Multisector Growth Models

Terry L. Roe · Rodney B.W. Smith ·
D. Şirin Saracoğlu

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Theory and Application

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Terry L. Roe
Department of Applied Economics
University of Minnesota
1994 Buford Ave.
St. Paul MN 55108
USA
troe@tc.umn.edu

Rodney B.W. Smith
Department of Applied Economics
University of Minnesota
1994 Buford Ave.
St. Paul MN 55108
USA
smith142@umn.edu

D. Şirin Saracoğlu
Department of Economics
Middle East Technical University (METU)
lnonu Bulv.
06531 Ankara
Turkey
ssirin@metu.edu.tr

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Preface

The primary objective of this book is to advance the state of the art in specifying and fitting to data structural multi-sector dynamic macroeconomic models, and empirically implementing them. The fundamental construct upon which we build is the Ramsey model. A most attractive feature of this model is the insights it provides into the dynamics of an economy in transition to long-run equilibrium. With some exceptions, Ramsey models are highly aggregated – typically single sector models. However, interest often lies in understanding the forces of economic growth across multiple sectors of an economy and on how policy impacts likely play out over time. Such analyses call for more disaggregated models that can be fit to country or regional data. This book shows how to: (i) extend the basic model to multiple sectors, (ii) how to adapt the basic model to account for policy instruments, and (iii) fit the model to data, and obtain equilibrium values both forward and backward in time from the data points to which the model is initially fit.

Although extremely helpful in understanding economic growth and structure, theory alone is not sufficient; we also need to confront theory with data. Fitting growth models to data has been greatly facilitated by advances in numerical algorithms and computer technology. The ease of obtaining numerical solutions using procedures that are relatively robust across a broad range of model specifications is important because the differential equations of even the single sector, two factor, closed economy Ramsey model are essentially analytically intractable. The methods advanced here, and illustrated with numerical examples, are easily used in the classroom. Our experience suggests this material is accessible to advanced undergraduate and beginning graduate students, and easily managed by those working in agencies and bureaus familiar with general equilibrium concepts. An understanding of the subtleties of control theory and numerical algorithms is not required, but familiarity with a programming language such as Mathematica is essential. Over the past several years, we have had students choose a country, conduct a growth accounting exercise, formulate the country data into an elementary social accounting matrix format, fit a model to data, and then obtain a numerical solution using off-the-shelf software. We found that such assignments greatly strengthens students' grasp of theoretical concepts and helps them link these concepts to real economies. Grasping the theory and knowing how to implement the theory to obtain empirical insights into real problems provides them a form of human capital that they are unlikely to attain so easily in other ways.

The book is organized by first reviewing the fundamentals of duality theory of the consumer and firm, which is then used to review the standard two-sector, two-factor Heckscher-Ohlin-Samuelson model of a small open economy. Using duality theory, Chapter 3 introduces the two sector closed economy Ramsey model in a rather structured fashion, and concludes with an empirical example. Chapter 4 develops a three-sector, open economy model with a non-traded good sector. Chapters 5 and 6 extend the three-sector model in several directions: intermediate factors of production; capital stock composed of the output of all sectors of the economy; government consumption with taxes and lump-sum transfers from households. Chapter 7 concludes with a two country "world" model. In each chapter, the model presentation follows a similar pattern and builds off the structure of the previous chapter. Each modeling chapter concludes with an empirical example using the same data set. The book concludes with two chapters that discuss how the data are organized to facilitate the fitting of models to data, and the strategy used to facilitate the solution of each model's system of differential equations.

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1

Introduction: Orientation and Focus

1.1 Introduction

Our general objective is to advance methodology for the analyses of economies using multi-sector dynamic general equilibrium models. We begin by building upon the static Heckscher-Ohlin-Samuelson framework expressed using duality theory as shown most clearly by Woodland (1982). This framework is then cast into a two-sector growth model in which infinitely lived households choose consumption and saving to maximize their dynastic utility. This specification of consumer behavior is a key element of the Ramsey growth model (as constructed by Ramsey (1928) and refined by Cass (1965) and Koopmans (1965)), and has now become the mainstay of growth models with endogenous savings behavior. We then extend this two-sector framework to less stylistic models over several chapters that culminate in a two country world model where each country produces two traded goods and a home-good.

These models are specified in continuous time and in a manner that proceeds from the specification of primitives, the definition of equilibrium, and the characterization of intra-temporal and inter-temporal equilibrium. This manner of presentation is desirable for several reasons. First, it greatly facilitates an understanding of the analytical features of each model, including "Stopler-Samuelson-like" and "Rybczynski-like" relationships similar to that of the static trade model. Second, it shows more clearly how a less stylized model follows from the former; thus providing insights into how further extensions of the framework not discussed here or envisioned by the authors might be developed. Third, this style of presentation facilitates the writing of code to empirically solve a model because it lays out clearly:

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(i) the key relationships among equations characterizing equilibrium that must be coded, (ii) the parameters to be estimated, and (iii) the necessary initial and terminal conditions for equilibrium. Moreover, since a challenge of writing code is one of finding coding errors, this process allows for virtually each line of code to be checked to determine whether it is reproducing the data corresponding to one or more of the model's structural equations.

