Solid Mechanics and Its Applications

Yuriy Povstenko

Fractional Thermoelasticity

Second Edition

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Solid Mechanics and Its Applications

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Preface to the Second Edition

If at first you don't succeed,
Try, try, try again.

W. E. Hickson

After the appearance of the first edition of the book *Fractional Thermoelasticity*, a growing interest in this subject has been observed, many new research articles have been published.

In comparison with the first edition, the book has been thoroughly revised and significantly extended. Although the general structure of the book resembles the first edition, most of the chapters have been enlarged, new material has been added. As the Mittag-Leffler functions and the Wright function appear in solutions of various types of equations with fractional operators, the main properties of these function have been presented in Chap. 1. The statement of the problems of fractional thermoelasticity is discussed in Chap. 2 in more details. The new problems of time-harmonic impact on the boundary have been solved in Chaps. 3 and 5. Two approaches to obtaining the space-time-fractional advection-diffusion equation are presented in Chap. 9. The second edition also contains two new chapters. In Chap. 10, cracks in the framework of fractional thermoelasticity are considered. Line cracks in a plane and circular cracks in a solid are investigated, the corresponding stress intensity factors are evaluated depending on the order of fractional derivative. Nonlocal elasticity assumes integral constitutive equation for the stress tensor, takes into account interatomic long-range forces, reduces to the classical theory of elasticity in the long wave-length limit and to the atomic lattice theory in the short wave-length limit. Often, the nonlocal kernel of a stress constitutive equation is selected as the Green function of the Cauchy problem for partial differential equation. Chapter 11 is devoted to the new theory of nonlocal elasticity in which the nonlocal modulus is the Green function of the Cauchy problem for the fractional heat conduction equation.

Several misprints have been corrected. Similarly to the first edition, the second edition of the book contains a large number of figures describing the influence of order of fractional derivatives on temperature and stress distribution in solids.

Częstochowa, Poland Yuriy Povstenko
May 2024

Preface to the First Edition

Thus, when God said
Let there be light, He implied,
Let there also be heat –
and there was heat.
I. McNeil

What would physics look like without gravitation?
Albert Einstein

What would physics look like without heat conduction?
Yuriy Povstenko

The famous Fourier law, which states the linear dependence between the heat flux vector and the temperature gradient, was formulated by Fourier in 1822 and marked the beginning of the classical theory of heat conduction. A few years later, Fourier's disciple Duhamel coupled the temperature field and the body deformation and pioneered studies on thermoelasticity.

The classical theory of heat conduction based on the phenomenological Fourier law, which ignores processes occurring at the microscopic level, is quite acceptable for different physical situations. However, many theoretical and experimental studies of transport phenomena testify that in media with complex internal structure (amorphous, porous, random and disordered materials, fractals, polymers, glasses, dielectrics and semiconductors, etc.) the classical Fourier law and the standard parabolic heat conduction equation are no longer accurate enough, and physical processes occurring at the microscopic level, in one way or another, should be taken into account. This leads to formulation of nonclassical theories, in which the Fourier law and the parabolic heat conduction equation are replaced by more general equations.

Each generalization of the heat conduction equation results in formulation of the corresponding generalized theory of thermal stresses. For example, thermoelasticity without energy dissipation proposed by Green and Naghdi [1] is based on the wave equation for temperature. Cattaneo's telegraph equation for temperature leads to the generalized thermoelasticity of Lord and Shulman [2]. This book is devoted to fractional thermoelasticity, i.e., thermoelasticity based on the heat conduction equation with differential operators of fractional order. Time-fractional differential operators describe memory effects, space-fractional differential operators deal with the long-range interaction. It should be emphasized that fractional calculus has been successfully used in physics, geology, chemistry, rheology, engineering, bioengineering, robotics, etc. The first paper on fractional thermoelasticity was published by the author in 2005. During the last decade, substantial literature on this subject has evolved, but there is no book which sums up investigations in this field. The present book, which for the major part is based on author's research, fills in such a blank.

The book is organized as follows. Chapter 1 presents essentials of fractional calculus. Different kinds of integral and differential operators of fractional order are discussed (the Riemann-Liouville fractional integrals, the Riemann-Liouville and Caputo fractional derivatives, and the Riesz fractional operators). Chapter 2 is devoted to time- and space-nonlocal generalizations of the Fourier law, the corresponding generalizations of the heat conduction equation and formulation of associated theories of fractional thermoelasticity. Different kinds of boundary conditions for the time-fractional heat conduction equation are analyzed including the conditions of perfect thermal contact and the moving interface boundary conditions at the solid-liquid interface. In Chaps. 3 and 4, the axisymmetric problems for the time-fractional heat conduction and associated thermal stresses are considered in polar and cylindrical coordinates, respectively. The central symmetric problem in spherical coordinates are studied in Chap. 5. It should be noted that the considered theory interpolates the classical theory of thermal stresses based on the parabolic heat conduction equation and the theory of thermoelasticty without energy dissipation proposed by Green and Naghdi and started from the hyperbolic wave equation for temperature. Chapter 6 presents thermoelasticity based on the space-time-fractional heat conduction equation. Chapter 7 is devoted to thermoelasticty which uses the fractional telegraph equation for temperature (fractional generalization of the well-known theory of Lord and Shulman). In Chap. 8, we formulate equations of fractional thermoelasticity of thin shells (solids with one size being small with respect to two other sizes). The generalized boundary conditions of nonperfect thermal contact for the time-fractional heat conduction in composite medium are also formulated. It is well-known that from mathematical viewpoint, the Fourier law and the theory of heat conduction and the Fick law and the theory of diffusion are identical. Chapter 9 deals with the theory of diffusive stresses caused by fractional advection-diffusion equation.

