Operator Theory
Advances and Applications
303

Sebastian Bechtel

Square Roots of Elliptic Systems in Locally Uniform Domains

L Linear
O Operators and
L Linear
S Systems

Birkhäuser

Birkhäuser

Operator Theory: Advances and Applications

Volume 303

Founded in 1979 by Israel Gohberg

Sebastian Bechtel

Square Roots of Elliptic Systems in Locally Uniform Domains

Birkhäuser

Sebastian Bechtel
Delft Institute of Applied Mathematics
Delft, The Netherlands

ISSN 0255-0156 ISSN 2296-4878 (electronic)
Operator Theory: Advances and Applications
ISSN 2504-3609 ISSN 2504-3617 (electronic)
Linear Operators and Linear Systems
ISBN 978-3-031-63767-4 ISBN 978-3-031-63768-1 (eBook)
https://doi.org/10.1007/978-3-031-63768-1

Mathematics Subject Classification: 35J47, 46E35, 46B70, 47A60, 26A33, 26D15

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

On the Euclidean space, the square root of an elliptic system is a well-studied object, in particular due to the solution of the Kato square root conjecture. However, as was remarked by Lions back in the 1960s, in applications it is likewise important to study systems that are posed on a rough open set equipped with mixed boundary conditions.

The present monograph describes the author's journey and contributions to the last raised question. It combines tools from harmonic analysis, interpolation theory, potential analysis, function spaces, and many more to investigate the square root of an elliptic system in so-called locally uniform domains. The Kato square root property is established and corresponding estimates are extended to Lebesgue spaces in an optimal range of exponents.

Even though the solution of the Kato square root problem in locally uniform domains is the central goal of this monograph, other tools and developments in the treatment of mixed boundary conditions find their spot in this book. Thus, this monograph is supposed to be a gentle starter for the reader to discover these topics.

Delft, Netherlands

Sebastian Bechtel

Contents

Chapter 1
Introduction

Taming non-smoothness has been a major theme in the analysis of partial differential equations and other branches of analysis in the past decades. This monograph grew out of the PhD thesis of the author and presents their contribution to the field in a streamlined way. The content of this book is strongly based on the publications [12–16] by the author together with his colleagues R. Brown, M. Egert, R. Haller, and P. Tolksdorf. Occasionally, the results in these articles are formulated in a more general manner. We tried to reduce technical difficulties as much as possible so that the reader gets an introduction to state-of-the-art results for square roots of elliptic systems in open sets with the least technical overhead possible.

Rough Geometry Let us begin with a glance on non-smooth geometry. A first fundamental question is the following: how could one measure the smoothness of an open set? Certainly, there is no single answer to this question. For instance, one may take into consideration the regularity of its boundary, either considered as being locally the graph of a function or as the boundary of a manifold with boundary [38]. Also, there are purely measure-theoretic concepts. For the set itself, a common condition is the *interior thickness condition*, which is tightly connected to the study of Sobolev spaces on an open set [41], but there are also conditions for the boundary like the notion of *Ahlfors–David regularity*. Besides that, there are involved metric conditions like the ε-cigar condition of Jones [47], corkscrew conditions, and many more.

As was just mentioned, there is a deep connection between concepts in rough geometry and the theory of Sobolev spaces. In the smooth case, many properties and constructions can be performed by "flattening the boundary" and working in the regular configuration of a halfspace, where simple reflection arguments are feasible. In the non-smooth case, a considerably more involved usage of the geometry is needed. Examples of this can be found in the works of Calderón [22] and Jones [47] on extension operators. The work of Jones already allows one to treat very irregular configurations like the Koch snowflake (Fig. 2.1). But there are limits, for instance,

S. Bechtel, *Square Roots of Elliptic Systems in Locally Uniform Domains*,
Linear Operators and Linear Systems 303,
https://doi.org/10.1007/978-3-031-63768-1_1

an open set has to be interior thick at least to allow for the construction of an extension operator for Sobolev spaces [41].

To establish further results in non-smooth geometry and in the theory of Sobolev spaces, it is often handy to have notions of (fractal) dimension at hand. In fact, there is a whole zoo of such dimensions, including that of Aikawa, Assouad, Hausdorff, and many more [33]. They are of different nature, for instance, the Aikawa dimension is Euclidean, and the Assouad dimension is purely metric. Furthermore, there are different purely metric dimensions, which emphasize different aspects, and hence might not coincide for certain sets. But there are also prominent examples where different notions do coincide [53]. Often, these bridges lead to deep insights! And even if different notions are not equivalent, they occasionally obey interesting relations, for example, the relation between porosity and the dimension of Assouad [55]: a set is porous if and only if its Assouad codimension is strictly positive.