Each chapter illustrates an application of the theory with an empirical example. The empirical examples draw upon a common data set, organized into a logically consistent structure and at a level of aggregation equivalent to each analytical model. This structure, which is well known to those familiar with static computable general equilibrium (CGE) analyses, is laid out in Chapter 8 and shows how to estimate many of the model's parameters in such a way that the empirical results approximate the base period values of an economy in transition to long-run growth. The empirical models can be solved both backward and forward in time, and thus the question naturally arises: Do model results replicate the data over some time interval? We thus discuss the need to validate model results with time series data, identify a number of challenges that remain in this regard, and contrast results from one of the less stylistic models to the underlying data.

Static CGE analysis was made possible by advancements in solution techniques like Scarf's fixed point algorithm, and later, software to solve collections of non-linear algebraic equations. Another facilitating factor was the development of the social accounting matrix as a way to reconcile the national income and product accounts with input-output accounts.[1] These developments, along with the more recent development of the Global Trade, Assistance, and Production (GTAP) data base,[2] among

[1] See Robinson (1989) for a review of literature of static computable general equilibrium models and a discussion of the structure of a social accounting matrix.

[2] This data base is supported by the Center for Global Trade Analysis, Dept. of Agricultural Economics, Purdue University with funding from numerous international agencies and government sources. The earlier versions of the data base built heavily

others, have made possible a plethora of empirical general equilibrium models. The proliferation of these models has been accompanied by critiques that the models lack empirical realism and are based on ad hoc algebraic structures with little or no supporting conceptual framework. For these and other reasons, Kehoe (2003) suggests ex-post performance evaluations of such models are essential if policy makers are to have confidence in the results produced by them.

We see a potential parallel with this history and the next generation of dynamic models as a challenge faced by this book. Two sets of developments help us face this challenge. One is the recent advances in computer software such as Mathematica, Matlab, GAUSS and others, each having built-in subroutines for solving systems of differential equations. The other is the development of solution strategies like the time elimination method of Mulligan and Sala-i-Martin (1991, 1993), the relaxation algorithm implemented by Trimborn et al. (2008), and the backward integration method of Brunner and Strulik (2002).[3] These developments suggest the feasibility of moving beyond the static models, and the ability to not repeat past mistakes. Part of this problem is rectified by the requirement that the empirical model to draw directly upon the inter-temporal equilibrium conditions of the analytical model, thus avoiding some of the criticisms related to ad hoc model specifications. Another rectifying effect that is not as prominent in static models is the natural tendency to contrast a dynamic model's predictions with time series data, thus pressuring the development of validation procedures.

It is generally accepted that the accumulation of physical and human capital are important to the economic growth of economies, but these effects only explain part of the variation across countries in income per capita and differences in rates of growth. Including exogenous total factor productivity, as we do in the models developed in this book, greatly improves the

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on the SALTER Project which was undertaken at the Australian Industry Commission during the 1980s and early 1990s. See Badri and Walmsley (2008).

[3]The Handbook of Computational Economics, volume 2, edited by Tesfatsion and Judd (2006), provides a thorough discussion on the solution of saddlepoint problems.

model's fit to data, but this measure remains an "ignorance" type parameter. Roughly, economic growth models of recent vintage, e.g., Romer (1990), and those presented by Aghion and Howitt (1998, 2009), question factor accumulation as the main engine of growth, focusing instead on the notion that growth is primarily driven by innovations that are themselves the result of profit motivated activities. Nevertheless, these models tend to remain highly stylized, and not particularly useful for the analysis of foreign trade and other policies of common importance to policy makers. We suggest that the topics and procedures addressed in this book are important precursors to the future empirical application of endogenous growth models.

1.2 Organization of the book

The chapters are laid out as follows. Chapter 2 focuses on static microeconomic theory and trade theory of the small open economy. The first part sets out the microeconomic principles and conditions used throughout the book with emphasis on duality theory of the consumer and the firm. These microeconomic relationships are then aggregated to the economy level where we specify the Heckscher-Ohlin-Samuelson (HOS) two sector, two factor, small open economy model. The Stopler-Samuelson and Rybczynski theorems are shown and the point made that the characterization of equilibrium of this model, and the theorems play a role in helping us specify equilibrium conditions and interpret the transition dynamics of models developed in later chapters. The chapter concludes with a static model of a three sector, small and open economy that produces a non traded good. This model establishes the endogenous price of the home-good as a function of the prices of traded goods and factor endowments. These and all other models are presented using duality theory.