The book contains a large number of Figures which show the characteristic features of temperature and stress distributions and represent the whole spectrum of order of fractional operators.

The corresponding sections of the book may be used by university lecturers of courses in heat and mass transfer, continuum mechanics, thermal stresses as well as in fractional calculus and its applications for graduate and postgraduate students. The book presents a picture of the state-of-the-art of fractional thermoelasticity and will also serve as a reference handbook for specialists in applied mathematics, physics, geophysics, elasticity, thermoelasticity and engineering sciences. The book provides information which puts the reader at the forefront of current research in the field of fractional thermoelasticity and is complemented with extensive references in order to stimulate further studies in this field as well as in the related areas.

Częstochowa, Poland Yuriy Povstenko
November 2014

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1. Green, A.E., Naghdi, P.M.: Thermoelasticity without energy dissipation. J. Elast. **31**, 189–208 (1993)
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About This Book

The book is devoted to fractional thermoelasticity, i.e. thermoelasticity based on the heat conduction equation with differential operators of fractional order. Time-fractional differential operators describe memory effects, space-fractional differential operators deal with the long-range interaction. The first paper on fractional thermoelasticity was published by the author in 2005. During the last two decades, substantial literature on this subject has evolved, but there is no book which sums up investigations in this field. The present book, which for the major part is based on author's research, fills in such a blank. The book contains a large number of Figures which show the characteristic features of temperature and stress distributions and represent the whole spectrum of order of fractional operators.

The corresponding sections of the book may be used by university lecturers of courses in heat and mass transfer, continuum mechanics, thermal stresses as well as in fractional calculus and its applications for graduate and postgraduate students. The book presents a picture of the state-of-the-art of fractional thermoelasticity and will also serve as a reference handbook for specialists in applied mathematics, physics, geophysics, elasticity, thermoelasticity and engineering sciences. The book provides information which puts the reader at the forefront of current research in the field of fractional thermoelasticity and is complemented with extensive references in order to stimulate further studies in this field as well as in the related areas.

Contents

Chapter 1
Essentials of Fractional Calculus

All the forces in the world are not so powerful as an idea whose time has come.

Victor Hugo

Abstract Essentials of fractional calculus are presented. Different kinds of integral and differential operators of fractional order are discussed. The notion of the Riemann–Liouville fractional integral is introduced as a natural generalization of the repeated integral written in a convolution-type form. The Riemann–Liouville fractional derivative is defined as left-inverse to the Riemann–Liouville fractional integral. The Caputo fractional derivative and the Riesz fractional operators (including the fractional Laplace operator) are considered. The cumbersome aspects of space-fractional differential operators disappear when one computes their Fourier integral transforms. In solutions of various types of equations with fractional operators, there appear the Mittag-Leffler functions and the Wright function. The main properties of the Mittag-Leffler functions and the Wright function are presented.

1.1 Riemann–Liouville Fractional Integrals

The primitive of a function $f(t)$ (the antiderivative of a function $f(t)$) will be denoted as

$$I^1 f(t) = \int_0^t f(\tau)\,\mathrm{d}\tau. \tag{1.1}$$

Next, consider the twofold primitive of a function $f(t)$

$$I^2 f(t) = \int_0^t \mathrm{d}\eta \int_0^\eta f(\tau)\,\mathrm{d}\tau. \tag{1.2}$$

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Integrating (1.2) by parts gives

$$I^2 f(t) = \int_0^t (t-\tau)\, f(\tau)\, \mathrm{d}\tau. \tag{1.3}$$

Similarly, integrating $n-1$ times by parts the n–fold primitive of a function $f(t)$

$$I^n f(t) = \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} f(t_n)\, \mathrm{d}t_n\,, \tag{1.4}$$

we obtain a single integral

$$I^n f(t) = \frac{1}{(n-1)\,!} \int_0^t (t-\tau)^{n-1} f(\tau)\, \mathrm{d}\tau = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} f(\tau)\, \mathrm{d}\tau, \tag{1.5}$$

where n is a positive integer, $\Gamma(n)$ is the Gamma function.

