Progress in Mathematics
352

Alberto Albano, Paolo Aluffi,
Michele Bolognesi, Cinzia Casagrande,
Elisabetta Colombo, Alberto Conte,
Antonella Grassi, Claudio Pedrini,
Gian Pietro Pirola, Alessandro Verra,
Editors

Perspectives on Four Decades of Algebraic Geometry, Volume 2

In Memory of Alberto Collino

Birkhäuser

Birkhäuser

Progress in Mathematics

Volume 352

Progress in Mathematics is a series of books intended for professional mathematicians and scientists, encompassing all areas of pure mathematics. This distinguished series, which began in 1979, includes research level monographs, polished notes arising from seminars or lecture series, graduate level textbooks, and proceedings of focused and refereed conferences. It is designed as a vehicle for reporting ongoing research as well as expositions of particular subject areas.

Alberto Albano • Paolo Aluffi •
Michele Bolognesi • Cinzia Casagrande •
Elisabetta Colombo • Alberto Conte •
Antonella Grassi • Claudio Pedrini •
Gian Pietro Pirola • Alessandro Verra
Editors

Perspectives on Four Decades of Algebraic Geometry, Volume 2

In Memory of Alberto Collino

Birkhäuser

Editors

Alberto Albano
Department of Mathematics
University of Turin
Torino, Italy

Paolo Aluffi
Department of Mathematics
Florida State University
Tallahassee, FL, USA

Michele Bolognesi
Institut A. Grothendieck
University of Montpellier
Montpellier, France

Cinzia Casagrande
Department of Mathematics
University of Turin
Torino, Italy

Elisabetta Colombo
Department of Mathematics
University of Milan
Milano, Milano, Italy

Alberto Conte
Department of Mathematics
University of Turin
Torino, Italy

Antonella Grassi
Department of Mathematics
University of Pennsylvania
Philadelphia, PA, USA

University of Bologna
Bologna, Italy

Claudio Pedrini
Department of Mathematics
University of Genoa
Genova, Italy

Gian Pietro Pirola
Department of Mathematics
University of Pavia
Pavia, Italy

Alessandro Verra
Department of Mathematics and Physics
Roma Tre University
Rome, Italy

ISSN 0743-1643 ISSN 2296-505X (electronic)
Progress in Mathematics
ISBN 978-3-031-66233-1 ISBN 978-3-031-66234-8 (eBook)
https://doi.org/10.1007/978-3-031-66234-8

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

A Short Memory of Our Father Alberto Collino in Someway Maybe a Keepsake

For his 70th birthday, among the gifts there was a t-shirt with a picture of the three of us in the mountains, we were trekking aiming what we call the blue lake, a place he used to love, and with a phrase:

we will always walk together

We remember our dad as a Math Professor with a pencil in his hand always ready to write down his geometrical thoughts, but we remember him as a father and, perhaps who knows, he was a father not only for us and an example for many…

Francesca and Giuseppe Collino

In Remembrance of Alberto Collino

The Alberto Collino that I came to know in the early part of both our careers was first and foremost a human being of high quality, quiet, modest, and with high standards, both for himself and others.

I was introduced to the Italian School of Algebraic Geometry by my advisor, Phillip Griffiths, and was immediately attracted by the intellectual humanity of its taste in problems and perspectives and the beauty of their solutions. Aldo Andreotti, Enrico Bombieri, and in my case in particular, Francesco Gherardelli, showed the way. I did not meet Alberto Collino in those early visits to Italy due to the fact that he was off at MIT writing his thesis "The Rational Equivalence Ring of a Symmetric Product of Curves" with Arthur Mattuck. But his Italian mathematical lineage was certainly reflected in the subject matter and content of that thesis. Significantly he bookended his entire career by returning to that very same subject matter in the final paper of his career, "The fundamental group of the open symmetric product of a hyperelliptic curve" some 40 years later.

The Italian School of that early period drew on its rich history to provide a very compatible complement to the powerful formalism and generality sweeping the subject in those years. For more than a few of us, that complement was our primary vehicle to deeper understanding, focusing as it did on the concrete examples lying in those days at the edges of our understanding. Alberto could see the bridges and connections between the two perspectives better than many of us—a fact that led him, for example, to be able to incorporate K-theoretic aspects into much of his work. But even there the footsteps of Castelnuovo and Enriques chatting about curves and surfaces on a stroll under the arched arcades of Bologna or Rome still echoed.

I did not know Alberto well, indeed he seemed to me to be a rather quiet private person, but I had enormous respect for him, both as a person and a mathematician. He contributed to our subject in an understated, yet very powerful way. His presence is greatly missed.

January 30, 2024 — Herb Clemens

Preface

The Roots of This Book

The origins of this book are rooted in the common aim of a group of people, working in the same research field, namely Algebraic Geometry, to commemorate a distinguished colleague, passed away in September 2020. This book is dedicated to the memory of Alberto Collino and to his exceptional spirit and talent in geometry, in mathematics, and in the life.

Conceiving the idea of this book, it was not difficult to realize that the scientific life of a person, which was deeply involved in the research work like he was, goes beyond a formal list of papers, nice results, and events. Behind all that one can often see an entire wide world of scientific relations and interchanges, between so much persons and scholars. From this point of view the arid curriculum vitae of a single person can offer, in the very best situations, an extremely interesting view on the punctual and global evolution of an entire discipline along the years.

This is specially true for the person to whom this volume is dedicated. Along four decades, Alberto Collino was scientifically in touch with many of the best algebraic geometers of his times. He was indeed very much aware of the importance of this exchange and almost considered it a moral duty with respect to his engagements as a man and as a scientist.

Since his MIT years, as a student of Arthur Mattuck, to the four decades spent as a full professor in the University of Turin, visiting back the USA sometimes, Collino's scientific life is dense of relations with the best actors of mathematical research. Rephrasing a well-known sentence, one could say: not only *to sit on the shoulders of giants* but dialoguing and learning from them as much as possible. Of course he was strong and capable enough to realize this task, which is a typical feature of all his scientific life.

However, we have to admit that, continuing in this vein, we are not respecting the severe attitude to understatement, of Subalpine type more than British one could say, of our colleague and friend. Nevertheless, he will forgive us if we freely sketch, in the next paragraph, a few short tracts of his life and of his scientific adventure.

The next paragraph is partially inspired, with a different elaboration and additional information, from the obituary written by Alberto Albano and Pietro Pirola for Notiziario UMI 2021.

Alberto Collino

Alberto Collino was born on April 1, 1947 in Verzuolo, a small town in Piedmont, Italy, where he has been resident during most of his life. His family was running the historical Hotel *Corona Grossa* in Saluzzo, the main city in the area, capital city of the Marquis in the Middle Age. He was married to Miranda Ferrato, prematurely dead in 1982, and they had two sons: Francesca, a physician, and Giuseppe, an engineer.

In Saluzzo he started to be known as a very brilliant student of Liceo classico *Giambattista Bodoni*. In particular, in 1966, he was awarded with the *Certificate of Honor*, by the Institution *Palazzo Civiltà del Lavoro*, as one of the best students in Italy.

In July 1970 he graduated in Mathematics at the University of Turin. His thesis is entitled *Behavior of the main topological properties as the topology varies in the lattice of topological structures on the same fundamental set R*. It was written under the guidance of Davide Carlo De Maria, Professor of Topology at the University of Turin.

