Springer Proceedings in Advanced Robotics 31
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Jadran Lenarčič
Manfred Husty *Editors*

Advances in Robot Kinematics 2024

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Springer Proceedings in Advanced Robotics 31

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Jadran Lenarčič · Manfred Husty
Editors

Advances in Robot Kinematics 2024

Springer

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Foreword

The field of robotics has attained a heightened level of maturity, riding on the waves of continual advancements in its foundational technologies. These breakthroughs significantly contribute to an unprecedented initiative aimed at seamlessly integrating robots into the fabric of human environments—from factories to home, hospitals, schools, and unstructured environments, extending into the expansive realms of firefighting, manufacturing, agriculture, and beyond. In the present days, robots hold the promise of delivering substantial impacts across a broad spectrum of real-world applications, spanning industrial manufacturing, healthcare, transportation, and the exploration of deep space and the sea. Anticipating how the future is being shaped, we envision a landscape where robots become an integral part of modern life, influencing diverse aspects of our daily existence.

In 2002, the inception of the *Springer Tracts in Advanced Robotics (STAR)* series marked a pivotal moment, driven by the overarching objective of presenting the research community with the latest and most significant advancements in the field of robotics. STAR series has showcased an array of both monographs and edited collections.

Motivated by the expansion of the robotic field and the emergence of novel research areas, recent years have seen an enlargement of the pool of proceedings within the STAR series. This evolution ultimately led to the parallel launch in 2016 of the *Springer Proceedings in Advanced Robotics (SPAR)* series, dedicated to promptly disseminating cutting-edge research results from selected symposia and workshops. Dissemination of the latest research results is presented in selected symposia and workshops.

This volume of the SPAR series is dedicated to the proceedings of the biennial edition of ARK on Advances in Robot Kinematics. Returning to Ljubljana in Slovenia where the whole story started back in 1988, ARK reaches its nineteenth gathering, confirming itself as a major anchor of research advances in robot kinematics serving the global robotics community.

The volume edited by Jadran Lenarčič, the father of ARK, with Manfred Husty contains 49 scientific contributions organized in 7 chapters. This collection spans a wide range of research developments in robot mechanisms, kinematics, analysis, design, planning, and control, a very fine addition to the SPAR series and a genuine tribute to ARK contributors and organizers.

April 2024

Bruno Siciliano
Oussama Khatib

Preface

After ten years (in 2020 the organization was prevented by COVID), the international symposium Advances in Robot Kinematics - ARK is returning to Ljubljana. ARK has been one of the most traditional robotics conferences for many years. It was organized for the first time in Ljubljana in 1988 being the first specialized scientific symposium in robot kinematics, and after 36 years it is still highly respected by the professional community. ARK's quality and contributions have shaped robotics science and laid the foundation for the theory of robot mechanisms and control.

The articles in this book deal with many traditional and emerging areas of robot kinematics and its use in design and planning of robot mechanisms. The book will be of particular interest to researchers and doctoral students who specialize in robot kinematics, but we believe that it will also attract readers from other fields of robotics.

In the ARK, contributions are selected according to a rigorous peer-review process. This year, we accepted 49 contributions for presentation at the symposium and published the same number in this book. We are pleased to say that our collaboration with Springer (originally with Kluwer Academic Publishers) goes back to the first half of the 1990s. We have already published the seventeenth book together. As far as we know, the ARK book was also the first in the SPAR series published in 2016. We would once again like to thank all the employees of this publishing house who have contributed to the publication of this book with their expertise and commitment.

We also would like to thank all the authors for their important contributions and all the reviewers who did an outstanding job in a very short time. Some of the authors have been with us all these years, without them the ARK conferences would not have been possible. This year we would also like to thank Dr. Tadej Petrič, Publishing Chair of the symposium, who has been an excellent support and help to the editors. It is thanks to him that the book was published in this form and on time.

Last year we received the shocking news that one of the initiators of our symposium and friend, Michael M. Stanisic of Notre Dame University, passed away far too soon. Mike was a two-time co-chair of the symposium and co-editor of two of our books. He will always be remembered for his scientific work and, above all, for his friendly attitude towards us all. When organizing a symposium like this, the most important thing is to create a community that can only be based on deep mutual respect and friendship. Mike's contribution in this area was invaluable.

After so many years, the time has slowly come for a generational change. This year, for the first time, we have divided the Scientific Committee into two parts, and we have formed the Honorary Steering Committee, which brings together pioneers and long-time supporters of ARK symposia in the past. The title of the Honorary Committee is not accidental, because the work of its members has been of great importance for ARK, for science, and for friendship, without which there can be no successful scientific cooperation and no serious scientific conference. There are no words to thank them adequately, but let us all say a sincere thank you and wish them many more good things in life. ARK 2024 continues the generational change with the hope that this important robotics conference will continue with scientific success in the future.