The notion of the Riemann–Liouville fractional integral is introduced as a natural generalization of the repeated integral $I^n f(t)$ written in a convolution-type form [20, 28, 57, 68]:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \qquad t>0, \quad \alpha>0. \tag{1.6}$$

The Laplace transform rule for the fractional integral reads

$$\mathcal{L}\{I^\alpha f(t)\} = \frac{1}{s^\alpha} f^*(s), \tag{1.7}$$

where the asterisk denotes the transform, s is the Laplace transform variable.

The convolution-type form of the Riemann–Liouville fractional integral (1.6) can be extended to [28]

$$I^\alpha_{(a+)} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi)\, \mathrm{d}\xi, \qquad x>a, \quad \alpha>0, \tag{1.8}$$

and

$$I^\alpha_{(b-)} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\xi-x)^{\alpha-1} f(\xi)\, \mathrm{d}\xi, \qquad x<b, \quad \alpha>0. \tag{1.9}$$

These integrals are sometimes called the left-sided and right-sided fractional integrals, respectively. It should be mentioned that, replacing t by x, we have changed notation in Eqs. (1.8) and (1.9) in comparison with Eq. (1.6) as the following consideration will concern space-fractional differential operators.

The left-sided and right-sided Liouville fractional integrals on the real axis have the form

$$I_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \qquad x \in \mathbb{R}, \quad \alpha > 0, \tag{1.10}$$

and

$$I_-^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi-x)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \qquad x \in \mathbb{R}, \quad \alpha > 0. \tag{1.11}$$

The Fourier transform rules for Liouville fractional integrals on the real axis are calculated according to the following formulae [28, 68]:

$$\mathcal{F}\left\{I_+^\alpha f(x)\right\} = \frac{1}{(-i\xi)^\alpha}\widetilde{f}(\xi), \qquad \alpha > 0, \tag{1.12}$$

$$\mathcal{F}\left\{I_-^\alpha f(x)\right\} = \frac{1}{(i\xi)^\alpha}\widetilde{f}(\xi), \qquad \alpha > 0, \tag{1.13}$$

where the tilde denotes the Fourier transform,

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{ix\xi}\,\mathrm{d}x, \tag{1.14}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{f}(\xi)\mathrm{e}^{-ix\xi}\,\mathrm{d}\xi, \tag{1.15}$$

ξ is the transform variable and $(i\xi)^\alpha$ means

$$(\pm i\xi)^\alpha = |\xi|^\alpha \exp\left[\pm\frac{1}{2}\, i\alpha\pi \,\operatorname{sign}\xi\right]. \tag{1.16}$$

1.2 Riemann–Liouville and Caputo Fractional Derivatives

The Riemann–Liouville derivative of the fractional order α is defined as left-inverse to the fractional integral I^{α}, i.e. [20, 28, 57]:

$$D_{RL}^{\alpha} f(t) = \begin{cases} \dfrac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau\right], \\ \qquad\qquad t>0, \quad n-1<\alpha<n, \\ \dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, \qquad t>0, \quad \alpha=n, \end{cases} \tag{1.17}$$

and for its Laplace transform requires knowledge of the initial values of the fractional integral $I^{n-\alpha} f(t)$ and its derivatives of the order $k = 1, 2, \ldots, n-1$

$$\mathcal{L}\left\{D_{RL}^{\alpha} f(t)\right\} = s^{\alpha} f^*(s) - \sum_{k=0}^{n-1} D^k I^{n-\alpha} f(0^+) s^{n-1-k}, \quad n-1<\alpha<n. \tag{1.18}$$

The Caputo fractional derivative [20, 28, 57]

$$D_C^{\alpha} f(t) = I^{n-\alpha} D^n f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \dfrac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\,\mathrm{d}\tau, \\ \qquad t>0, \quad n-1<\alpha<n, \\ \dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, \qquad t>0, \quad \alpha=n, \end{cases} \tag{1.19}$$

has the following Laplace transform rule

$$\mathcal{L}\left\{D_C^{\alpha} f(t)\right\} = s^{\alpha} f^*(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad n-1<\alpha<n. \tag{1.20}$$

The Caputo fractional derivative is a regularization in the time origin for the Riemann–Liouville fractional derivative by incorporating the relevant initial conditions [21]. In this book we shall use the Caputo fractional derivative omitting the index C

$$D_C^{\alpha} f(t) \equiv \frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}}.$$

The major utility of this type fractional derivative is caused by the treatment of differential equations of fractional order for physical applications, where the initial conditions are usually expressed in terms of a given function and its derivatives of integer (not fractional) order, even if the governing equation is of fractional order [42, 57].