After graduating he was accepted as a student at MIT (Massachusetts Institute of Technology). In 1974, after two years only, he earned his Ph.D. in Mathematics from this Institution. Arthur Mattuck (1930–2021), Professor of Mathematics at MIT, was the supervisor of his dissertation entitled *The Rational Equivalence Ring of Symmetric Product of Curves*.

Then, confirming all the best expectations of the beginning, his scientific and academic career found its home at the University of Turin. There he carried out his research and teaching over the course of 40 years, as assistant and then full professor since 1982.

Some older people, near to his generation, perhaps remember of his office in Istituto di Geometria, with the picture on the wall of a great Geometer from the city of Saluzzo, that is, Corrado Segre (1863–1924). It is possible that somebody also remembers of the dense footnotes Collino was writing, upon the research papers he was reading. This is the case, for instance, of some old preprints, arrived from Amsterdam, on the quartic double solids.

After four decades, Collino's commitment in Algebraic Geometry and his research legacy continue to be appreciated, for originality, top level of his expertise and the intelligent rigor when addressing a problem. Collino exerted a notable and credited scientific role in his field, being in touch, as remarked, with so many top scholars. This important point is well representing his personal and scientific gifts, exceptional even in life misfortune. Moreover this was made possible in spite of his self-effacing character, leading him to keep aloof from the main scene or academic controversies.

Author of 40 scientific publications, his research activity shows, starting from his doctoral thesis, a constant and fundamental interest in algebraic cycles. This led to profound results on rational and algebraic equivalence, and their links with Hodge's theory. Certainly influenced by Daniel Quillen, he developed his own initial interests in homotopy theories, by addressing problems of K-Theory, which is another field where he obtained remarkable results. Intersection Theory is a further and later line of research in his scientific activity, as we will see. Along this way he was led more recently to explore the links with Quantum Cohomology.

It is not possible for space reasons, to give a comprehensive view of the relationship of Alberto Collino to the world of Algebraic Geometry and his characters. We will add some considerations later. At this step, we only spend a few more words on the three mathematicians whose connections to Algebraic Geometry in Torino and to him were more continuous during the years.

These are the members of the Honorary Scientific Board of this book, namely *Herbert Clemens, William Fulton, Jacob Murre*. Collino contributed very much to the creation of these connections. Especially with Fulton, this opened to collaboration and a long friendship in the eighties of twentieth century. Before, Collino certainly shared his interest in algebraic cycles together with Clemens and Murre and, at MIT, with Spencer Bloch and Steve Kleiman.

From Torino he was in touch with Clemens and his student Giuseppe Ceresa, who was working on the cycle called nowadays Ceresa's cycle. With Conte and Murre, Collino started a program on the surface of conics in a quartic threefold, after his result on the lines in it. His footnoting to the old Dutch preprints we mentioned, likely started to be more intense, since these preprints are useful to study the surface of conics. Actually they describe the surface of lines of a quartic double solid and are due to Gerard Welters.

Enumerative Geometry is present above, e.g. the number of conics through a general point of a quartic threefold is rigorously determined: 972. Some visits of Fulton in Torino marked the beginning of a collaboration and the advent of Intersection Theory in Collino's scientific life. The academic year 1985/86 was scientifically fortunate and special, since he spent it at Brown University, invited by Fulton. A beautiful Mémoire of SMF, of classical flavor, was written by Collino and Fulton on the space of triangles and its compactification, building on it the Chow ring and the Schubert calculus.

These are episodes of an intense and fortunate mathematical life. Now he is no longer on the way, on the bus to Torino, or a train to Pavia, Milan, or Genoa for attending a talk, or for leaving, directed abroad, to a favorite collaboration in Geometry. We can close by some lines he wrote on a book, at the end of his doctoral period at MIT. They are verses by Lucretius:

Sweet it is, when on the great sea the winds are buffeting the waters, to gaze from the land on another's great struggles; not because it is pleasure or joy that anyone *should be distressed, but because it is sweet to perceive from what misfortune, you yourself are free*.[1]

Four Decades and More

Starting from his doctoral thesis, and along a period of more than four decades, Alberto Collino offered his contribution to the development of the discipline of Algebraic Geometry. His engagement and style in the field, built on the basis of his mathematical personality and the scientific education received, in Italy and at MIT, show up the highest quality.

This quality implies that he was fully aware of the scientific duty of not being closed in a self-referential area of research, disconnected from the rest of the world. He was instead very open and dialogued throughout all his life, with determination and confidence, with every person: from the very top mathematicians, with whom he was always into contact, accepting the relative disparity, to persons, young or not, asking him a scientific support.

Looking over the years to several and different aspects, like for instance seminars and research enterprises in which he had decided of taking part, his general style is constantly confirmed. The capability of opening a scientific dialogue with another scholar, so to reach, soon and keenly, the higher mathematical levels, was probably one of his best features.

In his life he was following closely the evolution of several different aspects of his favorite field, with a keen mind, sustained by a genuine interest in knowledge. Taking it with a grain of salt, one perhaps could say that, in the spirit of scientists at the beginning of modern history, it was important for him to perceive, psychologically even if not concretely, the overall movement of the great discipline that he was investigating.

Coherently with these and the previous remarks, we can infer that along his life, thanks to his position in the mathematical net of Science, Collino was present in the major developments and events in the history of his discipline, sometimes as an active witness. Certainly he was a serious, careful and rigorous witness of contemporary Geometry and, certainly, his personal and scientific life could become a starting point for a more in-depth investigation by the scholars. This is true for persons in geometry and mathematical sciences and for historians, not only of these disciplines. Therefore, at this latest step, we spend a few words, actually too few, on a second motivation of this book, of the same importance. This is, so to say, the non-memorial motivation, which is a natural consequence of the memorial one.

[1] *Suave, mari magno turbantibus aequora ventis, e terra magnum alterius spectare laborem; non quia vexari quemquamst iucunda voluptas, sed quibus ipse malis careas quia cernere suave est*. Lucretius, De Rerum Natura, l. II, ll. 1–4. Translation by Cyril Bailey, Oxford University Press.

This last motivation also aims to represent, with this book, a task performed and a service offered to scholars in Algebraic Geometry. In fact, as is evident several times in this Introduction, the scientific figure we have outlined has found himself to travel several important scientific roads in the field of Algebraic Geometry and has had a large number of contacts, even beyond his own themes of research.

Therefore, right from the beginning of the project for this book, it was natural to set the task of addressing, with this book in memory of Alberto Collino, a more unifying perspective, however partial the result achieved may be, on 40 years of Algebraic Geometry and on the schools which were flourishing along these years until today.

Let us point out that it was not only question of looking to the past. The two volumes of the book offer indeed, thanks to 29 research papers and their contents, an interesting wide view on nowadays Algebraic Geometry, with its diversities and its trends, with the participation of several leading figures in the field. An impressive testimonial of the participation to this book in memory of Alberto Collino, is the list of names of its authors all, the scientific editors of this book. *I am deeply grateful, together with all the other Scientific Editors, to all the people participating in this initiative*.