Jadran Lenarčič
Manfred Husty

Organization

Editors

Husty, Manfred	University Innsbruck, Unit Surveying and Geometry, Innsbruck, Austria
Lenarčič, Jadran	Jožef Stefan Institute, Slovenia

Contents

Ferdinand Freudenstein's Spatial Kinematics

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Abstract. In this paper we examine in detail an advanced graduate-level course taught by Prof. Freudenstein that bridged his first two cadres of doctoral students. In the autumn term of 1965, Prof. Freudenstein taught ME E8405 Spatial Kinematics at Columbia University. This would be Dr. Armand Dilpare's last term at Columbia University as he would graduate at the end of the calendar year. Dr. Dilpare was a non-traditional student, having worked in industry for several years before returning to university for doctoral studies. During his practice of engineering, Dr. Dilpare developed an appreciation for detail and advanced abilities in engineering graphics. He employed these skills when taking detailed and meticulous notes in his courses at Columbia University. Upon his retirement from the Florida Institute of Technology in 1994, Dr. Dilpare donated his course notes to the author. In this paper, the contents of Dr. Freudenstein's advanced graduate course in spatial kinematics are examined through the lens of Dr. Dilpare's detailed course notes. It is hoped that an elucidation of this course will be beneficial to current educators and practitioners in the field of kinematics.

1 Introduction

Prof. Ferdinand Freudenstein is widely recognized as the Father of Modern Kinematics [3,4,6], see Fig. 1. He made innumerable contributions to the field, one of which is the training of future researchers. By the end of 1965, Dr. Freudenstein had graduated his first cadre of doctoral students from Columbia University who would go on to become world-class researchers and leaders in kinematics. These doctoral graduates included George Sandor, Bernard Roth, Ronald Phillip, A.T. Yang, and Armand Dilpare. In 1966, and soon thereafter, a second cadre was studying with Prof. Freudenstein: Frank Buchsbaum, Leo Dobrjansky, Don Wallace, Ralph Dratch, Steve Dubowsky, and Mark Yuan. In this paper, we examine in detail an advanced graduate-level course in spatial kinematics taught by Prof. Freudenstein that bridged these first two cadres of doctoral students.

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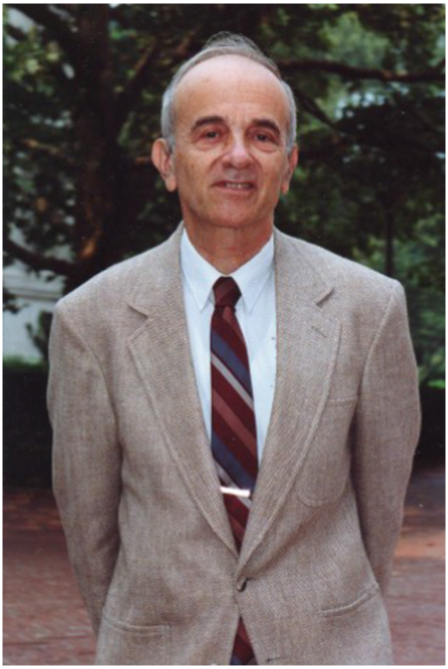
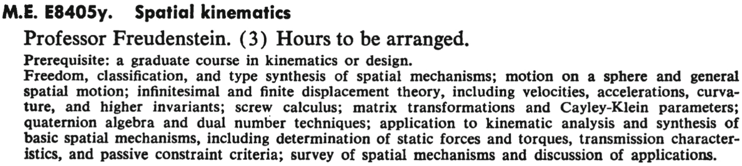


Fig. 1. Prof. Ferdinand Freudenstein (1926–2006)

2 Course Context

In the autumn term of 1965, Prof. Freudenstein taught ME E8405 Spatial Kinematics at Columbia University. Figure 2 shows the detailed description of this course from the Columbia University Bulletin [2].



M.E. E8405y. Spatial kinematics

Professor Freudenstein. (3) Hours to be arranged.

Prerequisite: a graduate course in kinematics or design.

Freedom, classification, and type synthesis of spatial mechanisms; motion on a sphere and general spatial motion; infinitesimal and finite displacement theory, including velocities, accelerations, curvature, and higher invariants; screw calculus; matrix transformations and Cayley-Klein parameters; quaternion algebra and dual number techniques; application to kinematic analysis and synthesis of basic spatial mechanisms, including determination of static forces and torques, transmission characteristics, and passive constraint criteria; survey of spatial mechanisms and discussion of applications.

Fig. 2. ME E8405 Course Description [2]

This would be Dr. Armand Dilpare's last term at Columbia University. Dr. Dilpare was a non-traditional student, having worked in industry for several years before beginning his doctoral studies. In his practice of engineering, Dr. Dilpare developed an appreciation for detail and advanced abilities in engineering graphics. He employed these skills when taking detailed and meticulous notes in his courses at Columbia University. Upon his retirement from the Florida Institute of Technology in 1994, Dr. Dilpare donated his course notes to the author. In this paper, we examine the contents of Dr. Freudenstein's advanced graduate course in spatial kinematics through the lens of Dr. Dilpare's detailed course notes.

Dr. Freudenstein's first cadre of doctoral students, 1965 graduates or earlier, are listed in Table 1 along with their year of graduation. In order to know the research focus areas of Dr. Freudenstein's research team during this period, see the dissertation titles of the first cadre students in Table 2.