If care is taken, the results concerning the Caputo derivative can be recast to the Riemann–Liouville version and vice versa according to the following formula [20]:

$$D_{RL}^{\alpha} f(t) = D_{C}^{\alpha} f(t) + \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0^{+}), \quad n-1 < \alpha < n. \tag{1.21}$$

It should be noted that in fractional calculus, where integrals and derivatives of arbitrary (not only integer) order are considered, there is no sharp boundary between integration and differentiation. For this reason, some authors [35, 57] do not use a separate notation for the fractional integral $I^{\alpha} f(t)$. The fractional integral of the order $\alpha > 0$ is denoted as $D_{RL}^{-\alpha}$. Sometimes, especially when describing boundary conditions for the time-fractional heat conduction equation, we will also use the notation

$$D_{RL}^{-\alpha} f(t) \equiv I^{\alpha} f(t), \qquad \alpha > 0. \tag{1.22}$$

The left-sided and right-sided Riemann–Liouville fractional derivatives of order $\alpha > 0$ are defined by [28]

$$D_{RL(a+)}^{\alpha} f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[\frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} f(\xi)\, \mathrm{d}\xi \right], \quad x > a, \quad n-1 < \alpha < n, \tag{1.23}$$

$$D_{RL(b-)}^{\alpha} f(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left[\frac{1}{\Gamma(n-\alpha)} \int_x^b (\xi-x)^{n-\alpha-1} f(\xi)\, \mathrm{d}\xi \right], \quad x < b, \quad n-1 < \alpha < n. \tag{1.24}$$

The left-sided and right-sided fractional derivatives corresponding to the left-sided and right-sided Liouville fractional integrals on the real axis have the form

$$D_{+}^{\alpha} f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[\frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^x (x-\xi)^{n-\alpha-1} f(\xi)\, \mathrm{d}\xi \right], \quad x \in \mathbb{R}, \quad n-1 < \alpha < n, \tag{1.25}$$

$$D_{-}^{\alpha} f(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left[\frac{1}{\Gamma(n-\alpha)} \int_x^{\infty} (\xi-x)^{n-\alpha-1} f(\xi)\, \mathrm{d}\xi \right], \quad x \in \mathbb{R}, \quad n-1 < \alpha < n. \tag{1.26}$$

The Fourier transform rules for the left-sided and right-sided Liouville fractional derivatives read:

$$\mathcal{F}\left\{D_{+}^{\alpha} f(x)\right\} = (-i\xi)^{\alpha} \widetilde{f}(\xi), \qquad \alpha > 0, \tag{1.27}$$

$$\mathcal{F}\left\{D_{-}^{\alpha} f(x)\right\} = (i\xi)^{\alpha} \widetilde{f}(\xi), \qquad \alpha > 0. \tag{1.28}$$

1.3 Riesz Fractional Operators

The Riesz form of the fractional derivative is a symmetric operator with respect to x [22, 53, 54] (we consider this operator for $0 < \beta < 2$):

$$\frac{\mathrm{d}^{\beta} f(x)}{\mathrm{d}|x|^{\beta}} = -\frac{1}{\sin(\beta\pi)}\left[\sin\left(\frac{\beta\pi}{2}\right) D_{+}^{\beta} f(x) + \sin\left(\frac{\beta\pi}{2}\right) D_{-}^{\beta} f(x)\right]. \tag{1.29}$$

This operator can be also written as [15, 43, 68]:

$$\frac{\mathrm{d}^{\beta} f(x)}{\mathrm{d}|x|^{\beta}} = \frac{\Gamma(1+\beta)}{\pi} \sin\left(\frac{\beta\pi}{2}\right)\int_{0}^{\infty} \frac{f(x+u) - 2f(x) + f(x-u)}{u^{1+\beta}}\,\mathrm{d}u. \tag{1.30}$$

For $\beta = 1$, the Riesz space-fractional derivative is related to the Hilbert transform (see [43]):

$$\frac{\mathrm{d}f(x)}{\mathrm{d}|x|} = -\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} \frac{f(u)}{x-u}\,\mathrm{d}u. \tag{1.31}$$

The Fourier transform rule for the Riesz derivative reads

$$\mathcal{F}\left\{\frac{\mathrm{d}^{\beta} f(x)}{\mathrm{d}|x|^{\beta}}\right\} = -|\xi|^{\beta}\mathcal{F}\{f(x)\}, \qquad 0 < \beta < 2, \tag{1.32}$$

which in the case $\beta = 2$ coincides with the standard formula

$$\mathcal{F}\left\{\frac{\mathrm{d}^{2} f(x)}{\mathrm{d}x^{2}}\right\} = -\xi^{2}\mathcal{F}\{f(x)\}. \tag{1.33}$$