Rome, Italy
July 7, 2023

Alessandro Verra

Contents

Contributors: Volume 1

Massimiliano Alessandro Mathematisches Institut der Universität Bayreuth, Bayreuth, Germany

Enrico Arbarello Accademia dei Lincei, Rome, Italy

Ingrid Bauer Mathematisches Institut, Universität Bayreuth, Bayreuth, Germany

Prakash Belkale Department of Mathematics, University of North Carolina, Chapel Hill, NC, USA

Michele Bolognesi Institut Montpellierain Alexander Grothendieck, Université de Montpellier, Montpellier, France

Fabrizio Catanese Mathematisches Institut der Universität Bayreuth, Bayreuth, Germany
Korea Institute for Advanced Study, Seoul, South Korea

Ivan Cheltsov School of Mathematics, University of Edinburgh, Edinburgh, Scotland

Ciro Ciliberto Dipartimento di Matematica, Università di Roma Tor Vergata, Rome, Italy

Jean-Louis Colliot-Thélène CNRS, Laboratoire de mathématiques d'Orsay, Université Paris-Saclay, Orsay, France

Alessio Corti Department of Mathematics, Imperial College London, London, UK

Olivier Debarre Institut de mathématiques de Jussieu, PRG, Université de Paris-Cité, Paris, France

Torsten Ekedahl Matematiska institutionen, Stockholms Universitet, Stockholm, Sweden

Najmuddin Fakhruddin School of Mathematics, Tata Institute of Fundamental Research, Mumbai, India

Paola Frediani Dipartimento di Matematica, Università di Pavia, Pavia, Italy

Christian Gleissner Mathematisches Institut, Universität Bayreuth, Bayreuth, Germany

Giulia Gugiatti Mathematics Section ICTP, Leonardo Da Vinci Building, Trieste, Italy

Elham Izadi Department of Mathematics, University of California San Diego, La Jolla, CA, USA

Steven L. Kleiman Department of Mathematics, 2-172 MIT, Cambridge, MA, USA

Jan O. Kleppe Faculty of Technology, Art and Design, Oslo Metropolitan University, Oslo, Norway

Julia Kotonski Mathematisches Institut, Universität Bayreuth, Bayreuth, Germany

Robert Laterveer Institut de Recherche Mathématique Avancée, CNRS, Université de Strasbourg, Strasbourg, France

Marc Levine Fakultät Mathematik, Universität Duisburg-Essen Essen, Germany

Emanuele Macrì Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d'Orsay, Orsay, France

Rick Miranda Department of Mathematics, Colorado State University, Fort Collins, CO, USA

Jihun Park Center for Geometry and Physics, Institute for Basic Science, Pohang, Korea

Sabrina Pauli Fakultät Mathematik, Universität Duisburg-Essen Essen, Germany

Giulia Saccà Department of Mathematics, Columbia University, New York, NY, USA

Federico Scavia Department of Mathematics, University of California, Los Angeles, CA, USA

Fernando Rodriguez Villegas Section ICTP, Leonardo Da Vinci Building, Trieste, Italy

Contributors: Volume 2

Fabrizio Catanese Mathematisches Institut der Universität Bayreuth, Bayreuth, Germany
Korea Institute for Advanced Study, Seoul, South Korea

François Charles DMA, École Normale Supérieure de Paris, Paris, France
Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay, Orsay, France

Mauricio Corrêa Università degli Studi di Bari, Bari, Italy

Christian Haesemeyer School of Mathematics and Statistics, University of Melbourne, Parkville, VIC, Australia

Andrew Kresch Institut für Mathematik, Universität Zürich, Zürich, Switzerland

Christian Lehn Fakultät für Mathematik, Technische Universität Chemnitz, Chemnitz, Germany

Angelo Felice Lopez Dipartimento di Matematica e Fisica, Università Roma Tre, Roma, Italy

Takuro Mochizuki Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan

Giovanni Mongardi Alma Mater Studiorum, Università di Bologna, Bologna, Italia

Juan Carlos Naranjo Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Barcelona, Spain

Johannes Nagel Institut de Mathématiques de Bourgogne, UMR 5584, Dijon, France

Madhav Nori Department of Mathematics, University of Chicago, Chicago, IL, USA

Kieran G. O'Grady Dipartimento di Matematica Guido Castelnuovo, Sapienza Università di Roma, Roma, Italia

Gianluca Pacienza Université de Lorraine, CNRS, IECL, Nancy, France

Deepam Patel Department of Mathematics, Purdue University, West Lafayette, IN, USA

Alena Pirutka Courant Institute of Mathematical Sciences, New York University, New York, NY, USA

Edoardo Sernesi Department of Mathematics and Physics, Universitá Roma Tre, Roma, Italy

Vasudevan Srinivas School of Mathematics, Tata Institute of Fundamental Research, Mumbai, India

Irene Spelta Centre de Recerca Matemàtica, Bellaterra, Spain

Orsola Tommasi Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, Padova, Italy

Yuri Tschinkel Courant Institute, New York, NY, USA
Simons Foundation, New York, NY, USA

Israel Vainsencher UFMG, Belo, Horizonte, Brazil

Claire Voisin Institut de Mathématiques de Jussieu-Paris, Paris, France

Charles A. Weibel Mathematics Department, Rutgers University, New Brunswick, NJ, USA

On Varieties with Ulrich Twisted Normal Bundles

Angelo Felice Lopez

Dedicated to the memory of Alberto Collino[1]

Mathematics Subject Classification Primary 14J60; Secondary 14H50

1 Introduction

In this note, we consider the problem of understanding when the normal bundle of a projective variety has a twist that makes it an Ulrich vector bundle.

Recall that if $X \subseteq \mathbb{P}^N$ is a smooth irreducible variety of dimension $n \geq 1$, we say that a vector bundle $\mathcal{E}$ on X is Ulrich if $H^i(\mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$.

The study of Ulrich vector bundles is closely related with several areas of commutative algebra and algebraic geometry and often gives interesting consequences on the geometry of X and on the cohomology of sheaves on X (see for example in [ES, B, CMRPL] and references therein).

Now, the normal bundle $N_{X/\mathbb{P}^N}$ is associated with the embedding, and it is therefore natural to ask when it has a twist that makes it an Ulrich vector bundle. If so, since Ulrich vector bundles are semistable, this would connect with the long-studied problem of varieties with semistable normal bundle. On the other hand, since $N_{X/\mathbb{P}^N}(-1)$ is globally generated, the natural question to be asked is: for which $k \geq 1$ we have that $N_{X/\mathbb{P}^N}(-k)$ is an Ulrich vector bundle?

[1] A personal note is written in Sect. 5.