Dr. Freudenstein's doctoral students, at the time of the Fall 1965 offering of ME E8405 Spatial Kinematics are listed in Table 3 along with their year of graduation. The most senior doctoral student at this time was Dr. Armand Dilpare (see Fig. 3). Throughout his career, Dr. Dilpare used multiple spelling variations of his family name, such as Dilpare, Dil Pare, and DilPare. During the later stages of his career, he consistently used the Dilpare spelling.

Table 1. Dr. Freudenstein's First Cadre of Doctoral Students

George Sandor (1959)	Bernard Roth (1962)
Ronald Philipp (1964)	A.T. Yang (1964)
Armand Dilpare (1965)	

Table 2. First Cadre Dissertation Titles

G. Sandor	A general complex-number method for plane kinematic synthesis with applications
B. Roth	A generalization of Burmester theory: nine-point path generation of geared five-bar mechanisms with gear ratio plus and minus one
R. Philipp	On the synthesis of two degree of freedom linkages for the maximum number of precision positions
A.T. Yang	Application of quaternion algebra and dual numbers to the analysis of spatial mechanisms
A. Dilpare	Kinematic synthesis of the six-link Watt mechanism by numerical methods

Table 3. Dr. Freudenstein's Cadre of Doctoral Students in Fall 1965

Armand Dilpare (1965)	Frank Buchsbaum (1967)
Leo Dobrjansky (1967)	Don Wallace (1968)
Ralph Dratch (1970)	Steven Dubowsky (1971)
Mark Yuan (1972)	

Fig. 3. Prof. Armand Dilpare circa 1986 [5]

3 Course Topics

Dr. Dilpare's course notes, from the initial class meeting in Fall 1965, include the following detailed list of topics to be covered in the course.

ME E8405 Spatial Kinematics
Fall 1965
Course Topics

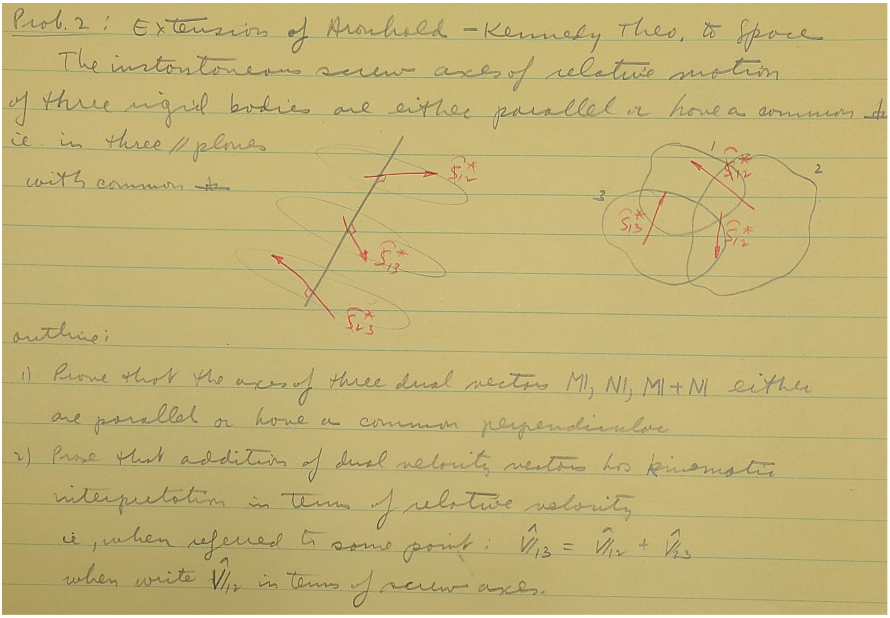
- General Theory
  - Structure
  - Type Synthesis
  - Classification
- Kinematic Analysis
  - Displacements, Velocities, Accelerations, Higher Accelerations
  - Static Force & Torque Analysis
  - Instantaneous Invariants & Curvature Theory
  - Mathematical Tools (vectors, matrices, quaternions, dual quaternions)
- Synthesis
  - Finite Positions of the Plane (Burmester Theory for spatial mechanisms is yet to be done)
  - Input-Output Motion by Algebraic Methods (yet to be done)
- Others
  - Mobility or Rotatability (skew four-bar)
  - Path, Function, & Motion Generation
  - Spatial Coupler Curves
  - Harmonic Analysis

  - Transmission Angles
  - Specific Applications

The course required students to complete a term paper project, and on the first day of class, Dr. Freudenstein provided the following list of suggested term paper topics.