Fractional dimensions are also tied to the study of Hardy's inequality [28, 31, 40, 52] and fractional variants thereof [25, 29], the study of characteristic functions as pointwise multipliers [34, 66] and their regularity as functions [67], or the existence of traces of almost everywhere defined functions [1, 48].

Boundary Conditions All of the three mentioned tools—Hardy's inequality, characteristic functions as pointwise multipliers, and trace operators—can be used to introduce homogeneous *Dirichlet boundary conditions*, which is to say that a function vanishes "in some sense" on the boundary. Observe that for the existence of a trace operator, some regularity of the function and the boundary are *a priori* needed, whereas it is always possible to write down a Hardy's term and ask for its finiteness. This already highlights that different concepts for the treatment of boundary conditions might not even be comparable, yet coincide in general, and each of them has advantages and disadvantages.

We will use two out of the three methods in the interpolation of Sobolev spaces with boundary conditions in Chap. 10. For the usage of pointwise multipliers, we refer the reader to [15]. The boundary conditions of the spaces under consideration in the chapter on interpolation are always formulated using a trace operator. We will see that this allows to apply simple functorial arguments in the treatment of these spaces. Nevertheless, Hardy's inequality is a handy way to encode a vanishing trace condition in a manner that is accessible to direct computations, and we are going to exploit this in the interpolation of Sobolev spaces in Sect. 10.4. Another example of such an application of Hardy's inequality is the "special" Calderón-Zygmund decomposition shown in Chap. 18. In Chap. 15, a fractional Hardy term is even turned into the definition of a "vanishing trace" for functions with a fractional order of Sobolev regularity, and a fairly general extension result without usage of localization techniques is established.

Besides homogeneous Dirichlet boundary conditions, there are other boundary conditions that one could impose. For example, one could demand that a function does not vanish at the boundary, but attains a prescribed function defined on the boundary, which corresponds to non-homogeneous Dirichlet boundary conditions.

One could also require that the gradient of the function satisfies some condition on the boundary. As with the trace operator, it is again a nontrivial question how such a condition even has to be understood. In the further course of this monograph, questions of this kind will not be addressed. Instead, a third type of boundary condition is in the spotlight: we speak of *mixed boundary conditions* if a homogeneous Dirichlet boundary condition is imposed on a portion of the boundary and *natural boundary conditions* are imposed on the rest of the boundary. In fact, mixed boundary conditions are one of the driving motives of this work and play a role in (almost) every chapter of this book.

The framework of Brewster, Mitrea, Mitrea, and Mitrea [21] could have been considered the state of the art in the treatment of mixed boundary problems. They use the (ε, δ)-domains introduced by Jones in a clever way as "charts" around the Neumann part to localize the mixed boundary constellation. This way, they can, for example, craft an extension operator for their geometric framework. In Chap. 4, we also use Jones' ideas to build an extension operator in the case of mixed boundary conditions. However, this operator is not based on localization but modifies the original construction of Jones. To get good estimates for his extension operator, Jones uses *connecting chains of cubes* between so-called *interior cubes*. In our construction, there are interior cubes, which are not connected to other interior cubes, but "escape" the underlying set through the Dirichlet boundary part. The whole construction is highly technical, but it allows to consider constellations which are irregular arbitrarily close to the interface between Dirichlet and Neumann part, and hence are not feasible by localization methods.

The central concept in the construction of this extension operator, but also for all other topics of this monograph, are *locally uniform domains* near the Neumann boundary part N. They are so important for us that we dedicate Chap. 2 to them. They were introduced in [16] by the author together with M. Egert and R. Haller to study Kato's square root property (we are going to come back to this in a minute). Roughly, these are open sets in which point close to N that are nearby to each other can be connected by an ε-cigar that is taken with respect to N.