The Riesz–Feller fractional derivative of order $0 < \beta < 2$ and skewness ϑ with $\vartheta = \min\{\beta, 2-\beta\}$ modifies the Riesz fractional derivative introducing asymmetry [22, 53, 54]

$$D_{\vartheta}^{\beta} f(x) = -\frac{1}{\sin(\beta\pi)} \left\{ \sin\left[\frac{(\beta-\vartheta)\pi}{2}\right] D_{+}^{\beta} f(x) + \sin\left[\frac{(\beta+\vartheta)\pi}{2}\right] D_{-}^{\beta} f(x) \right\} \tag{1.34}$$

and has the following Fourier transform rule

$$\mathcal{F}\left\{D_{\vartheta}^{\beta} f(x)\right\} = -|\xi|^{\beta} \exp\left[\frac{1}{2} i\pi\vartheta \operatorname{sign}\xi\right] \mathcal{F}\{f(x)\}, \qquad 0 < \beta < 2. \tag{1.35}$$

The one-dimensional Riesz derivative is the first step in the direction of defining fractional partial operators in higher dimensions. For example, the negative powers of the Laplace operator $(-\Delta)^{-\beta/2}$ with $\beta > 0$ are called the Riesz potentials (integrals), and their Fourier transforms are defined as [28]

$$\mathcal{F}\left\{(-\Delta)^{-\beta/2} f(\mathbf{x})\right\} = \frac{1}{|\boldsymbol{\xi}|^{\beta}} \mathcal{F}\{f(\mathbf{x})\}, \qquad \beta > 0, \tag{1.36}$$

where $\mathbf{x}$ is a vector of variables, $\boldsymbol{\xi}$ is a vector of transform variables.

The positive powers $(-\Delta)^{\beta/2}$, $\beta > 0$, are called the Riesz derivatives, having the Fourier transforms

$$\mathcal{F}\left\{(-\Delta)^{\beta/2} f(\mathbf{x})\right\} = |\boldsymbol{\xi}|^{\beta} \mathcal{F}\{f(\mathbf{x})\}, \qquad \beta > 0. \tag{1.37}$$

It is obvious that (1.37) is a fractional generalization of the standard formula for the Fourier transform of the Laplace operator:

$$\mathcal{F}\{(-\Delta) f(\mathbf{x})\} = |\boldsymbol{\xi}|^{2} \mathcal{F}\{f(\mathbf{x})\}. \tag{1.38}$$

If the considered function of two space variables $f(x, y)$ depends only on the radial coordinate $r = \left(x^2 + y^2\right)^{1/2}$, then the twofold Fourier transform

$$\widetilde{f}(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, \mathrm{e}^{i(x\xi + y\eta)}\, \mathrm{d}x\, \mathrm{d}y \tag{1.39}$$

can be simplified.

Introducing the polar coordinates

$$x = r\sin\theta, \qquad y = r\cos\theta,$$
$$\xi = \varrho\sin\vartheta, \qquad \eta = \varrho\cos\vartheta, \tag{1.40}$$

we obtain

$$\widetilde{f}(\xi,\eta) = \frac{1}{2\pi}\int\limits_0^{\infty} r f(r)\,\mathrm{d}r \int\limits_0^{2\pi} \mathrm{e}^{ir\varrho\cos(\theta-\vartheta)}\,\mathrm{d}\theta, \tag{1.41}$$

where we have used the same letter for the functions $f(x, y)$ and $f(r)$. Due to the periodic character of the second integrand, the following formula

$$\int\limits_0^{2\pi} \mathrm{e}^{ir\varrho\cos(\theta-\vartheta)}\,\mathrm{d}\theta = \int\limits_0^{2\pi} \mathrm{e}^{ir\varrho\cos\theta}\,\mathrm{d}\theta \tag{1.42}$$

is valid. Using the integral representation of the Bessel function of the first kind of the zeroth order [1]

$$\int\limits_0^{2\pi} \mathrm{e}^{iz\cos\theta}\mathrm{d}\theta = 2\pi J_0(z), \tag{1.43}$$

we get [69]

$$\mathcal{F}\left\{f(x,y)\right\} = \widetilde{f}(\xi,\eta) = \mathcal{H}_{(0)}\left\{f(r)\right\} = \widehat{f}(\varrho) = \int\limits_0^{\infty} r f(r)\, J_0(r\varrho)\,\mathrm{d}r, \tag{1.44}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\xi,\eta)\right\} = f(x,y) = \mathcal{H}_{(0)}^{-1}\left\{\widehat{f}(\varrho)\right\} = f(r) = \int\limits_0^{\infty} \varrho \widehat{f}(\varrho)\, J_0(r\varrho)\,\mathrm{d}\varrho. \tag{1.45}$$

Hence, in the case of axial symmetry the twofold Fourier transform of the classical and fractional Laplace operators with respect to the Cartesian coordinates x and y is reduced to the Hankel transform with respect to the radial coordinate r and

$$\mathcal{F}\left\{(-\Delta)\, f(x,y)\right\} = \mathcal{H}_{(0)}\left\{-\left[\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right]\right\} = \varrho^2\,\mathcal{H}_{(0)}\left\{f(r)\right\}, \tag{1.46}$$

$$\mathcal{F}\left\{(-\Delta)^{\beta/2}\, f(x,y)\right\} = \mathcal{H}_{(0)}\left\{(-\Delta)^{\beta/2}\, f(r)\right\} = \varrho^{\beta}\,\mathcal{H}_{(0)}\left\{f(r)\right\}. \tag{1.47}$$