ME E8405 Spatial Kinematics
Fall 1965
Suggested Term Paper Topics

- Analysis of any Spatial Mechanism not found in Literature
- Instantaneous Invariants on a Sphere
  - Euler-Savory Equation for Spherical Motion
  - Curves of Stationary Curvature
- Finite Position Theory
  - Burmester Theory for Centerpoint, Circlepoint, etc. 2, 3, 4, . . . , n positions
  - Ref. Schoenflies "Synthetic Geometry"
- Matrix Methods



Prob. 2: Extension of Aronhold-Kennedy Theo. to Space

The instantaneous screw axes of relative motion of three rigid bodies are either parallel or have a common ⊥
i.e. in three // planes
with common ⊥

outline:

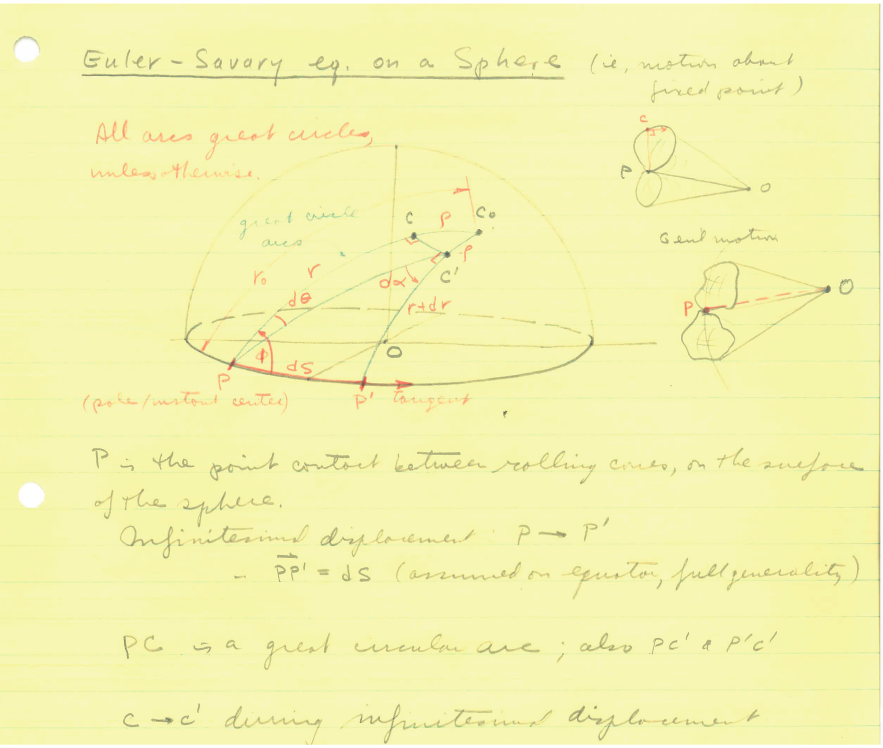
1) Prove that the axes of three dual vectors $\mathbb{M}$, $\mathbb{N}$, $\mathbb{M}+\mathbb{N}$ either are parallel or have a common perpendicular

2) Prove that addition of dual velocity vectors has kinematic interpretation in terms of relative velocity
i.e., when referred to some point: $\hat{\mathbb{V}}_{13} = \hat{\mathbb{V}}_{12} + \hat{\mathbb{V}}_{23}$
when write $\hat{\mathbb{V}}_{12}$ in terms of screw axes.

Fig. 4. Example HW Problem from ME E8405 Course Notes [1]

- Denavit & Hartenberg at Northwestern University
  - “Iterative Methods”
  - “Double Hooke Joint”
- Spherical or Spatial Coupler Curves (to date spherical four-bar)
- Classification of Spatial Mechanisms
- Study Patent Literature
- Study of Analogs Between Planar, Spherical, & Spatial. Ref. F.M. Dimentberg has done planar four-bar - spherical four-bar - spatial four-bar
- Mobility Criteria
- Force & Torque Analysis. Known for spherical four-bar and certain spatial mechanisms
- Overclosed Mechanisms. Matrix criteria, mathematical analysis, geometric criterion
- Transmission Angles & Criteria

An example homework problem from Dr. Dilpare’s course notes, on the spatial generalization of the Aronhold - Kennedy Theorem to spatial kinematics is shown in Fig. 4.



Euler - Savary eq. on a Sphere (ie, motion about fixed point)

All arcs great circles, unless otherwise.

P is the point contact between rolling cones, on the surface of the sphere.

Infinitesimal displacement : P → P'

- $\vec{PP'} = dS$ (assumed on equator, full generality)

PC is a great circular arc ; also PC' & P'C'

C → C' during infinitesimal displacement

Fig. 5. Example from ME E8405 Course Notes [1]

