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Data Analysis and Related Applications 4

New Approaches

Edited by
Yiannis Dimotikalis
Christos H. Skiadas

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Contents

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On the First-Passage Area of a One-Dimensional Diffusion Process with Stochastic Resetting

For a one-dimensional diffusion process with stochastic resetting $\mathcal{X}(t)$, obtained from an underlying diffusion $X(t)$, we study the statistical properties of its first-passage time through zero, when starting from $x > 0$, and its first-passage area, i.e. the random area swept out by $\mathcal{X}(t)$ until its first-passage time through zero. By making use of solutions of certain associated ODEs, we find explicit expressions for the Laplace transforms of the first-passage time and the first-passage area, and their single and joint moments.

1.1. Formulation of the problem and general results

This note deals with the first-passage area (FPA) of a diffusion process with stochastic resetting. It is a continuation of Abundo (2013, 2023b), Abundo and Del Vescovo (2017) and Abundo and Furia (2019), regarding the FPA of jump-diffusions, drifted Brownian motion, Lèvy process and Ornstein–Uhlenbeck process. In fact, here we considered a one-dimensional diffusion process in the presence of stochastic resetting $\mathcal{X}(t)$, obtained from an underlying diffusion $X(t)$; this kind of process is treated, for example, in den Hollander et al. (2019) and Evans et al. (2020) (see also the references in Abundo 2023a). We studied the statistical properties of the first-passage time (FPT) through zero of $\mathcal{X}(t)$, starting from $x > 0$, and its FPA, namely the random area swept out by $\mathcal{X}(t)$ until its FPT through zero. In some cases, we explicitly obtained the Laplace transform of the FPT and FPA, and their single and joint moments. Moreover, we provided the distribution of the maximum displacement of $\mathcal{X}(t)$.

Chapter written by Mario ABUNDO.

In the case that the underlying diffusion $X(t)$ is Brownian motion without drift, the FPA was already studied in Singh and Pal (2022), although the results found therein were obtained using special functions. In contrast, here we used nothing but elementary functions: our arguments were based on classical results for one-dimensional diffusions. In fact, the study of the distributions of the FPT and FPA was carried out via solutions of certain associated ODEs. We focused on the case when the underlying diffusion $X(t)$ is a Wiener process, i.e. a Brownian motion with or without drift, but the results can be extended to other processes.

The FPT and FPA of a diffusion process with stochastic resetting have important applications in many areas, for example, in biology, in the ambit of stochastic models for the activity of a neuron subject to resetting (see, for example, Nobile et al. 1985 and the references contained therein). Other important applications are found in queuing theory, where the first hitting time to zero can be identified with the busy period, i.e. the first instant at which the queue is empty, and the FPA is the total waiting time of the "customers" during a busy period (see, for example, Kearney 2004).

Now, we precisely describe the process $\mathcal{X}(t)$ with stochastic resetting.

We consider a one-dimensional temporally homogeneous diffusion process $X(t)$ driven by the SDE:

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB_t, \qquad [1.1]$$

and starting from the position $X(0) = x > 0$, where the drift $\mu(\cdot)$ and diffusion coefficient $\sigma(\cdot)$ are regular enough functions, such that there exists a unique strong solution of the SDE [1.1] for a given starting point, and B_t is the standard Brownian motion. We also assume that the FPT of the diffusion $X(t)$ below zero is finite with probability one.

We assume that resetting events can occur according to a homogeneous Poisson process with rate $r > 0$. Until the first resetting event $\mathcal{R}$, the process $\mathcal{X}(t)$ coincides with $X(t)$, and it evolves according to [1.1] with $\mathcal{X}(0) = X(0) = x > 0$. When the reset occurs, $\mathcal{X}(t)$ is set instantly to a position $x_R > 0$. After that, $\mathcal{X}(t)$ evolves again according to [1.1], starting afresh (independently of the past history) from x_R, until the next resetting event occurs, etc. The inter-resetting times turn out to be independent and exponentially distributed random variables with parameter r. In other words, in any time interval $(t, t + \Delta t)$, with $\Delta t \to 0^+$, the process can pass from $\mathcal{X}(t)$ to the position x_R with probability $r\Delta t + o(\Delta t)$, or it can continue its evolution according to [1.1] with probability $1 - r\Delta t + o(\Delta t)$. The process $\mathcal{X}(t)$ so obtained is called diffusion with stochastic resetting. For any C^2 function $f(x)$, its infinitesimal generator is given by (see, for example, Abundo 2013):

$$\begin{aligned}\mathcal{L}f(x) &= \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) + r(f(x_R) - f(x)) \\ &\equiv Lf(x) + r(f(x_R) - f(x)),\end{aligned} \qquad [1.2]$$

where $Lf(x) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x)$ is the "diffusion part" of the generator, i.e. regarding the diffusion process $X(t)$.

For an initial position $x > 0$, we are concerned with the FPT of $\mathcal{X}(t)$ through zero, namely:

$$\tau(x) = \inf\{t > 0 : \mathcal{X}(t) = 0 \mid \mathcal{X}(0) = X(0) = x\}, \tag{1.3}$$

and the corresponding FPA

$$A(x) = \int_0^{\tau(x)} \mathcal{X}(t)dt, \tag{1.4}$$

which is the area enclosed between the time axis and the path of the process $\mathcal{X}(t)$ up to the FPT through zero. We assume that both $\tau(x)$ and $A(x)$ are finite with probability one, for any $x > 0$. Note that

$$\tau(x) = \tau_X(x), \text{ if } \tau_X(x) < \sigma; \ \tau(x) = \sigma + \tau(x_R), \text{ if } \tau_X(x) \geq \sigma, \tag{1.5}$$

where $\tau_X(x)$ is the first-hitting time to zero of $X(t)$ starting from $x > 0$ and σ is an exponentially distributed random variable with parameter $r > 0$.