Differential Operators with Rough Coefficients and the Kato Square Root Problem We leave the geometric aspects aside for a moment to have a look at differential operators with rough coefficients. In a series of articles [49, 50], T. Kato asked the question when for an (at least maximal accretive) operator L on a Hilbert space, one has the identity $\mathsf{D}(\sqrt{L}) = \mathsf{D}(\sqrt{L^*})$. After a series of examples and, in particular, counterexamples [54, 56], the question was refined to the case where L is a second-order elliptic operator in divergence form with bounded, measurable, complex coefficients—in the first place as an operator on $\mathbb{R}^d$. In this case, L can be defined using a sesquilinear form $\mathfrak{a}\colon V \times V \to \mathbb{C}$, in which case the question can be reformulated as whether or not the identity $\mathsf{D}(\sqrt{L}) = V$ holds. This identity is called the *Kato square root property*. In the case of smooth coefficients, the operator L itself has optimal elliptic regularity, and the square root property is an easy application of complex interpolation. Also, if the operator is self-adjoint, the square root property follows readily from Kato's so-called *second representation* *theorem* [51]. We also present the relevant special case in Proposition 6.8. In the rough and non-selfadjoint situation, the square root property means that at least $\sqrt{L}$ has optimal elliptic regularity, even though this might not be the case for L itself. It turned out that the fractional exponent $1/2$ is the critical exponent for optimal elliptic regularity. For exponents strictly below $1/2$, optimal elliptic regularity follows from abstract arguments and was already known to Kato [49]. On the other hand, it is easy to construct counterexamples against optimal regularity for exponents above $1/2$ in dimension one [2].

Kato's motivation for this question came from applications to elliptic and hyperbolic equations, see [57] for more information. These ideas are nowadays successfully used in what is called the *first-order approach* [4, 5]. This underlines the relevance of Kato's question, and in particular the deviation from his original question, which was ruled out by the counterexamples of Lions and McIntosh.

From the viewpoint of harmonic analysis, Kato's square root problem asks to bound certain singular integrals. Besides the square root problem, there were other challenging problems of the same kind, like the boundedness of the Cauchy integral on Lipschitz curves, which were summarized under the name *Calderón program* [11, p. 463]. Armed with many novel techniques, Kato's square root problem was eventually solved in 2002 by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian in their seminal paper [9]. For more information, we refer the reader to the excellent surveys of McIntosh in [59, 60] and to the introduction of [9].

To close the loop to rough geometry and mixed boundary conditions, we take a look at a quote by Lions taken from a remark in [54], where he says the following:

> [...] par exemple, pour un opérateur elliptique A du 2ème ordre, non auto-adjoint, avec condition aux limites de Dirichlet sur une partie de la frontière et condition aux limites de Neumann sur le reste de la frontière, on ignore si $D(A^{1/2}) = D(A^{*1/2})$. Même chose d'ailleurs avec le problème de Dirichlet et une frontière irrégulière.[1]

Phrased differently, Lions suggests to combine the challenges in rough geometry and mixed boundary conditions with those in harmonic analysis coming from rough coefficients of a differential operator.

The seminal work by Axelsson, Keith, and McIntosh [11] opened the door to this problem. In that article, the authors provide *quadratic estimates* for perturbed Dirac operators. This framework allows to solve several problems from the Calderón program at once, including the Kato square root problem. Thereby, it is flexible enough to also treat systems of equations and their multiplicative perturbations, and it could be adapted by the same authors to give a first answer [10] to the problem posed by Lions. However, the class of admissible geometries in there is not easily accessible.

[1] For example, if A is a second-order elliptic operator that is not self-adjoint and that is subject to Dirichlet boundary conditions on a part of the boundary and Neumann boundary conditions on the rest, then it is unknown if the identity $D(A^{1/2}) = D(A^{*1/2})$ holds. The same is true if the operator is subject to pure Dirichlet boundary conditions, but the boundary is irregular.

This is why Egert, Haller-Dintelmann, and Tolksdorf refined the ideas from [11]. They observed that it is possible to prove quadratic estimates in a way that decouples harmonic analysis from geometry [30]. In a second paper, they used their modified framework to prove a very general result concerning Lions' problem [29]. Compared to the application to Kato's problem in [11], the application here is far more involved: it requires hard work to check the assumptions for the perturbed Dirac operator framework, including interpolation theory for Sobolev spaces incorporating boundary conditions, extrapolated optimal regularity for the Laplacian, or the construction of extension operators for Sobolev spaces with boundary conditions.