If the considered function of three space variables $f(x, y, z)$ depends only on the radial coordinate $r = \left(x^2 + y^2 + z^2\right)^{1/2}$, then the threefold Fourier transform

$$\widetilde{f}(\xi,\eta,\zeta) = \frac{1}{(2\pi)^{3/2}}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x,y,z)\,\mathrm{e}^{i(x\xi+y\eta+z\zeta)}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1.48}$$

can also be simplified.

Introducing the spherical coordinates

$$\begin{aligned} x &= r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi, \\ \xi &= \varrho\sin\varphi\cos\vartheta, \quad \eta = \varrho\sin\varphi\sin\vartheta, \quad \zeta = \varrho\cos\varphi, \end{aligned} \tag{1.49}$$

we get

$$\begin{aligned} \widetilde{f}(\xi,\eta,\zeta) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty r^2 f(r)\,\mathrm{d}r \int_0^\pi \mathrm{e}^{ir\varrho\cos\phi\cos\varphi}\,\sin\phi\,\mathrm{d}\phi \\ \times \int_0^{2\pi} \mathrm{e}^{ir\varrho\sin\phi\sin\varphi\cos(\theta-\vartheta)}\,\mathrm{d}\theta, \end{aligned} \tag{1.50}$$

where the same letter is used for the functions $f(x, y, z)$ and $f(r)$.

Due to the periodic character of the third integrand, we have

$$\int_0^{2\pi} \mathrm{e}^{ir\varrho\sin\phi\sin\varphi\cos(\theta-\vartheta)}\,\mathrm{d}\theta = \int_0^{2\pi} \mathrm{e}^{ir\varrho\sin\phi\sin\varphi\cos\theta}\,\mathrm{d}\theta. \tag{1.51}$$

The integral representation of the Bessel function (1.43) gives

$$\begin{aligned} \widetilde{f}(\xi,\eta,\zeta) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty r^2 f(r)\,\mathrm{d}r \\ &\times \int_0^\pi \sin\phi\,\cos\left(r\varrho\cos\phi\cos\varphi\right)\,J_0\left(r\varrho\sin\phi\sin\varphi\right)\,\mathrm{d}\phi \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty r^2 f(r)\,\mathrm{d}r \int_0^1 \cos\left(r\varrho\,v\cos\varphi\right)\,J_0\left(r\varrho\sqrt{1-v^2}\sin\varphi\right)\mathrm{d}v. \end{aligned} \tag{1.52}$$

Next, we use the integral [62]

$$\int_0^1 \cos\left(av\right)\,J_0\left(b\sqrt{1-v^2}\right)\,\mathrm{d}v = \frac{1}{\sqrt{a^2+b^2}}\,\sin\sqrt{a^2+b^2}, \tag{1.53}$$

and for the threefold Fourier transform in the central symmetric case we arrive at the following pare of equations:

$$\mathcal{F}\{f(x,y,z)\} = \mathcal{F}\{f(r)\} = \widetilde{f}(\varrho) = \sqrt{\frac{2}{\pi}} \int_0^\infty r\, f(r)\, \frac{\sin(r\varrho)}{\varrho}\, \mathrm{d}r, \tag{1.54}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\varrho)\right\} = f(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty \varrho\, \widetilde{f}(\varrho)\, \frac{\sin(r\varrho)}{r}\, \mathrm{d}\varrho. \tag{1.55}$$

This result coincides with the particular case of the n-fold Fourier transform in the central symmetric case obtained by another method in the book of Sneddon [69].

Hence, in the case of central symmetry the threefold Fourier transform with respect to the Cartesian coordinates x, y and z is reduced to the sine-Fourier transform of the special type with respect to the radial coordinate r. In this case

$$\mathcal{F}\{-\Delta f(x,y,z)\} = \mathcal{F}\left\{-\left[\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{2}{r}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right]\right\} = \varrho^2\, \widetilde{f}(\varrho) \tag{1.56}$$

and

$$\mathcal{F}\left\{(-\Delta)^{\beta/2} f(x,y,z)\right\} = \varrho^\beta\, \widetilde{f}(\varrho)\,. \tag{1.57}$$

It should be emphasized that the cumbersome aspects of space-fractional differential operators disappear when one computes their Fourier integral transforms.