A. F. Lopez (✉)
Dipartimento di Matematica e Fisica, Università Roma Tre, Roma, Italy
e-mail: lopez@mat.uniroma3.it

A. Albano et al. (eds.), *Perspectives on Four Decades of Algebraic Geometry, Volume 2*, Progress in Mathematics 352,
https://doi.org/10.1007/978-3-031-66234-8_1

Certainly, there are some easy examples. If X is a linear space in $\mathbb{P}^N$, then $N_{X/\mathbb{P}^N}(-1) \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus(N-n)}$ is clearly an Ulrich vector bundle on X. Also, if $X \subset \mathbb{P}^3$ is a curve such that $H^0(N_{X/\mathbb{P}^3}(-2)) = 0$, then also $H^1(N_{X/\mathbb{P}^3}(-2)) = 0$, since $\chi(N_{X/\mathbb{P}^3}(-2)) = 0$. Thus again $N_{X/\mathbb{P}^3}(-1)$ is an Ulrich vector bundle. This is actually an occurrence of a bit less trivial family of examples, namely n-dimensional varieties $X \subset \mathbb{P}^{n+2}, 1 \le n \le 3$ such that $H^j(N_{X/\mathbb{P}^{n+2}}(-2-j)) = 0$ for $0 \le j \le n-1$, see Lemma 2.1. Interesting examples of this type, but not all of them (see below and Remark 2.3), are linear standard determinantal curves in $\mathbb{P}^3$, surfaces in $\mathbb{P}^4$ and threefolds in $\mathbb{P}^5$, see Proposition 2.2 and [KMR1, Thm. 3.6].

As it turns out, the aforementioned are the only examples, as we will show in the ensuing

Theorem 1 *Let $X \subset \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$ and let k be an integer.*

Then $N_{X/\mathbb{P}^N}(-k)$ is an Ulrich vector bundle if and only if $k = 1$ and X is one of the following:

(i) *$X = \mathbb{P}^n$ embedded linearly in $\mathbb{P}^N$, or*
(ii) *$1 \le n \le 3$, $N = n+2$ and $X \subset \mathbb{P}^{n+2}$ is a variety such that $N_{X/\mathbb{P}^{n+2}}(-1)$ is 0-regular, or, equivalently, $H^j(N_{X/\mathbb{P}^{n+2}}(-j-2)) = 0$ for $0 \le j \le n-1$.*

Moreover, in the latter case, if X does not contain a line, then $N_{X/\mathbb{P}^{n+2}}(-1)$ is very ample.

The very ampleness statement is an application of recent results in [LS].

We remark that the family of curves X in $\mathbb{P}^3$ satisfying $H^0(N_{X/\mathbb{P}^3}(-2)) = 0$ is very large (see for example [EH, BE]), and there is not much hope to classify them. Moreover $N_{X/\mathbb{P}^3}(-1)$ is semistable, but not always stable by Ellia [El, Thm. 10].

More generally, on any $X \subset \mathbb{P}^N$, one could investigate the question of whether there exists a line bundle L on $X \setminus \mathrm{Sing}(X)$ such that $N_{X/\mathbb{P}^N}(L)$ is an Ulrich sheaf. As far as we know, this has been considered for the first time by Kleppe and Miró-Roig, who proved it possible for linear standard determinantal schemes [KMR1, Thm. 3.6].

Aside for the normal bundle, given $X \subseteq \mathbb{P}^N$ a smooth irreducible variety, one can also consider when the various bundles $\Omega_X(k)$, $T_X(k)$, $\Omega_{\mathbb{P}^N|X}(k)$ and $T_{\mathbb{P}^N|X}(k)$ are Ulrich vector bundles. We show in Proposition 4.1, that, with the exception of $T_X(k)$, a simple classification can be obtained. The case $T_X(k)$ appears to be more challenging, even though, varieties with Ulrich tangent bundle have been recently classified in [BMPT], and varieties with $T_X(1)$ Ulrich have been classified in [LR].

Finally, one could consider the same type of problems for other classes of bundles similar to Ulrich bundles, such as instanton bundles as defined in [C2], where the author deals with the case $k = 0$.

2 Examples

As in [ES, Cor. 2.3], for rank 2 bundles with $c_1(\mathcal{E}) = K_X+(n+1)H$, less vanishings are needed to verify the property of being Ulrich.

Lemma 2.1 *Let $X \subseteq \mathbb{P}^N$ is a smooth irreducible variety of dimension $n \geq 1$. Let $\mathcal{E}$ be a rank 2 vector bundle on X with $c_1(\mathcal{E}) = K_X + (n+1)H$. Then $\mathcal{E}$ is an Ulrich vector bundle if and only if $\mathcal{E}$ is 0-regular if and only if $H^j(\mathcal{E}(-j-1)) = 0$ for $0 \leq j \leq n-1$.*

Proof As is well known, the definition of Ulrich vector bundle given in [ES] is equivalent to $H^i(\mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$ (this follows, for example, by Eisenbud and Schreyer [ES, Prop. 2.1] and Beauville [B, Thm. 2.3]).

Now we have that $\mathcal{E} \cong \mathcal{E}^*(K_X+(n+1)H)$, hence the first equivalence is just [ES, Cor. 2.3]. Similarly, by Serre's duality, $\mathcal{E}$ is 0-regular if and only if $h^{n-i}(\mathcal{E}(-(n-i)-1)) = 0$ for all $i > 0$, that is $H^j(\mathcal{E}(-j-1)) = 0$ for $0 \leq j \leq n-1$. □

We produce examples of varieties as in (ii) of Theorem 1. They are a linear standard determinantal schemes. It is well known that there are smooth and irreducible ones by Peskine and Szpiro [PS, Thm. 6.2].

Proposition 2.2 *Let $n \in \{1, 2, 3\}$, let $s \geq 2$ and let $X_s \subset \mathbb{P}^{n+2}$ be a smooth irreducible ACM n-dimensional variety with resolution*

$$0 \to \mathcal{O}_{\mathbb{P}^{n+2}}(-s-1)^{\oplus s} \to \mathcal{O}_{\mathbb{P}^{n+2}}(-s)^{\oplus(s+1)} \to \mathcal{J}_{X_s/\mathbb{P}^{n+2}} \to 0. \tag{2.1}$$

Then $N_{X_s/\mathbb{P}^{n+2}}(-1)$ is an Ulrich vector bundle and it is very ample if $s \geq 3$.

Proof The fact that $N_{X_s/\mathbb{P}^{n+2}}(-1)$ is an Ulrich vector bundle follows by Kleppe and Miró-Roig [KMR1, Thm. 3.6]. We offer a proof for completeness's sake.

Observe that, for any $0 \leq j \leq n-1$, we have by Kleppe [K, Remark 2.2.6] that

$$H^j(N_{X_s/\mathbb{P}^{n+2}}(-j-2)) \cong \operatorname{Ext}^{j+1}(\mathcal{J}_{X_s/\mathbb{P}^{n+2}}, \mathcal{J}_{X_s/\mathbb{P}^{n+2}}(-j-2)). \tag{2.2}$$

Applying $\operatorname{Hom}(-, \mathcal{J}_{X_s/\mathbb{P}^{n+2}}(-j-2))$ to (2.1) we get the exact sequence

$$\begin{aligned}&\operatorname{Ext}^j(\mathcal{O}_{\mathbb{P}^{n+2}}(-s-1)^{\oplus s}, \mathcal{J}_{X_s/\mathbb{P}^{n+2}}(-j-2))\\ &\quad\to \operatorname{Ext}^{j+1}(\mathcal{J}_{X_s/\mathbb{P}^{n+2}}, \mathcal{J}_{X_s/\mathbb{P}^{n+2}}(-j-2)) \to\\ &\quad\to \operatorname{Ext}^{j+1}(\mathcal{O}_{\mathbb{P}^{n+2}}(-s)^{\oplus(s+1)}, \mathcal{J}_{X_s/\mathbb{P}^{n+2}}(-j-2))\end{aligned}$$

that is

$$H^j(\mathcal{J}_{X_s/\mathbb{P}^{n+2}}(s-j-1))^{\oplus s} \to \operatorname{Ext}^{j+1}(\mathcal{J}_{X_s/\mathbb{P}^{n+2}}, \mathcal{J}_{X_s/\mathbb{P}^{n+2}}(-j-2)) \to$$
$$\to H^{j+1}(\mathcal{J}_{X_s/\mathbb{P}^{n+2}}(s-j-2))^{\oplus(s+1)}.$$