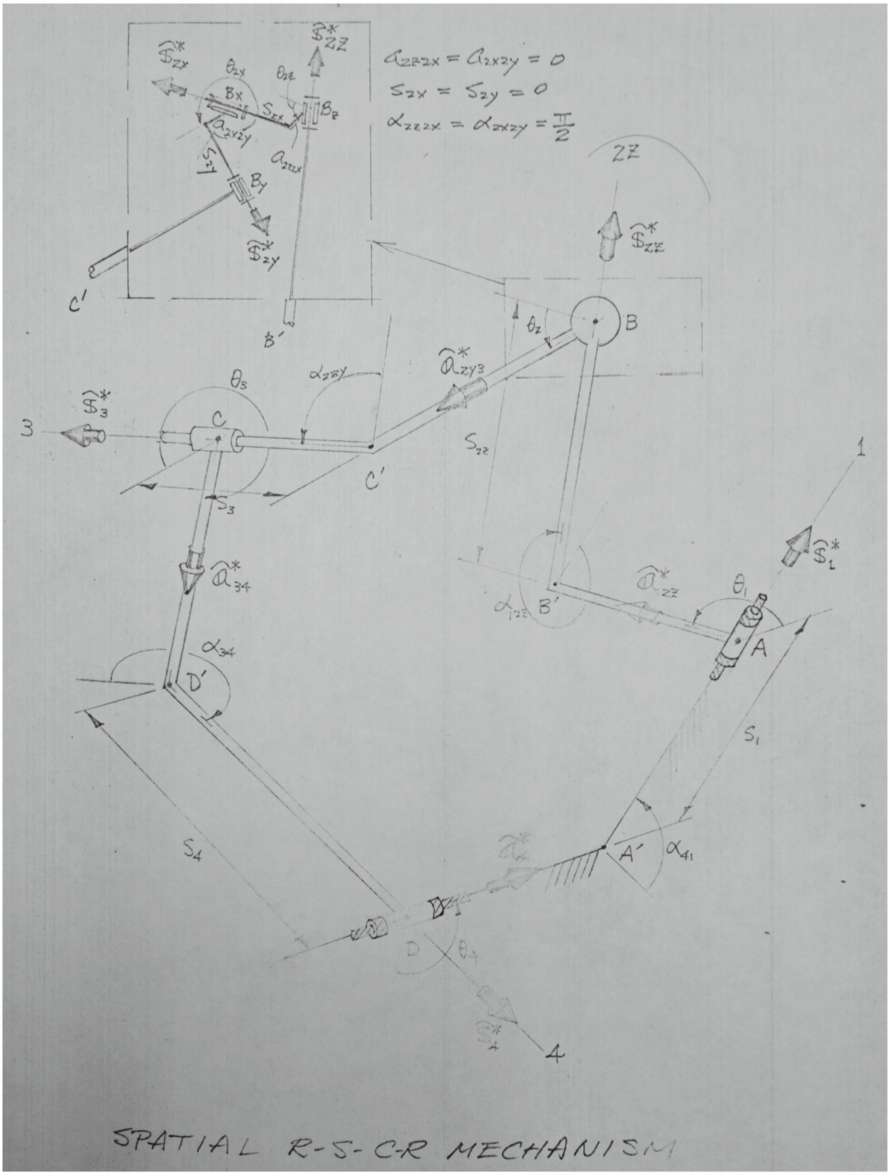


Fig. 6. Example from Term Paper Presentation by Ms. Tanaka [1]

An example of Dr. Dilpare's course notes, on the Euler-Savary Equation on a Sphere, is shown in Fig. 5.

Dr. Dilpare's course notes also include a few remarks regarding some of the term paper presentations of his classmates. On January 26, 1966, Mr. R.C Baccaglini presented *Displacement Analysis Using Dual Number Quaternions.*

Ms. Tanaka presented *Quaternion Algebra Applied to Spatial Mechanisms* and included as an example the analysis of the RSCR spatial mechanism, see Fig. 6. Mr. Donald Wallace presented *Analysis and Demonstration Model of an RRCRC Spatial Five-Bar Mechanism.* Mr. Mark Yuan's term paper presentation topic was *Analysis of a Skew Four-Bar Mechanism*, an R-R-S-Hooke spatial mechanism.

4 Summary

In this paper, the context in which Prof. Freudenstein taught the advanced graduate course ME E8405 Spatial Kinematics in Fall 1965 was summarized. The contents of Dr. Freudenstein's 1965 offering of the ME E8405 course, through the lens of Dr. Armand Dilpares detailed course notes, were examined. It is hoped that the dissemination of this important milestone in the development of modern kinematics will facilitate the learning and advancement of spatial kinematics.

Acknowledgements. The late Dr. Armand Leon Dilpare is gratefully acknowledged for giving his detailed and illustrated ME E8405 Spatial Kinematics fall 1965 course notes to the author. Dr. Ferdinand Freudenstein is whole-heatedly acknowledged for his founding of modern kinematics and for inspiring generations of kinematicians around the globe.

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The Inverse Kinematics of Cable-Driven Parallel Robot with More Than 6 Sagging Cables Part 1: From Ideal to Sagging Cables

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Abstract. A cable-driven parallel robot (CDPR) has n cables whose lengths are used to control the pose of the platform. We assume here that $n > 6$ and that the cables have elasticity and mass i.e. are sagging cables. In that case the inverse kinematics (IK, i.e. finding the cable lengths to reach a given platform pose) has usually an infinite number of solutions. It is then common to look for an IK solution that satisfies some optimality condition usually related to the cable tensions i.e. solving a very difficult constrained optimization problem as sagging cables leads to highly non linear constraints. To the best of the author knowledge this issue has never been addressed and we propose an algorithm for solving the IK without guaranteeing to obtain the minimum of the optimality criterion. Unfortunately it is time consuming with about 5 to 46 min of solving time for a CDPR with 8 cables.

1 Introduction

Cable-driven parallel robot (CDPR) are using cables going through a fixed point A_i in space and ending at a point B_i on the platform. A winching mechanism (either a rotary winch or one using linear actuators) allows one to modify the lengths of the cable. For a CDPR with at least 6 cables an appropriate adjustment of the cable lengths allows for the control of the 6 d.o.f of the platform.