In fact, we limit ourselves to study the case when the underlying process $X(t)$ is a Wiener process, i.e. Brownian motion with or without drift.

The main qualitative difference between the FPT of the process $\mathcal{X}(t)$ and the FPT of the underlying diffusion $X(t)$ is that, for the process $\mathcal{X}(t)$, the moments of the FPT are finite, while for the second one, they may be infinite. This is, for example, the case of Brownian motion starting from $x > 0$, where, as is well known, the first-hitting time to zero is finite with probability 1, but it has infinite expectation.

For $\lambda > 0$, let us consider the Laplace transform (LT) of $\int_0^{\tau(x)} U(\mathcal{X}(s))ds$, $U(x) = ax + b$ $(a, b \geq 0)$ being a polynomial of degree one, i.e.

$$M_\lambda(x) = E\left[e^{-\lambda \int_0^{\tau(x)} U(\mathcal{X}(s))ds}\right]. \tag{1.6}$$

Taking $U(x) = 1$, we obtain the LT of the FPT $\tau(x)$, while for $U(x) = x$, we get the LT of the FPA $A(x)$. The following holds (see Abundo 2023a):

PROPOSITION 1.1.– *The LT $M_\lambda(x)$ of $\int_0^{\tau(x)} U(\mathcal{X}(s))ds$ satisfies the differential problem:*

$$\begin{cases} LM_\lambda(x) - \lambda U(x)M_\lambda(x) - rM_\lambda(x) + rM_\lambda(x_R) = 0, \\ M_\lambda(0) = 1, \\ \lim_{x\to+\infty} M_\lambda(x) = M_\lambda(+\infty) < \infty, \end{cases} \tag{1.7}$$

where L denotes the infinitesimal generator of the underlying diffusion $X(t)$, which is given, for any C^2 function f, by

$$Lf(x) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x), \tag{1.8}$$

and f' and f'' denote the first and second derivatives of f. □

REMARK 1.1.– *Proposition 1.1 was already proved in Singh and Pal (2022) in the case when $X(t)$ is Brownian motion. Note that, for $r = 0$ (i.e. when no resetting is allowed), we obtain equation (2.12) of Abundo (2013), provided that the jump part in the infinitesimal generator is set to zero, while the second boundary condition is $M_\lambda(+\infty) = 0$.*

If the $n-$th order moment of $\int_0^{\tau(x)} U(\mathcal{X}(s))ds$ exists finite, it is provided by:

$$T_n(x) := E\left[\left(\int_0^{\tau(x)} U(X(s))ds\right)^n\right] = (-1)^n \left[\frac{\partial^n}{\partial\lambda^n} M_\lambda(x)\right]_{\lambda=0}, n = 1, 2, \ldots. \tag{1.9}$$

By calculating the $n-$th derivative with respect to λ, at $\lambda = 0$, of both members of [1.7], we obtain that, setting $T_0(x) = 1$, the $n-$th order moments $T_n(x)$ satisfy the ODEs:

$$LT_n(x) - rT_n(x) = -nU(x)T_{n-1}(x) - rT_n(x_R), \; x > 0, \tag{1.10}$$

with the constraint $T_n(0) = 0$ and the addition of an appropriate further condition (indeed, we need two conditions to obtain the unique solution of [1.10]). Note that for $r = 0$, [1.10] becomes equation (2.19) of Abundo (2013). In particular, for $U(x) \equiv 1$, [1.10] is nothing but the celebrated Darling and Siegert's equation (1953) for the moments of the FPT of a diffusion without resetting.

As regards the joint moments of $\tau(x)$ and $A(x)$, we consider the joint LT of $\tau(x)$ and $A(x)$, i.e. $E[e^{-\lambda_1\tau(x)-\lambda_2 A(x)}]$ ($\lambda_i > 0$):

$$M_{\lambda_1,\lambda_2}(x) = E\left[\exp\left(-\int_0^{\tau(x)} (\lambda_1 + \lambda_2\mathcal{X}(t))dt\right)\right]. \tag{1.11}$$

As easily seen, we get:

$$\frac{\partial M_{\lambda_1,\lambda_2}(x)}{\partial\lambda_1}|_{\lambda_1=\lambda_2=0} = -E[\tau(x)], \; \frac{\partial M_{\lambda_1,\lambda_2}(x)}{\partial\lambda_2}|_{\lambda_1=\lambda_2=0} = -E[A(x)], \tag{1.12}$$

and

$$\frac{\partial^2 M_{\lambda_1,\lambda_2}(x)}{\partial\lambda_1\partial\lambda_2}|_{\lambda_1=\lambda_2=0} = E[\tau(x)A(x)]. \qquad [1.13]$$

Applying the same reasoning as before and taking $U(x) = \lambda_1 + \lambda_2 x$, we obtain that $M_{\lambda_1,\lambda_2}(x)$ solves the problem

$$\begin{cases} LM_{\lambda_1,\lambda_2}(x) - rM_{\lambda_1,\lambda_2}(x) = (\lambda_1 + \lambda_2 x)M_{\lambda_1,\lambda_2}(x) - rM_{\lambda_1,\lambda_2}(x_R) \\ M_{\lambda_1,\lambda_2}(0) = 1 \\ \lim_{x\to+\infty} M_{\lambda_1,\lambda_2}(x) = M_{\lambda_1,\lambda_2}(+\infty) < \infty. \end{cases} \qquad [1.14]$$

Then, by applying $\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}$ and calculating it for $\lambda_1 = \lambda_2 = 0$, we obtain that $V(x) := E[\tau(x)A(x)]$ is the solution of the problem:

$$\begin{cases} LV(x) - rV(x) = -xE[\tau(x)] - E[A(x)] - rV(x_R) \\ V(0) = 0, \end{cases} \qquad [1.15]$$

with a suitable additional condition.