Extending the result of Egert, Haller-Dintelmann, and Tolksdorf is probably the deepest contribution of this monograph to the field. Their result is already fairly general, but there is some margin for improvement. For instance, they only treat bounded domains, which satisfy the interior thickness condition. Moreover, they assume that the whole boundary and not only the Dirichlet boundary part is Ahlfors–David regular. In their setup, the latter is no restriction because they require Lipschitz charts around the Neumann boundary part, which in turn implies that the full boundary is regular. Also, they do not treat multiplicative perturbations of elliptic systems. Our improvement in Chap. 14 is as follows: we allow the underlying set to be disconnected and unbounded. Only the Dirichlet boundary part is supposed to be Ahlfors–David regular, and regularity around the Neumann boundary only comes from O being locally uniform around N. Lastly, we allow multiplicative perturbations from either the right or left by an elliptic coefficient matrix.

It can be shown that N is *porous* when O is locally uniform near N. Porous sets are systematically studied in Chap. 7. Together with Ahlfors–David regularity of D, it follows that the full boundary of O is porous. Thus, we use porosity as a substitute for Ahlfors–David regularity for the full boundary in [30]. Besides Ahlfors–David regularity, Lipschitz charts around the Neumann part were needed for the existence of a Sobolev extension operator in their setting. This is not an issue for us since we can rely on the extension operator from Chap. 4 instead. The connectedness assumption was in fact not needed in [30] and could hence be easily eliminated.

The most severe challenge is to eliminate the interior thickness condition. In other words, this condition means that the underlying set is a space of homogeneous type. For this type of spaces, Christ managed to craft "dyadic grids" [23, 61], which can be used as a substitute for dyadic cubes in $\mathbb{R}^d$. The existence of such a cube structure is essential for the dyadic harmonic analysis in the proof of quadratic estimates and cannot be circumvented. Instead, we employ an *a posteriori* argument in Chap. 14. This works in two steps. First, we fatten the underlying set near the Dirichlet boundary part, thereby ensuring the interior thickness condition without losing geometric quality. Note that even if one starts with a connected set, this fattened set will be disconnected, which shows that the deviation from domains opens the road for a richer toolbox, even if one is only interested to apply the result to domains in the end. On this auxiliary set, we solve the Kato problem for an "extension" of the elliptic operator. Second, we decompose the functional calculus of this extended elliptic operator to transfer regularity of the square root back to the original elliptic operator.

Besides this, we also have to redo the arguments from [29] in the more complex geometric constellation. A lot of this is already done in the chapters on interpolation theory and Sobolev space, but we also need some more involved potential theory due to the lack of Ahlfors–David regularity for the full boundary.

Beyond Calderón-Zygmund Theory Another consequence of the rough nature of the coefficients is that extrapolation to L^p-spaces is much harder compared to classical Calderón-Zygmund theory. In particular, it is in general not possible to show $\mathrm{W}^{1,p} \to \mathrm{L}^p$ estimates for $\sqrt{L}$ for *all* $1 < p < \infty$. Nevertheless, extrapolation to $p \neq 2$ is possible and was pioneered by Blunck and Kunstmann [20]. In the situation of the classical Kato problem on $\mathbb{R}^d$, the L^p-extrapolation theory is well-understood [3]. These techniques go under the name "beyond Calderón-Zygmund theory". Extensions to the situation of mixed boundary conditions were first obtained for real equations [6] and later extended to complex systems [27] in a natural range when $p < 2 + \varepsilon$.

The geometric assumptions in [6, 27] were essentially dictated by the L^2-theory in [29, 30]. Hence, it is only natural to generalize these results to the situation from Chap. 14. This is performed in Chap. 19. Since we include multiplicative perturbations, arguments have to be changed compared to earlier work. Certainly, the recent monograph [8] paved the road for many of these changes, but some arguments like our approach to extrapolate the H^∞-calculus in Sect. 17.2 seem nevertheless to be new in the literature. The most innovative contribution concerns the case $p > 2$. In the reference works, only exponents up to $2 + \varepsilon$ were considered, where ε is an abstract parameter. A quantifiable interval was only used in the work of Auscher on $\mathbb{R}^d$ in [3] using gradient families, but these are not suitable to work in general geometric constellations. Instead, we quantify an upper endpoint for the extrapolation range for the perturbed system L by extrapolation properties of the unperturbed Lax–Milgram isomorphism $\mathcal{L}_0$. In particular, this shows that the extrapolation range is independent of a possible perturbation of the system.