On the other hand, $H^j(\mathcal{J}_{X_s/\mathbb{P}^{n+2}}(s-j-1)) = H^{j+1}(\mathcal{J}_{X_s/\mathbb{P}^{n+2}}(s-j-2)) = 0$ for $0 \le j \le n-1$ by (2.1). Hence $\mathrm{Ext}^{j+1}(\mathcal{J}_{X_s/\mathbb{P}^{n+2}}, \mathcal{J}_{X_s/\mathbb{P}^{n+2}}(-j-2)) = 0$ and therefore $N_{X_s/\mathbb{P}^{n+2}}(-1)$ is an Ulrich vector bundle by (2.2) and Lemma 2.1.

Now assume that $s \ge 3$. To see that $N_{X_s/\mathbb{P}^{n+2}}(-1)$ is very ample it is enough, by Lopez and Sierra [LS, Thm. 1], to show that X_s does not contain a line. Let $d = \deg X_s$ and let g be the sectional genus of X_s. Since, when $n \ge 2$, a general hyperplane section of X_s has the same resolution, we get by Ellia [El, Rmk. 1] that

$$d = \frac{s(s+1)}{2}, g = 1 + \frac{s(2s^2+3s+1)}{6} - s(s+1). \tag{2.3}$$

Note that $d \ge 6$, hence we are done if $n = 1$. On the other hand, if $n = 3$ and X_s contains a line L, then picking a general hyperplane H containing L, we get a surface $X_s \cap H$ containing L. Therefore, we will be done if we show that when $n = 2$ there is no line on X_s. Assume therefore that $n = 2$ and that $S := X_s$ contains a line L. By Kleppe and Miró-Roig [KMR2, Thm. 4.1] we know that $\mathrm{Pic}(S)$ is generated by the hyperplane section H and by K_S, hence there are two integers a, b such that $L \sim aH + bK_S$. This gives the equations

$$H \cdot (aH + bK_S) = 1, -2 = (aH + bK_S)[aH + (b+1)K_S]. \tag{2.4}$$

Setting

$$\alpha = (H \cdot K_S)^2 - dK_S^2 \text{ and } \beta = 2d + 1 + H \cdot K_S$$

it easily follows from (2.4) that

$$\alpha b^2 + \alpha b - \beta = 0. \tag{2.5}$$

Using (2.3) and the standard relation between the invariants of surfaces in $\mathbb{P}^4$ [H2, Appendix A, Ex. 4.1.3], one gets that

$$H \cdot K_S = \frac{s(4s^2 - 9s - 13)}{6}, K_S^2 = \frac{s(21s^3 - 118s^2 + 87s + 226)}{24} \tag{2.6}$$

and therefore

$$\alpha = \frac{s^2(s^4 + 3s^3 + s^2 - 3s - 2)}{144}, \beta = \frac{4s^3 - 3s^2 - 7s + 6}{6}. \tag{2.7}$$

Solving (2.5) it is easily seen that $b \le 1$. On the other hand, $b = 0$ gives the contradiction $\beta = 0$. Therefore, $b \le -1$. Now $H - L$ is base-point-free, hence $(H-L)^2 \ge 0$, that is $-2 - L \cdot K_S = L^2 \ge 2 - d$, and then

$$d(aH + bK_S) \cdot K_S = dL \cdot K_S \le d(d-4).$$

Now, using that $ad = 1 - bH \cdot K_S$ by (2.4), we deduce that

$$\alpha b \ge -d(d-4) + H \cdot K_S$$

and therefore

$$-\alpha \geq \alpha b \geq -d(d-4) + H \cdot K_S.$$

Replacing the values in (2.6) and (2.7) we find a contradiction if $s \geq 5$. On the other hand, (2.5) has no integer solutions for $s = 3, 4$ and we are done. □

Remark 2.3 As observed in the introduction, for $n = 1$, it is clear that there are many examples of curves that satisfy (ii) of Theorem 1 but are not linear standard determinantal. In fact, there are even subcanonical ones, see Remark 3.2. On the other hand, for $n \in \{2, 3\}$, we do not know examples of varieties as in (ii) of Theorem 1, aside from linear standard determinantal ones. We have checked all papers with examples of surfaces X in $\mathbb{P}^4$ and 3-folds X in $\mathbb{P}^5$ of low degree (such as in [R1, R2, DP] and several other papers) and, aside from linear standard determinantal ones, none of them has $N_{X/\mathbb{P}^{n+2}}(-1)$ being an Ulrich vector bundle. An explanation is given as follows.

Let $n \in \{2, 3\}$ and let $X \subset \mathbb{P}^{n+2}$ be a smooth irreducible n-dimensional variety such that $N_{X/\mathbb{P}^{n+2}}(-1)$ is an Ulrich vector bundle (or, equivalently, by Lemma 2.1, X is as in (ii) of Theorem 1).

If $n = 3$ and S is a smooth hyperplane section of X, we have that $N_{S/\mathbb{P}^4}(-1) \cong N_{X/\mathbb{P}^5}(-1)_{|S}$ is also an Ulrich vector bundle by Beauville [B, (3.4)]. Thus, in the study of the degree d and the sectional genus g, of varieties $X \subset \mathbb{P}^{n+2}$ as earlier, we can restrict to the case $n = 2$. Now, using the formula for c_2 in [C1, (2.2)], we get that

$$\chi(\mathcal{O}_X) = \frac{1}{2}(d^2 - 4d - 5g + 5). \tag{2.8}$$

In all papers with examples of surfaces in $\mathbb{P}^4$ and 3-folds in $\mathbb{P}^5$ of low degree (such as in [R1, R2, DP] and several other papers), there is only one case satisfying (2.8) and not linear standard determinantal, described as follows. Let $T \subset \mathbb{P}^4$ be the elliptic quintic scroll and let X be a surface linked to T in the complete intersection of a cubic and a quintic hypersurface containing T. Then $d = 10$, $g = 11$ and $\chi(\mathcal{O}_X) = 5$ by Ranestad [R1, Lemma 9.20]. Let C be a general hyperplane section of X. To show that $N_{X/\mathbb{P}^4}(-1)$ is not an Ulrich vector bundle, since $N_{C/\mathbb{P}^3}(-1) \cong N_{X/\mathbb{P}^4}(-1)_{|C}$, it is enough, by Beauville [B, (3.4)], to prove that $H^0(N_{C/\mathbb{P}^3}(-2)) \neq 0$, for then $N_{C/\mathbb{P}^3}(-1)$ is not an Ulrich vector bundle. To this end, by semicontinuity, it is enough to show that $H^0(N_{C/\mathbb{P}^3}(-2)) \neq 0$ when X is a surface linked to T in the complete intersection of a general cubic F and a quintic hypersurface containing T. We claim that such an F has isolated singularities. First observe that $\mathcal{I}_{T/\mathbb{P}^4}(3)$ is 0-regular. In fact, consider the exact sequences

$$0 \to \mathcal{I}_{T/\mathbb{P}^4}(l) \to \mathcal{O}_{\mathbb{P}^4}(l) \to \mathcal{O}_T(l) \to 0.$$