Note that for $r = 0$, [1.15] becomes the analogous equation, respectively, obtained in Abundo and Del Vescovo (2017) and Abundo (2023b), in the case of drifted Brownian motion and Ornstein–Uhlenbeck process without resetting. Of course, now the boundary conditions are different.

1.2. Brownian motion with resetting

In this section, we consider Brownian motion with resetting $\mathcal{X}(t)$, and we find explicit expressions of the LT and single and joint moments of the FPT and FPA, as solutions of certain differential problems. Note that for $r = 0$ the moments of the FPT and FPA are infinite. This follows by the fact that the FPT density of Brownian motion decays as $t^{-3/2}$ at large time t. In contrast, both the FPT density and the FPA density of Brownian motion with resetting decay exponentially fast at large values (see, for example, Abundo 2023a and the references therein); and so, the moments of the FPT and FPA are finite, for $r > 0$.

Since the underlying process of Brownian motion with resetting is $X(t) = B_t$, for any C^2 function $f(x)$, the infinitesimal generator L of $X(t)$ is given by $Lf(x) = \frac{1}{2}f''(x)$ and from [1.7], we get that the LT of $\int_0^{\tau(x)} U(\mathcal{X}(s))ds$ solves the problem:

$$\begin{cases} \frac{1}{2}\frac{\partial^2 M_\lambda}{\partial x^2}(x) - \lambda U(x)M_\lambda(x) - rM_\lambda(x) + rM_\lambda(x_R) = 0, \\ M_\lambda(0) = 1, \\ \lim_{x\to+\infty} M_\lambda(x) = M_\lambda(+\infty) < \infty, \end{cases} \qquad [1.16]$$

Taking $U(x) = 1$, we obtain that the LT of the FPT $\tau(x)$ is the solution of:

$$\begin{cases} \frac{1}{2}\frac{\partial^2 M_\lambda}{\partial x^2}(x) - (\lambda + r)M_\lambda(x) + rM_\lambda(x_R) = 0, \\ M_\lambda(0) = 1, \\ \lim_{x\to+\infty} M_\lambda(x) = M_\lambda(+\infty) < \infty, \end{cases} \qquad [1.17]$$

By solving, we explicitly obtain (see Abundo 2023a for details) the LT of $\tau(x)$:

$$M_\lambda(x) = E\left[e^{-\lambda\tau(x)}\right] =$$

$$= e^{-x\sqrt{2(\lambda+r)}} + \frac{r}{\lambda + re^{-x_R\sqrt{2(\lambda+r)}}}\left(e^{-x_R\sqrt{2(\lambda+r)}} - e^{-(x+x_R)\sqrt{2(\lambda+r)}}\right). \qquad [1.18]$$

REMARK 1.2.– *Formula* [1.18] *extends to all $x > 0$ equation (44) of Singh and Pal (2022) that holds for $x = x_R$. For $r = 0$, namely when no resetting occurs,* [1.18] *provides $E[e^{-\lambda\tau(x)}] = e^{-x\sqrt{2\lambda}}$, i.e. the well-known formula of the LT of the FPT through zero of Brownian motion starting from $x > 0$; this LT corresponds to the inverse Gaussian density for the FPT.*

As regards the LT of $A(x)$, by taking $U(x) = x$ in [1.16], we obtain that the LT of $A(x)$ is the solution of:

$$\begin{cases} \frac{1}{2}\frac{\partial^2 M_\lambda}{\partial x^2}(x) - rM_\lambda(x) = \lambda x M_\lambda(x) - rM_\lambda(x_R), \\ M_\lambda(0) = 1, \\ \lim_{x\to+\infty} M_\lambda(x) = M_\lambda(+\infty) < \infty, \end{cases} \qquad [1.19]$$

Unfortunately, equation [1.19] cannot be solved in terms of elementary functions. For $x = x_R$, its solution, $M_\lambda(x_R) = E[e^{-\lambda A(x_R)}]$, can be written in terms of special functions, precisely Scorer's and Airy functions (see Singh and Pal 2022). For $r = 0$, i.e. for Brownian motion without resetting, equation [1.19] is the Schrodinger equation for a quantum particle moving in a uniform field and it can be solved in terms of the Airy function.

1.2.1. *Moments of the FPT*

Now we calculate the first two moments of $\tau(x)$ by solving the ODE [1.10] with $U(x) = 1$, and $T_n(x) = E[\tau^n(x)], n = 1, 2$.

As for the mean of $\tau(x)$, taking $n = 1$, we have that $T_1(x) = E[\tau(x)]$ is the solution of the problem:

$$\begin{cases} \frac{1}{2}T_1''(x) - rT_1(x) = -1 - rT_1(x_R) \\ T_1(0) = 0, \ T_1(+\infty) < +\infty, \end{cases} \qquad [1.20]$$

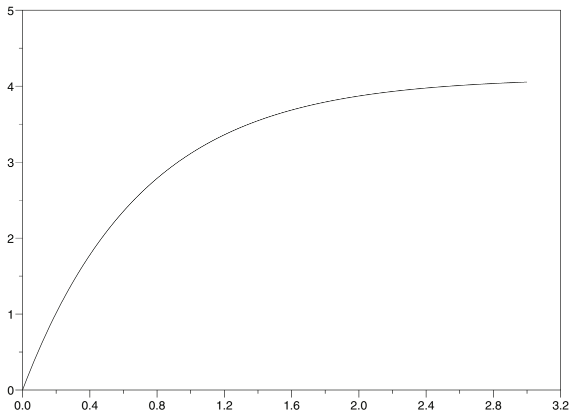

Figure 1.1. *Graph of $E[\tau(x)]$, given by [1.21], as a function of $x > 0$ for $r = x_R = 1$ (on the horizontal axes x)*

The appropriate additional condition is $T_1(\infty) < \infty$. By solving, we get (see Abundo (2023a) for details):

$$T_1(x) = E[\tau(x)] = \frac{1}{r} e^{x_R\sqrt{2r}} \left(1 - e^{-x\sqrt{2r}}\right), \qquad [1.21]$$

and for $x = x_R$:

$$T_1(x_R) = \frac{1}{r}(e^{x_R\sqrt{2r}} - 1), \qquad [1.22]$$

Note that $E[\tau(+\infty)] = \frac{1}{r} e^{x_R\sqrt{2r}} < +\infty$.