Additional information about various mathematical aspects of fractional calculus can be found in the pioneering book by Oldham and Spanier [51], in the remarkably comprehensive encyclopaedic-type treatise by Samko et al. [68], in the books by Diethelm [10], Miller and Ross [48] and Podlubny [57] devoted to fractional differential equations, and in the in-depth monograph by Kilbas et al. [28] (see also the extensive detailed survey of Gorenflo and Mainardi [20] and the beneficial paper of Valerio et al. [77] as well as the corresponding volumes of Handbook of Fractional Calculus [30, 31]).

The interested reader is also referred to numerous applications of fractional calculus in different areas of physics, chemistry, biology and engineering (see, for example, the books by Atanacković et al. [3], Herrmann [23], Leszczyński [32], Magin [35], Mainardi [40], Rabotnov [63], [64], Uchaikin [75], West et al. [78], Zaslavsky [82]; the monographs [5–7, 24, 27, 55, 56, 67, 71, 72]; the extensive overviews by Mainardi [38, 39], Metzler and Klafter [46, 47], Rossikhin and Shitikova [65, 66], Tenreiro Machado [73], Zaslavsky [81]; several papers [4, 8, 9, 12, 29, 37, 74], and references therein).

1.4 Mittag-Leffler Functions and Wright Function

The Mittag-Leffler functions and Wright function appear in solutions of various types of equations with fractional operators. According to Mainardi [41], the Mittag-Leffler function can be considered as "the Queen Function of the Fractional Calculus".

The Mittag-Leffler function in one parameter $E_\alpha(z)$ was introduced in [49, 50]. The generalized Mittag-Leffler function in two parameters $E_{\alpha,\beta}(z)$ was considered in [25, 26]. A comprehensive treatment of properties of the Mittag-Leffler functions can be found in [11, 16].

The Mittag-Leffler function in one parameter α

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \alpha > 0, \ z \in \mathbb{C}, \tag{1.58}$$

can be considered as the extension of the exponential function

$$E_1(z) = \mathrm{e}^z = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)}, \qquad z \in \mathbb{C}, \tag{1.59}$$

whereas the generalized Mittag-Leffler function in two parameters α and β is defined by the series representation

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha > 0, \ \beta > 0, \ z \in \mathbb{C}. \tag{1.60}$$

In the general case, the parameters α and β can be treated as complex numbers with some limitations on their real parts [16], but we restrict ourselves to positive values of α and β.

We present several particular cases of the Mittag-Leffler functions for negative values of argument used in this book

$$E_0(-x) = \frac{1}{1+x}, \tag{1.61}$$

$$E_1(-x) = \mathrm{e}^{-x}, \tag{1.62}$$

$$E_2(-x) = \cos\sqrt{x}, \tag{1.63}$$

$$E_{1/2}(-x) = \mathrm{e}^{x^2}\mathrm{erfc}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-u^2-2ux}\,\mathrm{d}u, \tag{1.64}$$

$$E_{1/2,1/2}(-x) = \frac{1}{\sqrt{\pi}} - x\,\mathrm{e}^{x^2}\mathrm{erfc}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-u^2-2ux}u\,\mathrm{d}u, \tag{1.65}$$

$$E_{1/2,3/2}(-x) = \frac{1}{x}\left[1 - e^{x^2}\operatorname{erfc}(x)\right], \tag{1.66}$$

$$E_{1/2,2}(-x) = \frac{1}{x^2}\left[\frac{2x}{\sqrt{\pi}} + e^{x^2}\operatorname{erfc}(x) - 1\right], \tag{1.67}$$

$$E_{0,2}(-x) = \frac{1}{1+x}, \tag{1.68}$$

$$E_{1,2}(-x) = \frac{1 - e^{-x}}{x}, \tag{1.69}$$

$$E_{2,2}(-x) = \frac{\sin\sqrt{x}}{\sqrt{x}}. \tag{1.70}$$

Here erfc (x) is the complementary error function.

The essential role of the Mittag-Leffler functions in fractional calculus is connected with the formula for the inverse Laplace transform [16, 20, 57]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha + b}\right\} = t^{\beta-1}\, E_{\alpha,\beta}(-bt^\alpha) \tag{1.71}$$

with the important particular cases for $\beta = 1$, $\beta = 2$ and $\beta = \alpha$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha + b}\right\} = E_\alpha(-bt^\alpha), \tag{1.72}$$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-2}}{s^\alpha + b}\right\} = t\, E_{\alpha,2}(-bt^\alpha), \tag{1.73}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha + b}\right\} = t^{\alpha-1}\, E_{\alpha,\alpha}(-bt^\alpha). \tag{1.74}$$

The following recurrence relations [11, 16]

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z). \tag{1.75}$$

$$E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z\,\frac{dE_{\alpha,\beta+1}(z)}{dz}, \tag{1.76}$$

$$\frac{d\left[z^{\beta-1}E_{\alpha,\beta}\left(z^\alpha\right)\right]}{dz} = z^{\beta-2}E_{\alpha,\beta-1}\left(z^\alpha\right) \tag{1.77}$$

are valid for the Mittag-Leffler function.