For $l = -1$ we get that $H^4(\mathcal{I}_{T/\mathbb{P}^4}(-1)) = 0$, for $l = 0$ we have that $H^3(\mathcal{I}_{T/\mathbb{P}^4}) = H^2(\mathcal{O}_T) = 0$, for $l = 1$ we see that $H^2(\mathcal{I}_{T/\mathbb{P}^4}(1)) = H^1(\mathcal{O}_T(1)) = 0$. Now, if E is the elliptic curve section of T, we know by Castelnuovo's theorem that $H^1(\mathcal{I}_{E/\mathbb{P}^3}(2)) = 0$, hence the exact sequence

$$0 \to \mathcal{I}_{T/\mathbb{P}^4}(1) \to \mathcal{I}_{T/\mathbb{P}^4}(2) \to \mathcal{I}_{E/\mathbb{P}^3}(2) \to 0$$

shows that $H^1(\mathcal{I}_{T/\mathbb{P}^4}(2)) = 0$, since T is linearly normal.

Thus, we have shown that $\mathcal{I}_{T/\mathbb{P}^4}(3)$ is 0-regular, hence globally generated (see for example [L, Thm. 1.8.5]). It follows by Otwinowska and Saito [OS, Thm. 1.2] that F has isolated singularities and therefore we have that C linked to E in the complete intersection of a smooth cubic F', hyperplane section of F, and a quintic hypersurface containing E. This gives an inclusion

$$0 \to \mathcal{O}_C(3H - E) \to N_{C/\mathbb{P}^3}(-2)$$

and we will prove that $H^0(\mathcal{O}_C(3H - E)) \neq 0$. To this end, we have the exact sequence

$$0 \to \mathcal{O}_{F'}(-2) \to \mathcal{O}_{F'}(3H - E) \to \mathcal{O}_C(3H - E) \to 0$$

that implies that $H^0(\mathcal{O}_C(3H - E)) = H^0(\mathcal{O}_{F'}(3H - E))$ and the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{I}_{E/\mathbb{P}^3}(3) \to \mathcal{O}_{F'}(3H - E) \to 0$$

gives that

$$h^0(\mathcal{O}_C(3H - E)) = h^0(\mathcal{O}_{F'}(3H - E)) = h^0(\mathcal{I}_{E/\mathbb{P}^3}(3)) - 1 = 4.$$

This proves that, when X is linked to T in the complete intersection of a cubic and a quintic hypersurface, $N_{X/\mathbb{P}^4}(-1)$ is not an Ulrich vector bundle.

3 Proof of Theorem 1

Before proving the theorem, we deal separately with the special case of codimension 2 subcanonical varieties.

Proposition 3.1 *Let $X \subset \mathbb{P}^{n+2}$ be a smooth irreducible variety of dimension $n \geq 1$, degree at least 2 such that $K_X = eH$ for some $e \in \mathbb{Z}$, $H \in |\mathcal{O}_X(1)|$. Then:*

(i) *Either $H^0(N_{X/\mathbb{P}^{n+2}}(-2)) \neq 0$ or there is an integer $t \geq -e - n - 2$ such that $H^1(N_{X/\mathbb{P}^{n+2}}(t)) \neq 0$.*
(ii) *If $n \geq 2$, then $N_{X/\mathbb{P}^{n+2}}(-1)$ is not an Ulrich vector bundle.*

Proof First, we see that (i) implies (ii). In fact, if $n \geq 2$ and $N_{X/\mathbb{P}^{n+2}}(-1)$ is an Ulrich vector bundle, then $H^0(N_{X/\mathbb{P}^{n+2}}(-2)) = 0$ and $H^1(N_{X/\mathbb{P}^{n+2}}(t)) = 0$ for every $t \in \mathbb{Z}$ since Ulrich vector bundles are ACM (see, for example, [B, (3.1)]), contradicting (i).

As for (i), assume now that $H^0(N_{X/\mathbb{P}^{n+2}}(-2)) = H^1(N_{X/\mathbb{P}^{n+2}}(t)) = 0$ for every integer $t \geq -e-n-2$.

Note that X is not a complete intersection, for otherwise $N_{X/\mathbb{P}^{n+2}} \cong \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$ for some integers $a \geq b \geq 1$ such that $ab \geq 2$, hence $a \geq 2$. But then, $H^0(N_{X/\mathbb{P}^{n+2}}(-2)) \neq 0$, a contradiction.

In particular X is non-degenerate.

Since X is subcanonical, in order to get a contradiction, it is enough to show that X is projectively normal: In fact, then the Evans-Griffith's theorem [EG, Thm. 2.4] would imply that X is a complete intersection (see for example [BC, Rmk. 13]).

Let $V = H^0(\mathcal{O}_{\mathbb{P}^{n+2}}(1))$, let $P^1(\mathcal{O}_X(1))$ be the sheaf of principal parts and consider, as in [Ei, Proof of Thm. 2.4], the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N^*_{X/\mathbb{P}^{n+2}}(1) & \xrightarrow{\cong} & N^*_{X/\mathbb{P}^{n+2}}(1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^1_{\mathbb{P}^{n+2}}(1)_{|X} & \longrightarrow & V \otimes \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \cong & & \\
0 & \longrightarrow & \Omega^1_X(1) & \longrightarrow & P^1(\mathcal{O}_X(1)) & \longrightarrow & \mathcal{O}_X(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Pick an integer $l \geq 0$. Tensoring the given diagram by $\mathcal{O}_X(l)$ and observing that

$$P^1(\mathcal{O}_X(1)) \otimes \mathcal{O}_X(l) \cong P^1(\mathcal{O}_X(l+1))$$

by Ein [Ei, (2.2)], we get the commutative diagram

$$\begin{array}{ccc}
V \otimes H^0(\mathcal{O}_X(l)) & & \\
\downarrow f_l & \overset{h_l}{\searrow} & \\
H^0(P^1(\mathcal{O}_X(l+1))) & \xrightarrow{g_l} & H^0(\mathcal{O}_X(l+1)) \\
\downarrow & & \\
H^1(N^*_{X/\mathbb{P}^{n+2}}(l+1)) & &
\end{array}$$

Now $H^1(N^*_{X/\mathbb{P}^{n+2}}(l+1)) \cong H^1(N_{X/\mathbb{P}^{n+2}}(l-e-n-2)) = 0$, hence f_l is surjective for every $l \geq 0$ and so is g_l by Ein [Ei, Prop. 2.3]. It follows that h_l is surjective for every $l \geq 0$ and the commutative diagram

$$\begin{array}{ccc} V \otimes H^0(\mathcal{O}_{\mathbb{P}^{n+2}}(l)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^{n+2}}(l+1)) \\ \downarrow{\scriptstyle \mathrm{Id}_V \otimes r_l} & & \downarrow{\scriptstyle r_{l+1}} \\ V \otimes H^0(\mathcal{O}_X(l)) & \xrightarrow{h_l} & H^0(\mathcal{O}_X(l+1)) \end{array}$$

shows by induction that $r_l : H^0(\mathcal{O}_{\mathbb{P}^{n+2}}(l)) \to H^0(\mathcal{O}_X(l))$ is surjective for every $l \geq 0$ and we are done. □

Remark 3.2 Instead, there are many subcanonical curves $X \subset \mathbb{P}^3$ with $H^0(N_{X/\mathbb{P}^3}(-2)) = 0$, hence with $N_{X/\mathbb{P}^3}(-1)$ an Ulrich vector bundle (see, for example, [BE]).