REMARK 1.3.– *Formula* [1.21] *extends to all $x > 0$ equation (45) of Singh and Pal (2022) that provides $E[\tau(x_R)]$ (see* [1.22]*). Note that, letting r go to zero in equation* [1.21]*, it follows $E[\tau(x)] = +\infty$, which matches the well-known result for Brownian motion (see, for example, Abundo and Del Vescovo 2017).*

Figure 1.1 shows an example of the shape of $E[\tau(x)]$, given by [1.21], as a function of $x > 0$ for $r = x_R = 1$.

As regards the second-order moment of $\tau(x)$, by taking $n = 2$ and $U(x) = 1$ in [1.10], we get that $T_2(x) = E[\tau^2(x)]$ is the solution of the problem:

$$\begin{cases} \frac{1}{2}T_2''(x) - rT_2(x) = -\frac{2}{r}e^{x_R\sqrt{2r}}(1 - e^{-x\sqrt{2r}}) - rT_2(x_R) \\ T_2(0) = 0,\ T_2(+\infty) < +\infty. \end{cases} \qquad [1.23]$$

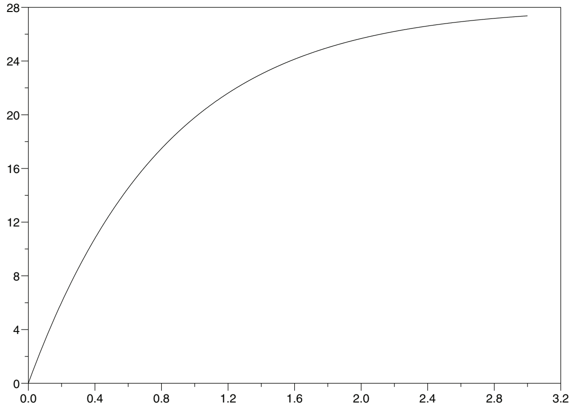

Figure 1.2. *Graph of $E[\tau^2(x)]$ as a function of $x > 0$ for $r = x_R = 1$ (on the horizontal axes x)*

By solving (see Abundo 2023a for details), we get:

$$T_2(x) = E[\tau^2(x)] = e^{x_R\sqrt{2r}} \left(\frac{2}{r^2} e^{x_R\sqrt{2r}} - \frac{2}{r^2} - \frac{2x_R}{r\sqrt{2r}} \right) \left(1 - e^{-x\sqrt{2r}} \right)$$

$$-\frac{2}{r} e^{(x_R - x)\sqrt{2r}} \left(\frac{1}{r} + \frac{x}{\sqrt{2r}} \right) + \frac{2}{r^2} e^{x_R\sqrt{2r}}, \qquad [1.24]$$

and for $x = x_R$:

$$E[\tau^2(x_R)] = e^{x_R\sqrt{2r}} \left(\frac{2}{r^2} e^{x_R\sqrt{2r}} - \frac{2}{r^2} - \frac{2x_R}{r\sqrt{2r}} \right). \qquad [1.25]$$

Figure 1.2 shows an example of the shape of $E[\tau^2(x)]$, given by [1.24], as a function of $x > 0$ for $r = x_R = 1$.

As regards the behaviors of the first two moments of the FPT $\tau(x)$ for $x \to 0^+$ and $x \to +\infty$, from [1.21] we get $E[\tau(x)] = a_1 x + o(x),$ as $x \to 0^+$, where the constant $a_1 = \frac{\sqrt{2r}}{r} e^{x_R\sqrt{2r}}$ depends on r and x_R, and $\lim_{x\to+\infty} E[\tau(x)] = a_1/\sqrt{2r}$.

From [1.24], we get that $E[\tau^2(x)] = a_2 x + o(x),$ as $x \to 0^+$, where $a_2 = \sqrt{2r} E[\tau^2(x_R)] + \frac{\sqrt{2}}{r\sqrt{r}} e^{x_R\sqrt{2r}}$ and $E[\tau^2(x_R)]$ is given by [1.25]. Moreover, $\lim_{x\to+\infty} E[\tau^2(x)] = E[\tau^2(x_R)] + \frac{2}{r^2} e^{x_R\sqrt{2r}}$. Therefore, $Var[\tau(x)] = a_2 x + o(x)$, as $x \to 0^+$, and $\lim_{x\to+\infty} Var[\tau(x)] = E[\tau^2(x_R)] + \frac{2}{r^2} e^{x_R\sqrt{2r}} - \frac{1}{r^2} e^{2x_R\sqrt{2r}}$.