For investigation of the convergence of integrals containing the Mittag-Leffler functions, their asymptotic representations for large negative values of argument are useful. For $x \to \infty$, we have

$$E_{\alpha}(-x) \sim \frac{1}{\Gamma(1-\alpha)x}, \tag{1.78}$$

$$E_{\alpha,2}(-x) \sim \frac{1}{\Gamma(2-\alpha)x}, \tag{1.79}$$

$$E_{\alpha,\alpha}(-x) \sim -\frac{1}{\Gamma(-\alpha)x^2}, \tag{1.80}$$

$$E_{\alpha,\beta}(-x) \sim \frac{1}{\Gamma(\beta-\alpha)x}. \tag{1.81}$$

To evaluate the Mittag-Leffler function the algorithms suggested in [17] were used; see also the MATLAB function [58] that implements these algorithms as well as the recent papers [14], [52, 76].

The Wright function was presented in [79, 80] and later on was discussed in [11, 18, 19, 28, 44, 57, 61], among others. The Wright function is defined by the series representation

$$W(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\alpha k + \beta)}, \qquad \alpha > -1, \quad \beta \in \mathbb{C}, \quad z \in \mathbb{C}. \tag{1.82}$$

In particular [36, 57, 59],

$$W(0, 1; z) = \mathrm{e}^z, \tag{1.83}$$

$$W\left(-\frac{1}{2}, \frac{1}{2}; -z\right) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right), \tag{1.84}$$

$$W\left(-\frac{1}{2}, 1; -z\right) = \operatorname{erfc}\left(\frac{z}{2}\right), \tag{1.85}$$

$$W\left(1, \nu+1; -\frac{z^2}{4}\right) = \left(\frac{2}{z}\right)^{\nu} J_{\nu}(z), \tag{1.86}$$

$$W\left(1, \nu+1; \frac{z^2}{4}\right) = \left(\frac{2}{z}\right)^{\nu} I_{\nu}(z). \tag{1.87}$$

Here $J_\nu(z)$ is the Bessel function of the first kind, $I_\nu(z)$ is the modified Bessel function of the first kind.

The Wright function satisfies the recurrence equations [11]

$$\alpha z W(\alpha, \alpha+\beta; z) = W(\alpha, \beta-1; z) + (1-\beta) W(\alpha, \beta; z), \tag{1.88}$$

$$\frac{\mathrm{d}W(\alpha, \beta; z)}{\mathrm{d}z} = W(\alpha, \alpha+\beta; z). \tag{1.89}$$

The Mainardi function $M(\alpha; z)$ [36, 37, 45, 57] is a particular case of the Wright function

$$M(\alpha; z) = W(-\alpha, 1-\alpha; -z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\,\Gamma[-\alpha k + (1-\alpha)]},$$
$$0 < \alpha < 1, \quad z \in \mathbb{C}, \tag{1.90}$$

$$M(\alpha; z) = \frac{1}{\alpha z} W(-\alpha, 0; -z), \quad 0 < \alpha < 1, \quad z \in \mathbb{C}. \tag{1.91}$$

The Wright function occurs in the expression for the inverse Laplace transform [13, 70]

$$\mathcal{L}^{-1}\left\{s^{-\beta} \exp\left(-\lambda s^{\alpha}\right)\right\} = t^{\beta-1} W\left(-\alpha, \beta; -\lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0, \tag{1.92}$$

in particular [36, 37]:

$$\mathcal{L}^{-1}\left\{\exp\left(-\lambda s^{\alpha}\right)\right\} = \frac{\alpha\lambda}{t^{\alpha+1}} M\left(\alpha; \lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0, \tag{1.93}$$

$$\mathcal{L}^{-1}\left\{s^{\alpha-1} \exp\left(-\lambda s^{\alpha}\right)\right\} = \frac{1}{t^{\alpha}} M\left(\alpha; \lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0. \tag{1.94}$$

The Wright function and the Mittag-Leffler function are interconnected by the Laplace transform [11, 28, 57]

$$\mathcal{L}\left\{W(\alpha, \beta; t)\right\} = \frac{1}{s} E_{\alpha,\beta}\left(\frac{1}{s}\right), \quad \alpha > 0, \quad \beta > 0, \tag{1.95}$$

and [18]

$$\mathcal{L}\left\{W(\alpha, \beta; -t)\right\} = E_{-\alpha,\beta-\alpha}\left(-s\right), \quad -1 < \alpha < 0, \quad \beta > 0, \tag{1.96}$$

whereas for the Mainardi function the corresponding relation takes the form

$$\mathcal{L}\left\{M\left(\alpha; t\right)\right\} = E_{\alpha}\left(-s\right), \quad 0 < \alpha < 1. \tag{1.97}$$

The Mittag-Leffler function and the Wright function are also related by the Fourier cosine transform [59–61]