})(x, y, z_{1}) = \sum_{\pm} (E^{\pm}, Z_{0}H^{\pm})(x, y, z_{1} \mp 0^{\pm}), \qquad (1.23)$$

where  $(E^{\pm}, H^{\pm})$  are the field radiated by the sources  $J_{z_1}^{\pm}$  and  $M_{z_1}^{\pm}$ , following  $(E^{\pm}, Z_0 H^{\pm}) = \mathcal{L}(\widehat{z}\mathcal{E}^{\pm}, \widehat{z}\mathcal{H}^{\pm})$  with  $(\mathcal{E}^{\pm}, \mathcal{H}^{\pm}) = \frac{\widehat{z}}{8\pi k_0^2} \mathcal{L}(Z_0 J_{z_1}^{\pm}, M_{z_1}^{\pm}) * \mathcal{W}$ , from (1.2) and (1.19).

#### **1.4.** Scattered potentials $(\mathcal{E}_s, \mathcal{H}_s)$ for arbitrary primary sources

#### **1.4.1.** A global expression of $(\mathcal{E}_s, \mathcal{H}_s)$ for a multimode plane

Considering the potentials  $(\mathcal{E}_i, \mathcal{H}_i)$  attached to the radiation of arbitrary sources from (1.19), we can now express the scalar potentials  $\mathcal{E}_s$  and  $\mathcal{H}_s$ , which satisfy the multimode conditions (1.9) or (1.14), as  $z \ge 0$ . From the method developed in [1]–[2],

we have for conditions (1.9) (for odd numbers  $N_{e(h)}$ ):

$$\mathcal{E}_{s}(x, y, z) = \pm \mathcal{E}_{i}(x, y, -z)$$

$$+ \left( \left( \frac{\widehat{z}}{\omega \varepsilon_{0}} \frac{\operatorname{grad}(\operatorname{div}(J)) + k_{0}^{2}J}{8\pi k_{0}} + \frac{\widehat{z}}{k} \frac{(-ik_{0}\operatorname{curl}(M))}{8\pi k_{0}} \right) \right)$$

$$* \sum_{\epsilon'=-1,1} \left( \left( \prod_{j=1}^{N_{e}} \frac{\epsilon' + g_{j}^{e}}{\epsilon' - g_{j}^{e}} \mp 1 \right) \mathcal{V}_{\epsilon'} + \sum_{j=1}^{N_{e}} \frac{\epsilon' a_{j}^{e} \mathcal{K}_{g_{j}^{e}}}{(g_{j}^{e} - \epsilon')} \right) \right) (x, y, -z), \quad (1.24)$$

and

$$\mathcal{H}_{s}(x, y, z) = \mp \mathcal{H}_{i}(x, y, -z)$$

$$+ \left( \left( \frac{\hat{z}}{\omega \varepsilon_{0}} \frac{(ik_{0} \operatorname{curl}(J))}{8\pi k_{0}} + \frac{\hat{z}}{k} \frac{(\operatorname{grad}(\operatorname{div}(M)) + k_{0}^{2}M)}{8\pi k_{0}} \right) \right)$$

$$* \sum_{\epsilon'=-1,1} \left( \left( \prod_{j=1}^{N_{h}} \frac{\epsilon' + g_{j}^{h}}{\epsilon' - g_{j}^{h}} \pm 1 \right) \mathcal{V}_{\epsilon'} + \sum_{j=1}^{N_{h}} \frac{\epsilon' a_{j}^{h} \mathcal{K}_{g_{j}^{h}}}{(g_{j}^{h} - \epsilon')} \right) \right) (x, y, -z),$$

$$(1.25)$$

while we have, for extended conditions (1.14) (for odd or even numbers  $N_{e(h)}$ ):

$$\begin{aligned} \mathcal{E}_{s}(x,y,z) &= a_{0\tau_{e}}^{e} \mathcal{E}_{i}(x,y,-z) \\ &+ \left( \left( \frac{\widehat{z}}{\omega \varepsilon_{0}} \frac{\operatorname{grad}(\operatorname{div}(J)) + k_{0}^{2}J}{8\pi k_{0}} + \frac{\widehat{z}}{k} \frac{(-ik_{0}\operatorname{curl}(M))}{8\pi k_{0}} \right) \right. \\ &+ \sum_{\epsilon'=-1,1} \left( \left( (R_{e,e}|_{\cos\beta=-\epsilon'}) - a_{0\tau_{e}}^{e}) \mathcal{V}_{\epsilon'} + \sum_{j=1}^{N_{e}} \frac{\epsilon' a_{j}^{e} \mathcal{K}_{g_{j}^{e}}}{(g_{j}^{e} - \epsilon')} \right) \right) (x,y,-z), \end{aligned}$$
(1.26)

and

$$\mathcal{H}_{s}(x,y,z) = a_{0\tau_{h}}^{h} \mathcal{H}_{i}(x,y,-z) + \left( \left( \frac{\widehat{z}}{\omega\varepsilon_{0}} \frac{(ik_{0} \operatorname{curl}(J))}{8\pi k_{0}} + \frac{\widehat{z}}{k} \frac{(\operatorname{grad}(\operatorname{div}(M)) + k_{0}^{2}M)}{8\pi k_{0}} \right) * \sum_{\epsilon'=-1,1} \left( ((R_{h,h}|_{\cos\beta=-\epsilon'}) - a_{0\tau_{h}}^{h}) \mathcal{V}_{\epsilon'} + \sum_{j=1}^{N_{h}} \frac{\epsilon' a_{j}^{h} \mathcal{K}_{g_{j}^{h}}}{(g_{j}^{h} - \epsilon')} \right) \right) (x, y, -z), \quad (1.27)$$

where  $\mathcal{V}_{\epsilon'}, \mathcal{K}_g$  are given by,

$$\mathcal{V}_{\epsilon'}(x, y, -z) = e^{\epsilon' i k_0 z} (E_1(i k_0 (|r| + \epsilon' z)) + (1 - \epsilon') \ln \rho),$$
  
$$\mathcal{K}_g(x, y, -z) = e^{i k_0 g z} \mathcal{J}_g(\rho, -z),$$
 (1.28)