1.2.2. *Moments of the FPA*

We recall that the FPA density of Brownian motion with resetting (i.e. for $r > 0$) decays exponentially fast at large values, so the moments of the FPA are finite (see, for example, Abundo 2023a and the references therein). The LT of the FPA, $E[\exp(-\lambda A(x)]$, was obtained in Singh and Pal (2022) for $x = x_R$ in terms of special functions, and the first two moments of the FPA at $x = x_R$ were there obtained, by calculating the first and second derivatives of $E[\exp(-\lambda A(x_R)]$ with respect to λ at $\lambda = 0$ (see equations (58) and (59) therein). Precisely, it results:

$$E[A(x_R)] = \frac{x_R}{r} e^{x_R\sqrt{2r}} := \alpha_1, \tag{[1.26]}$$

$$E[A^2(x_R)] = \frac{2e^{x_R\sqrt{2r}}}{r}\left[\frac{x_R^2}{r}\left(\frac{3}{4} + e^{x_R\sqrt{2r}}\right) + \frac{1}{r^2} - \frac{x_R^3}{2\sqrt{2r}}\right] - \frac{2}{r^3} := \alpha_2. \tag{[1.27]}$$

Now, we calculate the first two moments of the FPA $A(x)$ for every $x > 0$ by solving the ODE [1.10] with $U(x) = x$, and $T_n(x) = E[A^n(x)], n = 1, 2$.

As regards the mean of $A(x)$, taking $n = 1$, we have that $T_1(x) = E[A(x)]$ is the solution of the problem:

$$\begin{cases} \frac{1}{2}T_1''(x) - rT_1(x) = -x - rT_1(x_R) \\ T_1(0) = 0,\ T_1(x_R) = \alpha_1; \end{cases} \tag{[1.28]}$$

note that $A(+\infty) = +\infty$, so in contrast with the case of the mean FPT, the appropriate additional condition is $T_1(x_R) = \alpha_1$.

By solving (see Abundo 2023a), we get:

$$T_1(x) = E[A(x)] = \frac{x_R}{r} e^{x_R\sqrt{2r}}\left(1 - e^{-x\sqrt{2r}}\right) + \frac{x}{r}. \tag{[1.29]}$$

REMARK 1.4.– *Formula* [1.29] *extends to all $x > 0$ equation (58) of Singh and Pal (2022) which provides $E[A(x_R)]$ (see* [1.26]*). Note that, for $r = 0$, it follows $E[A(x)] = +\infty$, which matches the well-known result for Brownian motion (see, for example, Abundo and Del Vescovo 2017).*

For $x \to 0^+$, we have $E[A(x)] = \left(\frac{x_R\sqrt{2r}}{r} e^{x_R\sqrt{2r}} + \frac{1}{r}\right) x + o(x)$, while for large positive x, it holds $E[A(x)] \sim \frac{x}{r}$.

Figure 1.3, shows an example of the shape of $E[A(x)]$, given by [1.29], as a function of $x > 0$, for $r = x_R = 1$.

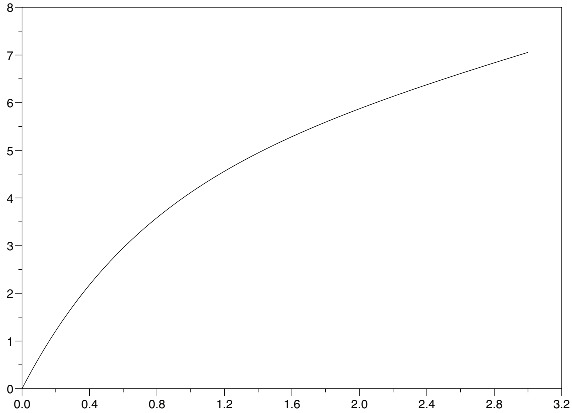

Figure 1.3. *Graph of $E[A(x)]$ as a function of $x > 0$ for $r = x_R = 1$ (on the horizontal axes x)*

As regards the second-order moment of $A(x)$, by taking $n = 2$ in [1.10] with $U(x) = x$, we get that $T_2(x) = E[A^2(x)]$ is the solution of the problem:

$$\begin{cases} \frac{1}{2}T_2''(x) - rT_2(x) = -2xE[A(x)] - rT_2(x_R) \\ T_2(0) = 0,\ T_2(x_R) = \alpha_2, \end{cases} \quad [1.30]$$

where $E[A(x)]$ is given by [1.29].

By solving, we obtain (see Abundo (2023a) for details):

$$T_2(x) = E[A^2(x)] = \left(1 - e^{-x\sqrt{2r}}\right)\left(\alpha_2 + \frac{2}{r^3}\right) +$$

$$+\frac{2x}{r^2}\left(x + x_R e^{x_R\sqrt{2r}}\right) - \frac{x_R x}{r}\left(\frac{x}{\sqrt{2r}} + \frac{1}{2r}\right)e^{-(x-x_R)\sqrt{2r}}. \quad [1.31]$$

Equation [1.31] extends to all values of $x > 0$, the formula found in Singh and Pal (2022) for the expectation of $A^2(x_R)$ (see equation (59) therein).

Figure 1.4 shows the shape of $E[A^2(x)]$, given by [1.31], as a function of $x > 0$, for $r = x_R = 1$. Note that the graph of $E[A^2(x)]$ is not globally concave or convex, but it presents an inflection point. From [1.31], it follows that, for $x \to 0^+$, $E[A^2(x)] = \left[\sqrt{2r}\left(\alpha_2 + \frac{2}{r^3}\right) + \frac{3x_R}{2r^2}e^{x_R\sqrt{2r}}\right]x + o(x)$, and $Var[A(x)]$ has the same behavior, at the first order in x. Moreover, for large $x > 0$, $E[A^2(x)] \sim \frac{2}{r^2}x^2$, and $Var[A(x)] \sim \frac{x^2}{r^2}$.