Chapter 3 introduces the two-sector Ramsey model of a closed economy in a series of steps. Each step helps acquaint the reader with: (i) the specification of household and firm optimization,

(ii) the normalization of variables into effective labor units, and (iii) the characterization of intra-temporal and inter-temporal equilibrium using duality in much the same manner as the HOS model. This pattern of model development is followed in all chapters. The chapter concludes with an algebraic example, and an empirical example. All of the empirical examples from Chapters 3, 4, 5, 6 and 7 are purposefully designed to link the empirical example of one to that of the other. Mathematica is used to obtain numerical solutions in all cases. The examples draw upon the same data of the Turkish economy for the year 2001, with the level of aggregation chosen to fit the structure of the analytical model in each chapter. The data and numerical procedures to support and illustrate how the models are fit to data and solved are presented in the last two chapters.

Chapter 4 presents the three sector model with two traded goods, one non-traded good and three factors of production. The intra-temporal equilibrium conditions for this model closely parallel the conditions of the static three sector model of Chapter 2, so a number of comparative static properties of the static model carry over to the dynamic model. The numerical example suggests that about 27 years are required for income per Turkish worker to double, Rybczynski-like effects of transition growth are shown as is the decline in the share of labor in agriculture and a number of other features of the Turkish economy.

Chapter 5 considers a number of extensions to the basic three sector model of Chapter 4, each of which are considered separately. These include intermediate inputs, vertical market structures, composite capital, and government revenues and expenditures. The composite capital specification is motivated by data which show that a country's stock of capital is a composite of the outputs of almost all sectors of the economy. The empirical example focuses on the intermediate input extension to the model where changes in the internal terms of trade during transition growth are shown to affect the traded goods sectors of the economy. Then, since the home-good is an important intermediate input to all sectors, a simulated productivity shock to this sector is shown to have multiplier effects on the traded good sectors that can be interpreted as releasing resources over time from home-good production to increase the country's gains from trade.

Chapter 6 incorporates all of the extensions of Chapter 4 into a single model. A number of complexities not encountered in Chapter 4 are encountered and addressed. The chapter also discusses a number of issues related to validating the empirical model. The model is solved both backward and forward from the base period 2001 and the results contrasted with time series data on Turkish gross domestic output and sectoral value-added for the period 1995–2006. A simulation is also performed which shows the effect on transition growth from lowering the tariff rate protecting the industrial sector.

Chapter 7 draws upon the model of Chapter 4 and casts it into a two-country-world in which each country produces two traded goods and their own respective non-traded good. Two models are considered. In the first, residents of one country can hold claims to capital stock in the other country over time. The second model disallows foreign ownership of domestic assets, in which case the model has two state variables, the capital stock in each country. To maintain a sense of symmetry with the previous numerical examples, we show how the capital stock of one country can be expressed as a function of the stock of the other country. This observation reduces our system to a single state variable. However, the number of differential equations characterizing inter-temporal equilibrium are increased to four, which nevertheless, can be numerically solved using the time elimination method. This chapter also concludes with a numerical example which focuses on the model permitting capital flows between countries.

Chapter 8 links the data to the model. We show how the Turkish data are organized into a social accounting matrix and how the parameters of the various empirical examples are estimated. The chapter also provides a definition of the economy's economic sectors.

Chapter 9 addresses the question of how to use numerical methods to solve the empirical models. The time elimination method receives considerable attention since this method is used in all of the empirical examples. We show, using Mathematica, a few coding “tricks” to facilitate solution and mention a number of other methods that can also be used to solve these models.

2

The Preliminaries

This chapter discusses features of static trade theory that are important components of the dynamic, multi-sector models developed in later chapters. Most of the notation used throughout the text is introduced and the style used to state the model's primitives, and to define and characterize equilibrium is presented.

The first section reviews key concepts and results from individual consumer and producer theory relevant to neoclassical trade theory. The exposition is simplified by assuming production technologies and preferences are differentiable and homothetic functions. Throughout the text we draw heavily upon the so called dual or indirect functions that characterize the constrained optimization behavior of individual agents. Readers interested in a more rigorous exposition of consumer theory should consult Cornes (1992) or Mas-Colell et al. (1995). A more rigorous treatment of producer theory can be found in Chambers (1988), and Fare and Grosskop (2004).

Using the concepts developed in Sections 1 and 2 introduces the Heckscher-Ohlin-Samuelson (HOS) model of a small open and competitive economy. The basic features of equilibrium and comparative statics as provided by the Stopler-Samuelson and Rybczynski theorems are discussed. Woodland (1982) provides an excellent characterization of this model. Section 3 considers, briefly, some further generalizations of the comparative statics of the HOS model. Section 4 concludes this chapter and presents a model of two traded goods, a home-good and three factors of production. A dynamic version of this model follows in later chapters.