Statics plays an essential role on the workspace of a CDPR because cable can only pull. A direct consequence is that the projection of the platform on the ground roughly cannot be outside the convex hull of the projection of the A_i points on the ground, the vertical limit being ruled by the minimal/maximal length/tension of the cables. Consequently it may be interesting to have more than 6 cables for increasing the CDPR workspace. Beside the workspace key elements for the control are its inverse kinematics (IK) and direct kinematics (DK) which respectively consists in determining the cable lengths for a given pose and determining the platform poses for given cable lengths. In this paper we will focus on the IK problem and use as example the spatial CDPR Cogiro with 8 cables. For this robot the locations of the A_i points in a reference frame and the B_i in the mobile frame are provided in Table 1.

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Table 1. Coordinates of the A_i, B_i points of the considered CDPR

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
x	–7.17512	–7.31591	–7.30285	7.18206	–7.16098	7.32331	7.30156	7.16129
y	–5.24398	–5.10296	5.23598	5.3476	5.37281	5.20584	–5.13255	-5.26946
z	5.46246	5.47222	5.47615	5.4883	5.48539	5.49903	5.489	5.49707
	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
x	0.50321	–0.50974	–0.50321	–0.50321	0.49607	0.49964	0.50209	–0.50454
y	–0.49283	0.35090	–0.2699	0.49283	0.35562	-0.34028	0.2749	–0.34629
z	0	0.99753	0	0	0.99954	0.99918	–0.00062	0.99752

The equations governing the kinematics can be classified into two categories:

1. geometric in a broad sense: they provide the relationships between the cable lengths and the platform pose
2. statics: to establish the relationships between the cable tensions and platform pose to obtain mechanical equilibrium

Cable modeling plays an essential role for both categories and we will consider two models: *ideal* (no cable mass and no elasticity) and *sagging* (cable with mass and elasticity). IK plays a relatively minor role in control as it is used mostly for pick and place tasks which require only off-line calculation. At the opposite IK plays a major role in CDPR analysis and optimization e.g. to determine the CDPR workspace or to determine the maximal cable tensions over a given workspace determined by bounds on the generalized coordinates of the platform. These types of problem have been solved when assuming ideal cables but are much more complex to deal with for sagging cables. In that case we may have to resort to a discretization of the workspace, solving the IK in numerous poses to obtain an approximate result, which implies that the IK solving should be very fast.

2 Inverse Kinematics with Ideal Cables

We define a reference frame such that its z axis is the local vertical and the x, y axis are horizontal. The pose of the platform will be defined by the 3 coordinates in the reference frame of its center of mass G while its orientation will classically be defined by 3 angles (we are using the roll/pitch/yaw angles).

For a CDPR with n ideal cables the cables lengths for reaching a given pose is directly the distances between the A_i and B_i points. The coordinates of the B_i points in the reference frame are obtained directly from the pose parameters $\mathbf{X}$. However we have to consider two important factors:

1. the mechanical equilibrium condition provides 6 linear equations while we have as unknowns the n tensions τ in the cables. If $n > 6$ we have therefore

an underconstrained system with possibly an infinite number of solutions. We have thus to define an optimality condition for determining a single IK solution being given the constraint that all the τ must be positive as a cable can only pull and cannot push. The is the classical problem of *tension distribution* that has been addressed in numerous papers [1–4,6–8,10,14,16,18,19].

2. in practice it is almost impossible to ensure that all n cables are under tension. Indeed to be under tension each cable length must be **exactly** the distance between the A_i, B_i points. If a cable length is greater than this distance, then the cable will be slack and its tension equal to 0 while if the length is shorter than the distance, then the platform will be in a pose that differs from the prescribed one

For point 1 we assume in this paper that the optimality criterion is to minimize the norm of the τ vector (i.e. a single number) and the Maple `Minimize` function is able to solve this optimization problem under the linear constraints of the mechanical equilibrium condition The choice of this optimality condition is arbitrary and any other condition may be used as soon as it is continuous.

However point 2 emphasizes that the solution obtained in point 1 cannot be put in practice unless there is at most 6 cables under tension. Therefore to get an overview of the IK solution we will consider the optimality problem for $m \in [1, 8]$ cables under tension and all possible combinations of cables under tension. Note that for some combinations with $m \leq 6$ we may not find a solution with positive tension but when a solution is found we may enforce it as slack cables may be obtained just by having their lengths being much larger than the distances between the A_i, B_i.

In summary we obtain different types of IK solution satisfying the optimality condition:

- with 7 or 8 cables under tension but this IK solution cannot be realized in practice,
- we obtain a single IK solution for each combination of 6 cables under tension but we retain only the one having positive tensions
- in some specific cases we may obtain IK solutions with less than 6 cables under tension and these solutions are feasible

We have designed a `Maple` procedure that calculates all these IK solutions for a given platform pose. Note that the optimal tensions vary linearly with the platform mass M so that we may solve the IK for an arbitrary mass and deduce a solution for any other mass.