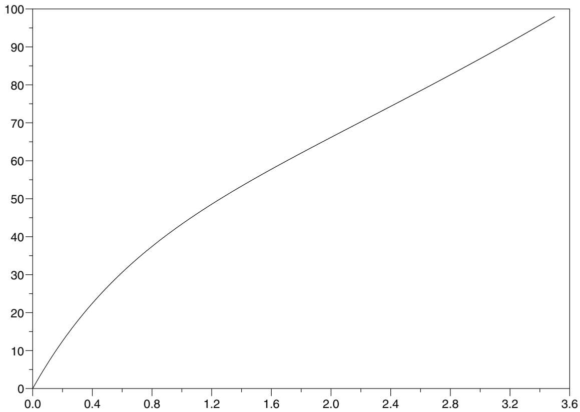

Figure 1.4. *Graph of $E[A^2(x)]$ as a function of $x > 0$ for $r = x_R = 1$ (on the horizontal axes x)*

REMARK 1.5.– *From the previous calculations, we conclude that the following growth conditions (upper bounds) hold:*

$$E[A(x)] \leq const \cdot x, \; E[A^2(x)] \leq const' \cdot x^2, \; \text{for large } x > 0. \qquad [1.32]$$

Similar bounds hold for Brownian motion with a negative drift μ (without resetting, i.e. $r = 0$), because in that case, the moments of $A(x)$ grow at most polynomially in x (see Abundo and Del Vescovo 2017).

1.2.3. *Joint moment of $A(x)$ and $\tau(x)$*

In this section, we find an explicit expression for $E[A(x)\tau(x)]$, i.e. the joint moment of $A(x)$ and $\tau(x)$. The joint LT of $\tau(x), A(x)$ is $M_{\lambda_1,\lambda_2}(x) = E\left[e^{-\lambda_1 \tau(x)} e^{-\lambda_2 A(x)}\right], \lambda_1, \lambda_2 > 0$. By reasoning as before, we find that it satisfies the differential equation:

$$(L - r)M_{\lambda_1,\lambda_2}(x) = (\lambda_1 + \lambda_2 x)M_{\lambda_1,\lambda_2}(x) - rM_{\lambda_1,\lambda_2}(x_R). \qquad [1.33]$$

Then, by taking $\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}$ in both members of equation [1.33] and calculating it at $\lambda_1 = \lambda_2 = 0$, we obtain that $V(x) := E[\tau(x)A(x)]$ satisfies the differential problem:

$$(L - r)V(x) = -xE[\tau(x)] - E[A(x)] - rV(x_R), \; V(0) = 0, \qquad [1.34]$$

with a suitable additional condition.

In fact, since $E[A(x)\tau(x)] \leq \sqrt{E[A^2(x)]}\sqrt{E[\tau^2(x)]}$, by taking into account the behaviors of the moments of the FPA at large x (see [1.32]), we obtain that the additional condition for the ODE [1.34] must be:

$$V(x) = E[\tau(x)A(x)] \leq const \cdot x, \text{ for large } x > 0. \qquad [1.35]$$

By solving, we obtain (see Abundo 2023a for details):

$$V(x) = -\left[\frac{x_R e^{x_R\sqrt{2r}}}{r^2} + V(x_R)\right] e^{-x\sqrt{2r}} + \bar{V}(x), \qquad [1.36]$$

where $V(x_R) = e^{x_R\sqrt{2r}}\left[\left(8e^{x_R\sqrt{2r}} - 1\right)\frac{x_R}{4r^2} - \frac{3x_R^2}{2r\sqrt{2r}}\right]$,
$\bar{V}(x) = V(x_R) + \frac{(e^{x_R\sqrt{2r}}+1)x}{r^2} + \frac{x_R e^{x_R\sqrt{2r}}}{r^2} - e^{-(x-x_R)\sqrt{2r}}\left(\frac{x^2}{2r\sqrt{2r}} + \frac{(1+2x_R\sqrt{2r})x}{4r^2}\right)$.
Therefore, the expressions of $Cov[A(x), \tau(x)] := V(x) - E[\tau(x)]E[A(x)]$ and of the correlation coefficient

$$\rho_{\tau(x),A(x)} := \frac{Cov[\tau(x), A(x)]}{\sqrt{Var[\tau(x)]Var[A(x)]}}, \qquad [1.37]$$

follow soon. As easily seen, $\rho_{\tau(x),A(x)}$ turns out to be positive for every $x > 0$, i.e. $\tau(x)$ and $A(x)$ are positively correlated. Moreover, $\lim_{x\to 0^+} \rho_{\tau(x),A(x)} = \rho_0 > 0$, and $\lim_{x\to+\infty} \rho_{\tau(x),A(x)} = \rho_\infty < \rho_0$. The graph of $\rho_{\tau(x),A(x)}$ increases from ρ_0 to a maximum value; after that, it decreases to $\rho_\infty < \rho_0$, as $x \to +\infty$. This kind of behavior for $\rho_{\tau(x),A(x)}$ was also observed for the drifted Brownian motion without reset (see Abundo and Del Vescovo 2017) and Ornstein–Uhlenbeck process without reset (see Abundo 2023b).

Figure 1.5 (top panel) shows the graph of $\rho_{\tau(x),A(x)}$, as a function of $x > 0$ for $r = x_R = 1$ (on the horizontal axes x); the existence of the maximum is revealed by an enlargement around $x = 0.1$ (bottom panel). Figure 1.6 shows another example of the graph of $\rho_{\tau(x),A(x)}$, obtained for $r = 1$ and $x_R = 2$.

1.2.4. *Maximum displacement*

We define the maximum displacement of Brownian motion with resetting $\mathcal{X}(t)$, starting from $x > 0$, as the random variables $\mathcal{M}_x = \max_{t\in[0,\tau(x)]} \mathcal{X}(t)$ (obviously, we have $\mathcal{M}_x \geq x$). Note that the event $\{\mathcal{M}_x \leq z\}$ occurs if and only if $\mathcal{X}(t)$ first exits the interval $(0, z)$ through the left end 0. Therefore, for $z \geq x$, we get that the distribution function $F_{\mathcal{M}_x}(z) = P(\mathcal{M}_x \leq z)$ is the solution of the differential equation (see, for example, Abundo 2013, 2023b):

$$\begin{cases} \mathcal{L}w(x) = \frac{1}{2}w''(x) + rw(x_R) - rw(x) = 0, \ x \in (0, z) \\ w(0) = 1, \ w(z) = 0, \end{cases} \qquad [1.38]$$

where $\mathcal{L}$ is the infinitesimal generator of $\mathcal{X}(t)$ (see [1.2]).