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2.1 Microeconomic foundations

Throughout the text, the following notation denotes factor endowments, factor rental rates and output prices. Sectors are indexed by $j \in \{1, ..., M\}$, and denote the quantity of sector-j's output by the scalar Y_j. Corresponding output prices are denoted $\mathbf{p} = (p_1, ..., p_M) \in \mathbb{R}^M_{++}$, with the scalar p_j representing the per-unit price of sector-j output. We index factor endowments by $i \in \{1, ..., N\}$, and denote the economy's level of endowment i by the scalar V_i and the vector of factor endowments by $\mathbf{V} \equiv (V_1, ..., V_N) \in \mathbb{R}^N_{++}$. Corresponding factor rental rates are denoted $\mathbf{w} = (w_1, ..., w_N) \in \mathbb{R}^N_{++}$, with the scalar w_i representing the rental rate of factor V_i. For simplicity, outputs are often given a sector specific designation, such as agriculture, a, manufacturing, m, and the home-good, s. Likewise, endowments are often given designations like labor, L, capital, K, and land H.

2.1.1 Consumer preferences

The economy is composed of a large number of atomistic households. Each household faces the same vector of prices $\mathbf{p}$ and the same vector of factor rental rates $\mathbf{w}$. Let $\boldsymbol{v}^h = \left(v^h_1, ..., v^h_N\right) \in \mathbb{R}^N_{++}$ denote the level of factor endowments held by household-h, with v^h_i representing the household's endowment of factor i. In most applications that follow we suppress the h superscript of $\boldsymbol{v}^h$ and v^h, and use instead $\boldsymbol{v}$ and v_i. Given factor rental rates $\mathbf{w}$, the household's income is given by $\mathbf{w} \cdot \boldsymbol{v}$, which is used to purchase q_j units of consumption good j at market price p_j, $j = 1, ..., M$. Then, the household's budget constraint is given by

$$\mathbf{w} \cdot \boldsymbol{v} \geq \mathbf{p} \cdot \mathbf{q}$$

where $\mathbf{q} = (q_1, ..., q_M) \in \mathbb{R}^M_{++}$. In other words, each household consumes a strictly positive level of each consumption good.

Consumer preferences over goods are represented by the utility function $u : \mathbb{R}^M_{++} \rightarrow \mathbb{R}_+$, defined as $u(\mathbf{q})$.

Assumption 1 $u(\mathbf{q})$ *satisfies the following properties:*

1. $u(\mathbf{q})$ is increasing and strictly concave in $\mathbf{q}$,

2. $u(\mathbf{q})$ is everywhere continuous, and everywhere twice differentiable,

3. $u(\mathbf{q})$ is homothetic.

Assumption 1.1 yields indifference curves that are convex, Assumption 1.2 ensures Marshallian demands are continuous functions, while Assumption 1.3 yields Marshallian demands that are separable in prices and income.

Two indirect functions emerge from the consumer's problem: the indirect utility function and the expenditure function. The *indirect utility function* gives the household's maximum attainable utility given income $\mathbf{w} \cdot \boldsymbol{\upsilon}$, defined as

$$\mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \boldsymbol{\upsilon}) \equiv \max_{\mathbf{q}} \{u(\mathbf{q}) : \mathbf{w} \cdot \boldsymbol{\upsilon} \geq \mathbf{p} \cdot \mathbf{q}\}$$

The indirect utility function inherits the following properties from the direct utility function (see Cornes, pp. 67–70):

V1. Homogeneous of degree zero in $\mathbf{p}$ and $\mathbf{w} \cdot \boldsymbol{\upsilon}$; $\mathcal{V}(\theta\mathbf{p}, \theta\mathbf{w} \cdot \boldsymbol{\upsilon}) = \mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \boldsymbol{\upsilon}), \theta > 0$,

V2. $\mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \boldsymbol{\upsilon})$ is convex in $\mathbf{p}$,

V3. $\mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \boldsymbol{\upsilon})$ is continuous and differentiable in $\mathbf{p}$ and $\mathbf{w} \cdot \boldsymbol{\upsilon}$,

V4. $\mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \boldsymbol{\upsilon}) = v(\mathbf{p})\, \mathbf{w} \cdot \boldsymbol{\upsilon}$: separable in $\mathbf{p}$ and $\mathbf{w} \cdot \boldsymbol{\upsilon}$,

By V4, the marginal utility of an additional unit of income is $v(\mathbf{p})$.

V5. Given differentiability, Marshallian demands follow from Roy's identity,

$$q^j(\mathbf{p})(\mathbf{w} \cdot \boldsymbol{\upsilon}) = -\frac{v_{p_j}(\mathbf{p})}{v(\mathbf{p})} \mathbf{w} \cdot \boldsymbol{\upsilon} \tag{2.1}$$

where, throughout the text, the subscript on a function indicates a partial derivative, e.g., $v_{p_j} = \partial v(\mathbf{p}) / \partial p_j$ and $v_{p_1 p_2} = \partial^2 v(\mathbf{p}) / \partial p_1 \partial p_2$.

Since consumers face the same prices and have identical preferences, the "community" indirect utility function is given by

$$\mathcal{V} = v(\mathbf{p})(\mathbf{w} \cdot \mathbf{V})$$

while the total domestic Marshallian demand for good j is

$$Q_j = q^j(\mathbf{p})(\mathbf{w} \cdot \mathbf{V}), \ \forall\, j \in \{1, \cdot \cdot \cdot, M\} \tag{2.2}$$

These functions are the simple aggregation of individual consumer welfare and demands. It also follows from V1 that (2.2) is homogeneous of degree minus one in prices $\mathbf{p}$ and of degree one in income.