For example we have considered the pose defined by $x = -3, y = 3, z = 2$ and the RTL angles all equal to –6°C. We have found the 9 following optimal criterion value and configurations for a mass $M = 1$: 103.19 (no slack cable), 109.388 (8 slack), 116.8276 (7 slack), 113.382 (2 slack), 123.9338 (1 slack), 125.3721 (7, 8 slack), 126.2566 (2, 7 slack), 127.511 (1, 8 slack), 135.8629 (1,2 slack).

3 Inverse Kinematics with Sagging Cables

Sagging cable model takes into account the proper mass of the cable together with its elasticity. The concept of slack cable does no more exists in that case as there will always be tension in the cable because of its mass. To the best of the author knowledge the IK of CDPR with more than 6 sagging problem has not been addressed. There are numerous sagging cable models [5,11,13,15,17,20] and we have adopted the classical model presented in the textbook of Irvine [9]. Irvine model is planar: the cable is attached at one of its extremity A and is submitted at its other extremity B to a force with F_x, F_z horizontal and vertical components. In the planar frame with origin at A, an horizontal axis x_i and vertical axis z_i the coordinates of B are x_b, z_b. The Irvine model relates x_b, z_b to F_x, F_z and to the cable length at rest L_0. We have:

$$x_b = F_x \left(\frac{L_0}{EA_0} + \frac{\sinh^{-1}\left(\frac{F_z}{F_x}\right) - \sinh^{-1}\left(\frac{-\mu g L_0 + F_z}{F_x}\right)}{\mu g} \right) \tag{1}$$

$$z_b = \frac{F_z L_0}{EA_0} - \frac{\mu g\, {L_0}^2}{2EA_0} + \frac{\sqrt{F_x^2 + F_z^2} - \sqrt{F_x^2 + (-\mu g L_0 + F_z)^2}}{\mu g} \tag{2}$$

where E is the Young modulus of the cable material (which characterizes its elasticity), μ the linear density of the material and A_0 the area of a cable cross-section. For the example we will assume that all cables are identical with $\mu = 0.079\,\text{kg/m}$, a diameter of $4\,\text{mm}$ and $E = 1e11$ Pa. Note that because of the arbitrary choice of the direction of the x axis we impose x_b, F_x to be positive.

For a CDPR with n cables we have the $2n$ Irvine equations and 6 equations that describes the mechanical equilibrium for a total of $2n+6$ equations. In terms of unknowns we have the n triplets (F_x, F_z, L_0) and therefore $3n$ unknowns. Consequently if $n > 6$, then we have an underconstrained system with less equations than unknowns and, like the ideal cables case an optimality condition has to be defined. To the best of the author knowledge the IK problem with more than 6 sagging cables has not yet been addressed. For the CDPR we are considering with 8 cables we have therefore 22 equations and 24 unknowns. For the sake of simplicity we denote by $u_1, u_2, \ldots u_{24}$ the 24 unknowns of the IK problem.

As for the ideal case an IK solution will be obtained by using an optimality condition. At the opposite of ideal cables the cable tension is not the same all over the cables. In this paper we use as criterion the norm of the cable tensions at the A_i, this tension being equal to $\sqrt{F_x^2 + (F_z - \mu g L_0)^2}$ but the approach proposed for solving the IK is generic and may be used with any other criterion as soon as it is continuous. However the non linear nature of the Irvine equations leads to a constrained optimization problem that is much more difficult to solve than for the ideal cables. Note also that at the opposite of the ideal cables case the non linearity of the equations imposes that we cannot deduce directly the IK solution for a given platform mass from an IK solution for a different mass.

4 Deriving a Sagging Solution from the Ideal Case Results

The underlying principle that leads to our approach is based on the continuity of the Irvine equations: if $E \to \infty$ and $\mu \to 0$, then these equations become identical to the one obtained for ideal cables. Consequently we may assume that if we have an IK solution with sagging cables and are able to increase E and decrease μ, then we will end up close to an IK solution with ideal cables. Reciprocally starting from an IK solution S_i with ideal cables if we are able to physically introduce elasticity and mass to the cables we will end up in an IK solution S_s for sagging cable.

Our first test was a direct application of this principle by starting from an ideal cable IK solution and using a continuation process on the E, μ starting from a large E and a very small μ while using a local minimizer to refine the solution at each step of the process, solution that must satisfy the 22 constraint equations. However our tests have shown that it is very difficult to find a starting point for E, μ and furthermore that even when starting values for E, μ have been obtained the minimizer was unable to find a solution after a few steps in the process. These problems appear whatever the continuation strategy is: we have tried modifying E, μ simultaneously, modifying E or μ alone while keeping constant the other parameter, having different continuation scheme for E and μ and all of them failed after a few steps. Our interpretation is that the optimality condition is very sensitive for these extreme values of E, μ.

Consequently we have decided to use another strategy. We will consider all the IK solutions obtained for the ideal cables case and apply a continuation scheme on E, μ to obtain a solution for sagging cables without considering the criterion value. We were quite confident about the feasibility of this approach as we have already used it for the DK [12]. But a big difference here is that the DK equations system is always square while for the IK we have 22 equations and 24 unknowns, which prohibits to use directly a continuation process.