We now prove our first result.

Proof of Theorem 1 If X is embedded linearly in $\mathbb{P}^N$, we saw in the introduction that $N_{X/\mathbb{P}^N}(-1)$ is an Ulrich vector bundle. If X is as in (ii) of Theorem 1, since $c_1(N_{X/\mathbb{P}^{n+2}}(-1)) = K_X + (n+1)H$, we see that $N_{X/\mathbb{P}^N}(-1)$ is an Ulrich vector bundle by Lemma 2.1. Moreover, if, in addition, X does not contain a line, then $N_{X/\mathbb{P}^{n+2}}(-1)$ is very ample by Lopez and Sierra [LS, Thm. 1].

Vice versa assume that $X \subset \mathbb{P}^N$ is a smooth irreducible variety of dimension $n \geq 1$, degree $d \geq 1$ and such that $N_{X/\mathbb{P}^N}(-k)$ is an Ulrich vector bundle. In particular $H^0(N_{X/\mathbb{P}^N}(-k-1)) = 0$ and therefore $k \geq 1$, since, as is well known, $N_{X/\mathbb{P}^N}(-1)$ is globally generated.

If $d = 1$ we have that $X = \mathbb{P}^n$ embedded linearly in $\mathbb{P}^N$.

Assume from now on that $d \geq 2$ and let $H \in |\mathcal{O}_X(1)|$.

We first show that $k = 1$ and $\mathrm{codim}_{\mathbb{P}^N} X = 2$.

Note that $N - n \geq 2$. In fact, if $N - n = 1$ then $N_{X/\mathbb{P}^N} = \mathcal{O}_X(d)$ and to have $\mathcal{O}_X(d-k)$ Ulrich implies that $H^0(\mathcal{O}_X(d-k-1)) = H^n(\mathcal{O}_X(d-k-n)) = 0$, that is $k \geq d \geq 2$ and $H^0(\mathcal{O}_X(k-2)) = 0$, a contradiction.

Let C be a general curve section of X and let g be its genus, so that

$$2g - 2 = [K_X + (n-1)H] \cdot H^{n-1} = K_X \cdot H^{n-1} + (n-1)d$$

and therefore

$$K_X \cdot H^{n-1} = 2g - 2 - (n-1)d. \tag{3.1}$$

Since $c_1(N_{X/\mathbb{P}^N}(-k)) = K_X + (N + 1 - kN + kn)H$ we get, using (3.1) that

$$\begin{aligned} \deg(N_{X/\mathbb{P}^N}(-k)_{|C}) &= [K_X + (N + 1 - kN + kn)H] \cdot H^{n-1} \\ &= 2(d + g - 1) - (k-1)(N-n)d. \end{aligned}$$

Now [CH, Lemma 2.4(iii)] implies that

$$\deg(N_{X/\mathbb{P}^N}(-1)_{|C}) = (N-n)(d+g-1)$$

and we deduce that

$$(N-n-2)(d+g-1) + (k-1)(N-n)d = 0$$

so that $k = 1$ and $N - n = 2$.

If $1 \leq n \leq 3$, we have that X is as in (ii) of Theorem 1 by Lemma 2.1.

To finish the proof, we show that the case $n \geq 4$ does not occur. In fact, if $n \geq 4$, then Barth–Larsen-type theorems (see for example [H1, Thm. 2.2]) give that X is subcanonical and now Proposition 3.1(ii) implies that $N_{X/\mathbb{P}^N}(-1)$ is not Ulrich vector bundle, a contradiction. □

4 Other Standard Bundles Associated to *X*

In this section, we give a very simple and complete answer to the question, for a smooth irreducible variety $X \subseteq \mathbb{P}^N$, of when $\Omega_X(k)$, $\Omega_{\mathbb{P}^N|X}(k)$ and $T_{\mathbb{P}^N|X}(k)$ are Ulrich vector bundles for some integer k.

Proposition 4.1 *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$ and let k be an integer. Then*

(i) $\Omega_X(k)$ *is an Ulrich vector bundle if and only if $k = 2$ and X is a line.*
(ii) $\Omega_{\mathbb{P}^N|X}(k)$ *is an Ulrich vector bundle if and only if $k = 2$ and X is a rational normal curve in $\mathbb{P}^N$.*
(iii) $T_{\mathbb{P}^N|X}(k)$ *is an Ulrich vector bundle if and only if either $N = 2, k = -1$ and X is a conic or $N = 1, k = -2$ and $X = \mathbb{P}^1$.*

Proof We let g to be the sectional genus of X and d its degree. To see (i), the assertion being obvious when $k = 2$ and X is a line, assume vice versa that $\Omega_X(k)$ is an Ulrich vector bundle. If $n \geq 2$, since Ulrich vector bundles are ACM by Beauville [B, (3.1)], we get the contradiction $H^1(\Omega_X) = 0$. If $n = 1$, then the Ulrich condition $H^i(\Omega_X(k-1)) = 0$ for $i \geq 0$ gives that either $g = 0$ and $d = 1, k = 2$ or $g \geq 1$. In the latter case, we have that $H^0(\mathcal{O}_X(1-k)) = 0$, hence $k \geq 2$, but then $H^0(\Omega_X(k-1)) \neq 0$, a contradiction. This proves (i).

Now we deal with (ii) and (iii).

If X is a rational normal curve in $\mathbb{P}^N$, then, as is well known, $\Omega_{\mathbb{P}^N|X} \cong \mathcal{O}_{\mathbb{P}^1}(-N-1)^{\oplus N}$, and therefore

$$H^i(\Omega_{\mathbb{P}^N|X}(1)) \cong H^i(\mathcal{O}_{\mathbb{P}^1}(-1))^{\oplus N} = 0$$

for every $i \geq 0$, so that $\Omega_{\mathbb{P}^N|X}(2)$ is an Ulrich vector bundle.

If $N = 2, k = -1$ and X is a conic or if $N = 1, k = -2$ and $X = \mathbb{P}^1$, it is easily checked that $T_{\mathbb{P}^N|X}(k)$ is an Ulrich vector bundle.