The *expenditure function* gives the minimum cost of achieving utility level $q \in \mathbb{R}$ at given prices $\mathbf{p}$, and is defined as

$$E(\mathbf{p}, q) \equiv \min_{\mathbf{q}} \{\mathbf{p} \cdot \mathbf{q} : q \leq u(\mathbf{q})\}$$

The expenditure function inherits from $u(\cdot)$, the following properties:

E1. $E(\mathbf{p}, q) > 0$ for any $\mathbf{p}$ and $q > 0$,

E2. $E(\mathbf{p}, q)$ is non-decreasing in $\mathbf{p}$ and q,

E3. $E(\mathbf{p}, q)$ is concave and continuous in $\mathbf{p}$,

E4. $E(\lambda\mathbf{p}, q) = \lambda E(\mathbf{p}, q), \ \lambda > 0$*:* homogeneous of degree 1 in $\mathbf{p}$,

E5. $E(\mathbf{p}, q) = \mathcal{E}(\mathbf{p})\, q$*:* separable in $\mathbf{p}$ and q,

E6. Shephard's lemma: If $E(\mathbf{p}, q)$ is differentiable in $\mathbf{p}$, then

$$q_j = E_{p_j}(\mathbf{p}, q) = \mathcal{E}_{p_j}(\mathbf{p})\, q, \ \ j = 1, ..., M$$

E1 says purchasing a strictly positive consumption bundle is costly. **E**2 says, all else equal, (i) if the price of a consumption good increases, then the cost of achieving the same level of utility increases, or (ii) increasing utility requires an increase in expenditures. By **E**3, the expenditure function is continuous and yields downward sloping Hicksian demand functions. Condition **E**4 implies demand functions are homogeneous of degree zero in $\mathbf{p}$. **E**5 results from Assumption 1.3 and implies demand functions are separable in $\mathbf{p}$ and q (see Chambers, 1988, Chapter 2). Later, we interpret the quantity q to be a composite consumption good, the unit cost of which is $\mathcal{E}(\mathbf{p})$.

2.1.2 Production technologies

Each sector j is composed of a large number of identical, atomistic firms. Each firm faces the same vector of input and output prices. Let y_j be the output of each firm in sector j and let $\boldsymbol{v}^j \equiv \left(v_1^j, ..., v_N^j\right) \in \mathbb{R}_+^N$ represent the vector of productive factors used by that firm, where v_i^j is the level of factor i used by the sector j firm. Represent the technology of a sector j firm by the production function $f^j : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$, defined as $y_j = f^j(\boldsymbol{v}^j)$.

Assumption 2 $f^j(\boldsymbol{v}^j)$ *satisfies the following properties:*

1. $f^j(\mathbf{0}) = 0$, and $f^j(\boldsymbol{v}^j) > 0$ for any $\boldsymbol{v}^j \gg \mathbf{0}^N$,
2. $f^j(\boldsymbol{v}^j)$ is linearly homothetic in $\boldsymbol{v}^j$,
3. $f^j(\boldsymbol{v}^j)$ is non-decreasing and strictly concave in $\boldsymbol{v}^j$,
4. $f^j(\boldsymbol{v}^j)$ is everywhere continuous and everywhere twice differentiable in $\boldsymbol{v}^j$.

Here $\mathbf{0}^N \in \mathbb{R}_+^N$ is a vector of N zeros and the notation $\boldsymbol{v}^j \gg \mathbf{0}$ means at least one element of $\boldsymbol{v}^j$ is strictly positive. Assumption 2.1 ensures it is not possible to produce a positive level of output with no input, and ensures there are no fixed costs. Assumption 2.2 says individual firm technologies satisfy constant returns to scale (CRS). An important implication of Assumption 2.2 is, when all firms face the same output and input prices, sectoral production levels and input demands are simple linear aggregations of individual firm choices. Another implication is the corresponding cost function is separable in input prices and output levels. Assumption 2.3 ensures the production technology is well-behaved and yields the familiar convex isoquants: it imposes diminishing marginal returns on individual input use. Assumptions 2.1 and 2.3 also ensure the existence of a cost and aggregate value-added (GDP) function defined later. Finally, Assumption 2.4 allows the use of differential calculus to derive corresponding cost and GDP functions.