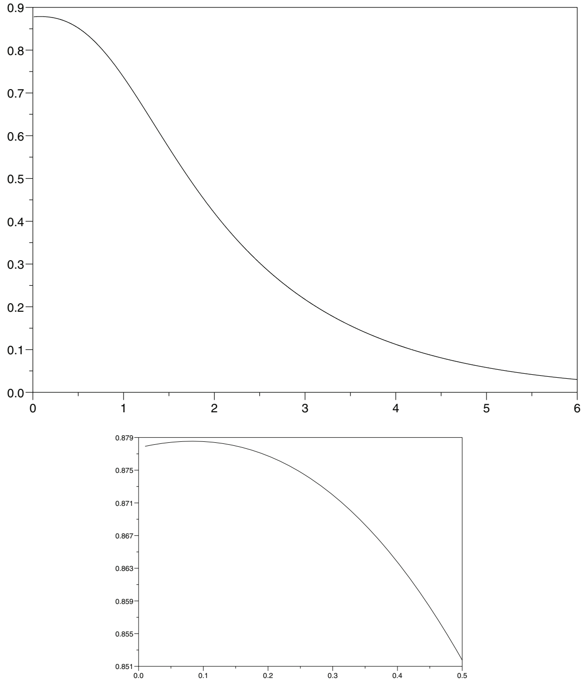

Figure 1.5. *Graph of $\rho_{\tau(x),A(x)}$ as a function of $x > 0$ for $r = x_R = 1$ (on the horizontal axes x); the second panel shows an enlargement around $x = 0.1$*

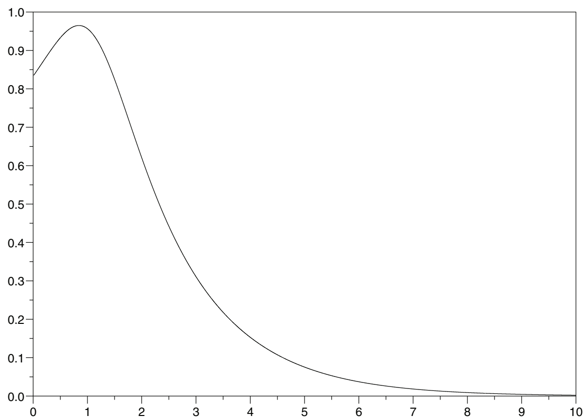

Figure 1.6. *Graph of $\rho_{\tau(x),A(x)}$ as a function of $x > 0$ for $r = 1$ and $x_R = 2$ (on the horizontal axes x)*

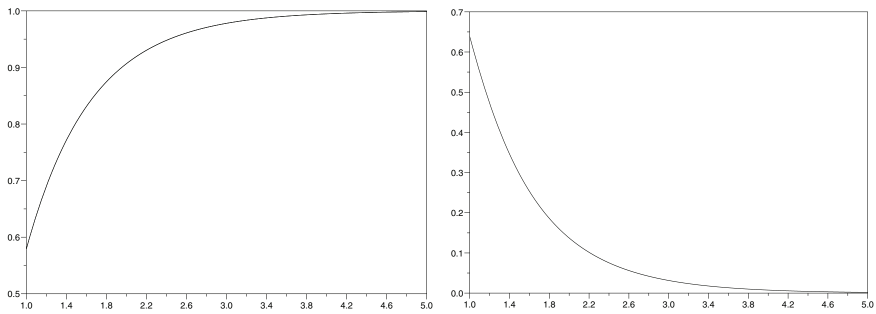

Figure 1.7. *Left panel: distribution function of the maximum displacement* $\mathcal{M}_x$ *of Brownian motion with resetting, for* $x = 1, x_R = 0.5$ *and* $r = 1$; *right panel: the corresponding density (on the horizontal axes* $z \geq x = 1$)

By solving, we get (see Abundo 2023a for details):

$$F_{\mathcal{M}_x}(z) = P(\mathcal{M}_x \leq z) = \begin{cases} 0, & z < x \\ c_1(z)e^{-x\sqrt{2r}} + c_2(z)e^{x\sqrt{2r}} + a(z), & z \geq x, \end{cases} \qquad [1.39]$$

where

$$\begin{cases} c_1(z) = (e^{z\sqrt{2r}} - 1)^{-1}(e^{-z\sqrt{2r}} + e^{-2x_R\sqrt{2r}})^{-1} \\ c_2(z) = -e^{-2x_R\sqrt{2r}}c_1(z) \\ a(z) = 1 - c_1(z) - c_2(z). \end{cases} \qquad [1.40]$$

As seen, the distribution function of $\mathcal{M}_x - x$ appears to have a tail that decays exponentially fast, and so the expectation $E[\mathcal{M}_x]$ results to be finite. By calculating the solution of [1.38] for $r = 0$, we obtain that, for Brownian motion without resetting, the distribution of the maximum displacement is:

$$F_{\mathcal{M}_x}(z) = 1 - \frac{x}{z},\ z \geq x, \qquad [1.41]$$

and it is zero for $z < x$. This implies that the expectation of the maximum displacement of Brownian motion without resetting is infinite. Figure 1.7 shows an example of the distribution function of the maximum displacement $\mathcal{M}_x$ of Brownian motion with resetting, and the corresponding probability density, obtained by taking the derivative in [1.39], for $x = 1, x_R = 0.5$ and $r = 1$.