Vice versa let $\mathcal{E}$ be either $\Omega_{\mathbb{P}^N|X}(k)$ or $T_{\mathbb{P}^N|X}(k)$ and assume that $\mathcal{E}$ is an Ulrich vector bundle. As is well known, since $\Omega_{\mathbb{P}^N|X}(2)$ and $T_{\mathbb{P}^N|X}(-1)$ are globally generated, we get that $k \le 2$ in the first case and $k \le -1$ in the second case. Let C be a general curve section of X, let g be its genus and d its degree. Set

$$\varepsilon = \begin{cases} -1 & \text{if } \mathcal{E} = \Omega_{\mathbb{P}^N|X}(k), \\ 1 & \text{if } \mathcal{E} = T_{\mathbb{P}^N|X}(k) \end{cases}$$

so that $c_1(\mathcal{E}) = [\varepsilon(N+1) + kN]H$. Then [CH, Lemma 2.4(iii)] implies that

$$[\varepsilon(N+1) + kN]d = N(d + g - 1)$$

that is

$$N(g-1) + [N(1-\varepsilon-k) - \varepsilon]d = 0. \tag{4.1}$$

If $d = 1$ (hence in particular if $N = 1$), we deduce that $N = 1, X = \mathbb{P}^1$ and either $\varepsilon = -1, k = 2$ or $\varepsilon = 1, k = -2$, giving rise to the cases in (ii) and (iii). Suppose then that $d \ge 2$, hence also $N \ge 2$. It follows easily that $N(1-\varepsilon-k) - \varepsilon > 0$, hence (4.1) implies that $g = 0$ and either $\varepsilon = -1, k = 2, d = N$ or $\varepsilon = 1, k = -1, d = N = 2$. In the latter case we have the case of a conic in (iii). To finish the proof, consider the case $\varepsilon = -1, k = 2, d = N$. If $n \ge 2$, then the Ulrich condition for $\Omega_{\mathbb{P}^N|X}(2)$ gives in particular that $H^1(\Omega_{\mathbb{P}^N|X}) = 0$. But then the Euler sequence

$$0 \to \Omega_{\mathbb{P}^N|X} \to \mathcal{O}_X(-1)^{\oplus(N+1)} \to \mathcal{O}_X \to 0$$

gives a contradiction. Hence, $n = 1$ and if X spans a $\mathbb{P}^r$ we have that

$$0 = H^0(\Omega_{\mathbb{P}^N|X}(1)) = H^0(\Omega_{\mathbb{P}^r|X}(1) \oplus \mathcal{O}_X^{N-r})$$

so that $r = N$ and X is a rational normal curve in $\mathbb{P}^N$. □

5 A Personal Note About Alberto Collino

I first met Alberto while I was a graduate student at Brown University in the mid-1980s. He was a visiting professor, and I learned then the way, I think, he always has been: a very pleasant person in human relationships and one always available. He made me feel confident. This was confirmed over the years (even recently), when, after coming back to Italy, I visited Torino. He was a complete mathematician and, among his merits, it is not to be underestimated the contribution that he gave to the Italian algebraic geometry. He was part of the generation of important mathematicians that went to the East Cost to follow a Ph.D. program and then brought back to Italy the way of working learned there.

Acknowledgments Research partially supported by PRIN "Advances in Moduli Theory and Birational Classification" and GNSAGA-INdAM.

References

[B] A. Beauville. *An introduction to Ulrich bundles*. Eur. J. Math. **4** (2018), no. 1, 26–36.

[BC] E. Ballico, L. Chiantini. *On smooth subcanonical varieties of codimension 2 in $\mathbb{P}^n$, $n \geq 4$*. Ann. Mat. Pura Appl. (4) **135** (1983), 99–117 (1984).

[BE] E. Ballico, Ph. Ellia. *Some more examples of curves in $\mathbb{P}^3$ with stable normal bundle*. J. Reine Angew. Math. **350** (1984), 87–93.

[BMPT] V. Benedetti, P. Montero, Y. Prieto Montañez, S. Troncoso. *Projective manifolds whose tangent bundle is Ulrich*. Preprint 2021, arXiv:2108.13944.

[C1] G. Casnati. *Special Ulrich bundles on non-special surfaces with $p_g = q = 0$*. Internat. J. Math. **28** (2017), no. 8, 1750061, 18 pp.

[C2] G. Casnati. *Tangent, cotangent, normal and conormal bundles are almost never instanton bundles*. Preprint 2022, in preparation.

[CH] M. Casanellas, R. Hartshorne. *Stable Ulrich bundles*. With an appendix by F. Geiss, F.-O. Schreyer. Internat. J. Math. **23** (2012), no. 8, 1250083, 50 pp.

[CMRPL] L. Costa, R. M. Miró-Roig, J. Pons-Llopis. *Ulrich bundles*. De Gruyter Studies in Mathematics, **77**, De Gruyter 2021.

[DP] W. Decker, S. Popescu. *On surfaces in $\mathbb{P}^4$ and 3-folds in $\mathbb{P}^5$*. In: Vector bundles in algebraic geometry (Durham, 1993), 69–100, London Math. Soc. Lecture Note Ser., **208**, Cambridge Univ. Press, Cambridge, 1995.

[Ei] L. Ein. *Vanishing theorems for varieties of low codimension*. In: Algebraic geometry (Sundance, UT, 1986), 71–75, Lecture Notes in Math., 1311, Springer, Berlin, 1988.

[El] Ph. Ellia. *Exemples de courbes de $\mathbb{P}^3$ à fibré normal semi-stable, stable*. Math. Ann. **264** (1983), no. 3, 389–396.

[EG] E. G. Evans, P. Griffith. *The syzygy problem*. Ann. of Math. (2) **114** (1981), no. 2, 323–333.

[EH] G. Ellingsrud, A. Hirschowitz. *Sur le fibré normal des courbes gauches*. C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), no. 7, 245–248.

[ES] D. Eisenbud, F.-O. Schreyer. *Resultants and Chow forms via exterior syzygies*. J. Amer. Math. Soc. **16** (2003), no. 3, 537–579.

[H1] R. Hartshorne. *Varieties of small codimension in projective space*. Bull. Amer. Math. Soc. **80** (1974), 1017–1032.

[H2] R. Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics **52**. Springer-Verlag, New York-Heidelberg, 1977.

[K] J. O. Kleppe. *The Hilbert-flag scheme, its properties and its connection with the Hilbert scheme. Applications to curves in the 3-space*. Thesis, University of Oslo (1981).

[KMR1] J. O. Kleppe, R. M. Miró-Roig. *On the normal sheaf of determinantal varieties*. J. Reine Angew. Math. **719** (2016), 173–209.

[KMR2] J. O. Kleppe, R. M. Miró-Roig. *The representation type of determinantal varieties*. Algebr. Represent. Theory **20** (2017), no. 4, 1029–1059.

[L] R. Lazarsfeld. *Positivity in algebraic geometry, I*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge **48**, Springer-Verlag, Berlin, 2004.

[LR] A. F. Lopez, D. Raychaudhury. In preparation.

[LS] A. F. Lopez, J. C. Sierra. *A geometrical view of Ulrich vector bundles*. Preprint 2021, arXiv:2105.05979. To appear on Int. Math. Res. Not..

[OS] A. Otwinowska, M. Saito. *Monodromy of a family of hypersurfaces containing a given subvariety*. Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 3, 365–386.

[PS] C. Peskine, L. Szpiro. *Liaison des variétés algébriques. I*. Invent. Math. **26** (1974), 271–302.

[R1] K. Ranestad. *On smooth surfaces of degree ten in projective four space*. Ph.D. thesis, University of Oslo, 1988. https://silo.tips/download/on-smooth-surfaces-of-degree-10-in-the-projective-fourspace

[R2] K. Ranestad. *Surfaces of degree 10 in the projective fourspace*. In: Problems in the theory of surfaces and their classification (Cortona, 1988), 271–307, Sympos. Math., XXXII, Academic Press, London, 1991.