Two indirect functions associated with the producer's problem are the cost function and the sectoral value-added function. The *cost function* is defined as:

$$c^j(\mathbf{w}, y_j) \equiv \min_{\boldsymbol{v}^j} \left\{ \mathbf{w} \cdot \boldsymbol{v}^j : y_j \leq f^j\left(\boldsymbol{v}^j\right)\right\}, \ j = 1, 2, ..., M$$

and is the firm's analog of the household's expenditure function. It inherits from Assumption 2, the following properties:

C1. $c^j(\mathbf{w}, y_j) > 0$ for any $\mathbf{w}$ and $y_j > 0$,

C2. $c^j(\mathbf{w}, y_j)$ is non-decreasing in $\mathbf{w}$ and y_j,

C3. $c^j(\mathbf{w}, y_j)$ is concave and continuous in $\mathbf{w}$,

C4. $c^j(\theta\mathbf{w}, y_j) = \theta c^j(\mathbf{w}, y_j)$: homogeneous of degree one in $\mathbf{w}$,

C5. $c^j(\mathbf{w}, y_j) = C^j(\mathbf{w})\, y_j$: separable in $\mathbf{w}$ and y_j,

where $C^j(\mathbf{w})$ is the unit cost of producing output j. Finally, we have

C6. Shephard's lemma: If $c^j(\mathbf{w}, y_j)$ is differentiable in $\mathbf{w}$, then

$$v_i^j = C_{w_i}^j(\mathbf{w})\, y_j, \ i = 1, ..., N,$$

where $C_{w_i}^j(\cdot)$ is the derived unit demand for input i from sector j.

C1 says producing a strictly positive level of output is costly. **C**2 says, all else equal, if the price of an input increases production cost increases, or increasing output increases production costs. By **C**3, the cost function is continuous and yields conditional input demand functions that are decreasing in own prices. Condition **C**4 implies input demand functions are homogeneous of degree zero in $\mathbf{w}$. **C**5 results from Assumption 2.2 and implies constant marginal and average costs. Furthermore, given **C**5, the output supply and input demand functions are both separable in $\mathbf{w}$ and y_j (see Chambers, 1988, Chapter 2).

Since all firms in a sector employ the same technology and face the same output and input prices, characterizing the aggregate technology for the sector is straightforward. Let $\mathbf{V}^j \equiv \left(V_1^j, ..., V_N^j\right) \in \mathbb{R}_+^N$ denote the vector of factors employed in producing output Y_j, where V_i^j is the aggregate level of factor i used by sector j firms. While the total number of firms in a sector are indeterminate, their identical nature implies if each firm produces a share, Υ_j^o, of total sectoral output j, then the firm also employs the same $\bar{\Upsilon}_j$ share of factor inputs, i.e.,

$$\begin{aligned} y_j^o &= \Upsilon_j^o Y_j, \\ v_i^o &= \Upsilon_j^o V_i^j, \ \forall i \in \{1, \cdots, N\} \end{aligned}$$

Hence, the sector level production function is a linear expansion of individual firm production functions. That is,

$$\Upsilon_j^o Y_j = f^j\left(\boldsymbol{v}^j\right) = f^j\left(\Upsilon_j^o \mathbf{V}^j\right)$$

which implies the sector level production function is

$$Y_j = f^j\left(\mathbf{V}^j\right)$$

To distinguish between firm level and aggregate sectoral production however, it is convenient to represent the aggregate technology for sector j by the production function $\mathcal{F}^j : \mathbb{R}_+^N \to \mathbb{R}_+$, defined as

$$Y_j = \mathcal{F}^j\left(\mathbf{V}^j\right) \tag{2.3}$$

Then, the corresponding sectoral cost function, denoted TC_j, is given by

$$TC_j = C^j\left(\mathbf{w}\right) Y_j \tag{2.4}$$

The economy-wide gross national product function is obtained by maximizing aggregate sectoral income subject to the technology (2.3) and the endowment constraints. In this case we have:

$$G(\mathbf{p},\mathbf{V}) \equiv \max_{\mathbf{V}^1,\ldots,\mathbf{V}^M} \left\{ \sum_{j=1}^{M} p_j \mathcal{F}^j\left(\mathbf{V}^j\right) : V_i \geq \sum_{j=1}^{M} V_i^j,\ i = 1,\ldots,M \right\} \tag{2.5}$$

Woodland (1982, p. 123) shows the function $G(\cdot)$ satisfies the following properties:

G1. $G(\mathbf{p},\mathbf{V}) \geq 0$ for all $\mathbf{p}$ and $\mathbf{V}$,

G2. $G(\lambda\mathbf{p},\mathbf{V}) = \lambda G(\mathbf{p},\mathbf{V})$, $\lambda > 0$: linearly homogeneous in $\mathbf{p}$,

G3. $G(\mathbf{p},\lambda\mathbf{V}) = \lambda G(\mathbf{p},\mathbf{V})\ \lambda > 0$: linearly homogeneous in $\mathbf{V}$,

G4. $G(\mathbf{p},\mathbf{V})$ is continuous, non-decreasing, and convex in $\mathbf{p}$,

G5. $G(\mathbf{p},\mathbf{V})$ is continuous, non-decreasing, and concave in $\mathbf{V}$,

G6. Hotelling's lemma. If $G(\cdot)$ is everywhere differentiable in $\mathbf{p}$ and $\mathbf{V}$, then

$$\begin{aligned} Y_j &= G_{p_j}(\mathbf{p},\mathbf{V}) \\ w_i &= G_{V_i}(\mathbf{p},\mathbf{V}) \end{aligned}$$