To solve this issue we will set 2 unknowns (u_i, u_j) to their ideal cables case values and therefore obtain a square system on which the continuation scheme can be applied. This process of fixing the values of 2 unknowns will be applied for each possible pairs among the 24 unknowns, which amounts to 276 cases. If the continuation scheme converges with positive F_x, then we have obtained a sagging cable IK solution that we use as starting point for a constrained minimizer where all the unknowns are free to evolve. We retain as optimal IK solution the one having the lowest criterion over all ideal cables IK solutions.

This computation is rather intensive. For our example

- we obtain the sagging solution from the ideal cables solutions in a mean time of about 326 s per initial ideal cable solution.
- the result provided by the minimizer is established in a mean time of 117.36 s (minimum: 13.8, maximum: 282.388).

As an example we consider the pose defined by $x = -3, y = 3, z = 2$ and the RTL angles all equal to -6°C, the platform mass being set to $M = 50$ kg.

We have obtained 9 initial configurations for the IK with ideal cables that lead to 132 IK solutions with sagging cables in a computation time of 2683.778 s. When running the minimizer on all these 132 solutions we determine the global minimal criterion as 542.3467 in 131.092 s, the cable lengths being 9.5961, 9.7308, 5.7093, 11.3642, 5.565, 10.5546, 13.3204, 13.7295 m. The total computation time for establishing the IK solution is therefore 46 min 54 s (note that the algorithm has been implemented in `Maple` and a C implementation will be much faster). Note that in all the cases we have considered the IK solution with sagging cables is always derived from the IK ideal cables solution where all 8 cables are taught. Under the assumption that this is always the case the solving time will be reduced to 5 min, 12 s but we cannot prove the assumption. Furthermore we cannot prove that we have reached the global minimum but we will propose in the next section a strategy to check several solutions and possibly improve them.

4.1 Checking and Improving the IK Solution

Clearly the strategy presented in the previous section cannot guarantee to find the IK solution having the global lowest criterion. We may however perform a check to determine if a better solution may be found under the assumption that we have determine the IK solution for different poses. Assume that we have found IK solution S_1, S_2 for the poses $\mathbf{X_1}$ and $\mathbf{X_2}$. We define a continuation scheme on the platform pose by setting it to $\mathbf{X} = \mathbf{X_1} + \lambda(\mathbf{X_2} - \mathbf{X_1})$. As in the previous section we have to fix the values of a pair of unknowns (u_i, u_j) for applying the continuation scheme and we will consider all 276 possible pairs of unknowns. The algorithm uses the following steps, starting with $\lambda = 0$, a small ϵ, $S = S_1$ and u_i, u_j being set to their values at S_1:

1. if ϵ is lower than a very small threshold we are close to a system singularity and exit
2. we set λ_i to λ, λ to $\lambda + \epsilon \leq 1$. S is used as guess for the Newton scheme for the new system
3. a. if Newton does not converge or one of the F_x of the solution is negative we divide ϵ by 2, set $\lambda = \lambda_i$ and restart at step 1
   b. if Newton converge to a solution S with all $F_x > 0$ and a minimizer is used to determine a solution S' around S that minimizes the criterion, considering the 24 unknowns so that the values of u_i, u_j may change at each step. If the minimizer failed we divide ϵ by 2, set λ to λ_i and restart at step 1. If it is successful we set $S = S'$, we increase ϵ (e.g. by 10%). and go to step 1 unless $\lambda = 1$ (in that case the platform is at pose $\mathbf{X_2}$ and we exit)

For each pair (u_i, u_j) such that the procedure has exited with $\lambda = 1$ we consider the criterion value C_2^n obtained at S and compare it to the value of the criterion C_2 that has been established previously for $\mathbf{X_2}$. If C_2^n is lower than C_2 we update C_2 by C_2^n. Note that as soon as such a case occur we will use the new IK solution obtained for $\mathbf{X_2}$ as starting point and repeat the continuation process toward all the other poses as a change in the criterion at $\mathbf{X_2}$ may lead to a better criterion for $\mathbf{X_j}, j \neq 2$. As an example we consider the pose $\mathbf{X_1}$ defined by $x = -3, y = -3, z = 2$ and the RTL angles all equal to -6 degrees while $\mathbf{X_2}$ differs from $\mathbf{X_1}$ by $y = 3$. Among the 276 pairs of (u_i, u_j) 115 pairs lead to an IK solution at $\mathbf{X_2}$, 111 leading to the same IK solution than the one that was originally found while 4 solutions were different but with a higher value of the optimality criterion. The computation time of this check is 50 min and 47 s.

5 Conclusion

We have considered the very difficult problem of solving the IK problem for CDPR with more than 6 sagging cables, based on the Irvine textbook cable model. As this an underconstrained problem it is necessary to look for a solution that minimizes an optimality criterion and we have proposed a generic scheme that allows one to deal with any continuous criterion and proposes an IK solution. Although we cannot guarantee to have reached the global minima for the criterion we still have a good solution that is however obtained in a relatively large computation time. A companion paper will investigate the use of neural networks to speed up the solving.

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