1.3. Drifted Brownian motion with resetting

In this section, we consider drifted Brownian motion with resetting $\mathcal{X}(t)$, namely $X(t) = x + B_t + \mu t$, and we find explicit expressions of the LT, the moments of the FPT and FPA, and the maximum displacement of $\mathcal{X}(t)$, as solutions of differential problems. We omit the details, and we limit ourselves to report only the results which can be obtained without using special functions (for more details, see Abundo 2023a).

Note that the FPT through zero of Brownian motion with non-zero drift μ (without resetting), starting from $x > 0$ is finite with probability one, only if the drift is negative. Instead, the FPT through zero of drifted Brownian motion with resetting $\mathcal{X}(t)$ is finite, irrespective of the sign of the drift μ and the moments of the FPT are also finite, for any $0 < r < +\infty$ (see, for example, Pal et al. 2019). The infinitesimal generator of $X(t)$ is now given by $Lf(x) = \frac{1}{2}\frac{df^2}{dx^2} + \mu\frac{df}{dx}$; by proceeding as in the case of undrifted Brownian motion with resetting, the solutions of the various equations can be obtained by taking $\mu + \sqrt{\mu^2 + 2(\lambda + r)}$ in place of $\sqrt{2(\lambda + r)}$ in the corresponding formulae.

1.3.1. *The Laplace transform of* $\tau(x)$

The LT of $\tau(x)$ turns out to be

$$M_\lambda(x) = E\left[e^{-\lambda\tau(x)}\right] = e^{-x\left(\mu+\sqrt{\mu^2+2(\lambda+r)}\,\right)} +$$

$$+M_\lambda(x_R)\frac{r}{\lambda+r}\left(1 - e^{-x\left(\mu+\sqrt{\mu^2+2(\lambda+r)}\,\right)}\right), \text{ where}$$

$$M_\lambda(x_R) = \frac{(\lambda+r)e^{-x_R\left(\mu+\sqrt{\mu^2+2(\lambda+r)}\,\right)}}{\lambda+re^{-x_R\left(\mu+\sqrt{\mu^2+2(\lambda+r)}\,\right)}}.$$

REMARK 1.6.– *The above formula is new, because only the undrifted Brownian motion with resetting was studied in Singh and Pal (2022).*

1.3.2. *Moments of the FPT*

The mean of the FPT is:

$$E[\tau(x)] = \frac{1}{r}e^{x_R(\mu+\sqrt{\mu^2+2r})}\left(1 - e^{-x(\mu+\sqrt{\mu^2+2r})}\right). \qquad [1.42]$$

For $\mu = 0$, we again obtain [1.21].

As for the second-order moment of $\tau(x)$, we have:

$$E[\tau^2(x)] =$$

$$= e^{x_R(\mu+\sqrt{\mu^2+2r})} \left(\frac{2}{r^2} e^{x_R(\mu+\sqrt{\mu^2+2r})} - \frac{2}{r^2} - \frac{2x_R}{r\sqrt{\mu^2+2r}}\right)\left(1 - e^{-x(\mu+\sqrt{\mu^2+2r})}\right)$$

$$- \frac{2}{r} e^{(x_R-x)(\mu+\sqrt{\mu^2+2r})} \left(\frac{1}{r} + \frac{x}{\sqrt{\mu^2+2r}}\right) + \frac{2}{r^2} e^{x_R(\mu+\sqrt{\mu^2+2r})}. \quad [1.43]$$

For $\mu = 0$, we again obtain [1.24].

1.3.3. *Mean of the FPA*

The mean of the FPA turns out to be:

$$E[A(x)] = E[A(x_R)] \left(1 - e^{-x\left(\mu+\sqrt{\mu^2+2r}\right)}\right) + \frac{x}{r} + \frac{\mu}{r^2}\left(1 - e^{-x\left(\mu+\sqrt{\mu^2+2r}\right)}\right), \quad [1.44]$$

where

$$E[A(x_R)] = e^{x_R\left(\mu+\sqrt{\mu^2+2r}\right)} \left[\frac{x_R}{r} + \frac{\mu}{r^2}\left(1 - e^{-x_R\left(\mu+\sqrt{\mu^2+2r}\right)}\right)\right]. \quad [1.45]$$

For $\mu = 0$, we again obtain $E[A(x_R)] = \frac{x_R}{r} e^{\sqrt{2r}x_R}$ and $E[A(x)] = \frac{x_R}{r} e^{\sqrt{2r}x_R}(1 - e^{-x\sqrt{2r}}) + \frac{x}{r}$ (see [1.26] and [1.29].

1.3.4. *Maximum displacement*

The distribution function of the maximum displacement $\mathcal{M}_x$ turns out to be:

$$F_{\mathcal{M}_x}(z) = P(\mathcal{M}_x \le z) = \begin{cases} 0, & z < x \\ c_1(z)e^{d_1 x} + c_2(z)e^{d_2 x} + a(z), & z \ge x. \end{cases} \quad [1.46]$$

where $d_1 = -\mu - \sqrt{\mu^2+2r} < 0$, $d_2 = -\mu + \sqrt{\mu^2+2r} > 0$, and the functions $c_i(z)$ and $a(z)$ are given by:

$$\begin{cases} c_1(z) = -\left[e^{-\left(\mu+\sqrt{\mu^2+2r}\right)z} - 1 + e^{-2x_R\sqrt{\mu^2+2r}}\left(1 - e^{-\left(\mu-\sqrt{\mu^2+2r}\right)z}\right)\right]^{-1} \\ c_2(z) = -c_1 e^{-2x_R\sqrt{\mu^2+2r}} \\ a(z) = 1 - c_1(z) - c_2(z). \end{cases} \quad [1.47]$$