The major implications of conditions G1 – G6 are that the gradients of $G(\cdot)$ yield aggregate sectoral supply functions, $G_{p_j}(\mathbf{p},\mathbf{V})$, that are non-decreasing in own-price, homogeneous of degree zero in prices $\mathbf{p}$, and homogeneous of degree one in endowments $\mathbf{V}$. The inverse factor demand functions $G_{V_i}(\mathbf{p},\mathbf{V})$ are downward sloping in own factor levels, homogeneous of degree one in prices and homogeneous of degree zero in endowments. The Hessian matrix of $G(\mathbf{p},\mathbf{V})$ is positive semi-definite.[1]

[1] Young's theorem implies that the second derivative matrix of $G(\mathbf{p},\mathbf{V})$ is symmetric, $G_{p\mathbf{V}}(\cdot) = G_{\mathbf{V}p}(\cdot)$. Thus, an increase in w_i due to a unit increase in p_j is equal to the increase in Y_j due to an increase in v_i. See Diewert (1973, 1974).

It is also convenient to specify a sectoral value-added function. For many of the models developed in this text, at least one productive sector is endowed with a factor specific to its production process. For example, we typically model land as a factor used only in producing agricultural products. Farmers can rent land in and out among themselves at some market determined land rental rate, but they do not rent land to producers in other sectors of the economy. Since each farmer's production function satisfies Assumption 2, there exists a corresponding sectoral agricultural production and cost function (2.3) and (2.4). However, in the case of the sectoral production function, the sector specific factor is pre-determined or fixed, and the sectoral level production function exhibits decreasing returns to scale in all the other factors employed in other sectors of the economy. This property gives rise to a sector level value-added function.

More formally, divide the input vector $\mathbf{V}^j$ into two subvectors, a vector of variable inputs and a vector of sector specific inputs. Let the first ς_j factors be variable and the remaining $N - \varsigma_j$ factors be sector specific. Denote the vector of variable factors by $\mathbf{V}^j = \left(V_1^j, ..., V_{\varsigma_j}^j\right) \in \mathbb{R}_+^{\zeta_j}$, and denote the vector of sector specific factors by $\bar{\mathbf{V}}^j = \left(\bar{V}_{\zeta_j+1}^j, ..., \bar{V}_N^j\right) \in \mathbb{R}_+^{N-\zeta_j}$. With fixed factors $\bar{\mathbf{V}}^j$, the j^{th} sector value-added function can be defined as:

$$\Pi^j\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) \equiv$$

$$\max_{y_j, \mathbf{V}^j} \left\{p_j Y_j - \mathbf{w} \cdot \left(\mathbf{V}^j, \mathbf{0}^{N-\zeta_j}\right) : Y_j \geq \mathcal{F}^j\left(\mathbf{V}_\varsigma^j, \bar{\mathbf{V}}^j\right)\right\} \tag{2.6}$$

where $\mathbf{0}^{N-\zeta_j} \in \mathbb{R}_+^{N-\zeta_j}$ is a vector of zeros. Given Assumption 2, the sectoral value-added function properties include:

Π1. $\Pi^j\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) \geq 0$ for all $p_j, \mathbf{w}$, and $\bar{\mathbf{V}}^j$,

Π2. $\Pi^j\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right)$ is nondecreasing in p_j and nonincreasing in $\mathbf{w}$,

Π3. $\lambda\Pi^j\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) = \Pi^j\left(\lambda p_j, \lambda\mathbf{w}, \bar{\mathbf{V}}^j\right), \lambda > 0$: linearly homogeneous in p_j and $\mathbf{w}$, Π4. $\lambda\Pi^j\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) = \Pi^j\left(p_j, \mathbf{w}, \lambda\bar{\mathbf{V}}^j\right)$, $\lambda > 0$: linearly homogeneous in $\bar{\mathbf{V}}^j$

Π5. $\Pi^j\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) = \pi^j\left(p_j, \mathbf{w}\right)\Phi\left(\bar{\mathbf{V}}^j\right)$: separable in fixed endowments,

Π6. Hotelling's lemma. If $\Pi^j(\cdot)$ is everywhere differentiable in $\mathbf{p}$, $\mathbf{w}$ and $\bar{\mathbf{V}}^j$, then sectoral supply Y_j and sectoral factor demand V_i^j are, respectively,

$$\begin{aligned} Y_j &= \Pi^j_{p_j}\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) \\ V_i^j &= -\Pi^j_{\tilde{w}_i}\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) \end{aligned}$$

The factor rental rate (or shadow price) of the sector specific factors is given by

$$w_i^j = \Pi^j_{\bar{V}_i^j}\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right)$$

For the case of a single sector specific factor, say land, that is rented in or out among farmers, $\pi^j\left(p_j, \mathbf{w}\right)$ is the rental rate that clears the land rental market. Moreover, it can be shown that the output price gradient of the economy-wide GDP function yields the same level of supply as the corresponding output price gradient of the sector value-added function,

$$Y_j = G_{p_j}\left(\mathbf{p}, \mathbf{V}\right) = \Pi^j_{p_j}\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right)$$

where the gradients are evaluated at values $(\mathbf{p}, \mathbf{w}, \mathbf{V})$ yielding an equilibrium to the economy. This property is particularly useful for decomposing effects into direct and indirect. For instance, the direct effect of a price change on Y_j is $\partial\Pi^j_{p_j}/\partial p_j$ while the indirect effects are transmitted through factor markets and are given by $\left(\partial\Pi^j_{p_j}/\partial w_i\right)\left(\partial w_i/\partial p_j\right)$. Together, they equal the total effect which can be shown to equal $\partial G_{p_j}\left(\mathbf{p}, \mathbf{V}\right)/\partial p_j$.

2.2 The Heckscher-Ohlin-Samuelson model

The optimizing behavior of producers and consumers embodied in expressions (2.2) and (2.4) provide the building blocks for specifying the well known Heckscher-Ohlin-Samuelson (HOS) model. The economy is small, open and competitive, endowed with two factors, and produces two outputs. Denote the endowment vector by $\mathbf{V} = (L, K)$, and interpret K as units of physical capital and interpret L as units of labor. Neither endowment is traded internationally. A main feature of the model is that the number of traded goods equal the number of factors, $M = N$.

2.2.1 The behavior of households

The individual household[2] is endowed with resources $\boldsymbol{\upsilon} = (\ell, k) \in \mathbb{R}^2_{++}$, where ℓ and k denote labor and capital, respectively. The household provides the services of these resources to firms in return for wages, w, and capital rents, r, yielding income $w\ell + rk$.

Given prices (p_1, p_2), the household's budget constraint is

$$w\ell + rk \geq p_1 q_1 + p_2 q_2$$

Consumer preferences are given by the utility function $u(q_1, q_2)$ satisfying Assumption 1. Consequently, the consumer's optimization problem yields the indirect utility function

$$\mathrm{v}(p_1, p_2)(w\ell + rk) \equiv \max_{q_1, q_2} \{u(q_1, q_2) : w\ell + rk \geq p_1 q_1 + p_2 q_2\}$$

where $\mathrm{v}(p_1, p_2)(w\ell + rk)$ satisfies properties V1 – V4. The corresponding Marshallian demands are ,

$$q^j(p_1, p_2)(w\ell + rk) = -\frac{\mathrm{v}_{p_j}(p_1, p_2)}{\mathrm{v}(p_1, p_2)}(w\ell + rk), \; j = 1, 2$$

Since consumers face the same prices and have identical preferences, the community indirect utility function is

$$\mathcal{V} = \mathrm{v}(p_1, p_2)(wL + rK)$$

while aggregate domestic Marshallian demand for good j is

$$Q_j = q^j(p_1, p_2)(wL + rK), \; j = 1, 2 \tag{2.7}$$

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[2] The term household is used instead of the consumer to reinforce the point that resource endowments are not owned by firms.

Given homothetic preferences, the community indirect utility function and Marshallian demands are simple linear aggregates of individual consumer welfare and demand. The marginal utility of income $v(p_1, p_2)$, and the good specific income effect $q^j(p_1, p_2)$ are common to all households.

2.2.2 *The price taking firm*

As in Section 2.1.2, both sectors are composed of a large number of identical, atomistic firms. All firms face the same input and output prices. Let y_j be the output of a firm in sector j and let $\boldsymbol{v}^j = (\ell_j, k_j) \in \mathbb{R}^2_{++}$ represent the level of labor ℓ_j and capital k_j employed by the firm. The technology for sector $j = 1, 2$ is represented by the production function $f^j : \mathbb{R}^2_{++} \to \mathbb{R}_+$, defined as $y_j = f^j(\ell_j, k_j)$, where $f^j(\cdot)$ satisfies Assumption 2. Recall from the previous discussion that production of either output requires a strictly positive level of capital and labor.

Inputs are chosen to maximize profits. Each firm can be viewed as maximizing profits in two steps. First, it chooses the input bundle (ℓ_j, k_j) that minimizes the cost of producing y_j units of output. The corresponding cost function is given by

$$C^j(w, r)\, y_j \equiv \min_{\ell_j, k_j} \left\{ w\ell_j + rk_j : y_j \le f^j(\ell_j, k_j) \right\}, \ \ j = 1, 2$$

and satisfies conditions **C**1 – **C**6. In the second step, given the cost function $C^j(\cdot)\, y_j$, the firm solves the optimization problem

$$\Pi^j(p_j, w, r) \equiv \max_{y_j} \left\{ p_j y_j - C^j(\cdot)\, y_j \right\}$$

The optimal choice of y_j must satisfy the following complementary slackness condition

$$y_j \ge 0; \ p_j - C^j(\cdot) \le 0; \ \text{and} \ \left[p_j - C^j(\cdot) \right] y_j = 0$$

Hence, in a competitive equilibrium only zero profits are possible.