1.1 Notation

We introduce notation that is used throughout the monograph. There is also an index, which contains relevant notation developed in the course of this book.

General For a discrete set J write $\#_J$ or $|J|$ for the number of elements of J. The indicator function of a set A is denoted by $\mathbf{1}_A$.

Euclidean Concepts The dimension of the ambient Euclidean space is $d \geq 2$. The Euclidean distance is $\mathrm{d}(x, y) = |x - y|$ and it induces the distance of sets $\mathrm{d}(A, B)$. Write $\mathrm{B}(x, r)$ for the Euclidean ball of radius r centered in x. The radius of a given ball B is denoted by $\mathrm{r}(B)$. For a number $c > 0$ and a ball B write cB for the concentric ball with radius $c\mathrm{r}(B)$. The norm $|\cdot|_\infty$ is the Euclidean infinity norm and induces the cubes. As for balls, if $c > 0$ is a scaling factor and Q a cube of sidelength $\ell(Q)$, write cQ for the concentric cube of sidelength $c\ell(Q)$. For a given cube Q define "annuli" by

$$C_1(Q) = 4Q \qquad \text{and} \qquad C_j(Q) = 2^{j+1}Q \setminus 2^j Q \qquad (j \geq 2).$$

Put $\operatorname{diam}(A)$ for the diameter of a set A. The Lebesgue measure of a measurable set A is $|A|$. The symbols $\mathrm{d}x$, $\mathrm{d}t$ and so on indicate integration with respect to the Lebesgue measure. If γ is a rectifiable curve in $\mathbb{R}^d$, write $\ell(\gamma)$ for the length of γ. The notation $z \in \gamma$ means $z = \gamma(t)$ for some t. When $f : \mathbb{R}^d \to \mathbb{C}$ is a measurable function, put

$$\mathrm{M}(f)(x) = \sup_B \fint_B f(y)\,\mathrm{d}y$$

for all $x \in \mathbb{R}^d$ for which the right-hand side exists, where the supremum is taken over all balls B with $x \in B$. Here, $\fint$ is the average over a given set. Say that $\mathrm{M}(f)$ is the maximal function of f and the mapping $f \mapsto \mathrm{M}(f)$ is the maximal operator.

By $\mathcal{H}^s(E)$, $s \in (0, d]$, we denote the s-dimensional *Hausdorff measure* of $E \subseteq \mathbb{R}^d$ defined as follows. For $\varepsilon > 0$ we put

$$\mathcal{H}^s_\varepsilon(E) := \inf\Big\{\sum_i \mathrm{r}(B_i)^s : \bigcup_i B_i \supseteq E, \mathrm{r}(B_i) \leq \varepsilon\Big\}$$

and since this value is increasing as $\varepsilon \to 0$ we define $\mathcal{H}^s(E) := \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(E)$.

Function Spaces Write $\mathrm{L}^p(\Xi)$ for the Lebesgue space on a measurable set Ξ. If Ξ is open, write $\mathrm{W}^{k,p}(\Xi)$ for the Sobolev space of order k. The space $\mathrm{H}^{s,p} := \mathrm{H}^{s,p}(\mathbb{R}^d)$ is the usual Bessel potential space of order $s \in \mathbb{R}$. It is defined via the fractional powers $(1 - \Delta)^{\frac{s}{2}}$ of the Laplacian. They can be expressed via (unbounded) Fourier multiplication operators.

On an open set U the class $\mathcal{M}(U)$ consists of the meromorphic functions on U and $\mathrm{H}^\infty(U)$ of the bounded holomorphic functions on U. Equip $\mathrm{H}^\infty(U)$ with the norm $\|\cdot\|_\infty$ so that it becomes a Banach algebra.

If f is a function on A and $B \supseteq A$ is another set clear from the context, then $\mathcal{E}_0 f$ is the zero extension of f to B.

Operators If T is a given operator, then $\mathsf{D}(T)$, $\ker(T)$ and $\mathsf{R}(T)$ denote its domain, null space and range. Its spectrum and resolvent set are denoted by $\sigma(T)$ and $\rho(T)$. If $(H_i)_i$ is a sequence of Hilbert spaces and T_i is an operator in H_i for each i, then $\bigotimes_i H_i$ is the Hilbert space of sequences $(U_i)_i$ with $U_i \in H_i$ and $\|(U_i)_i\| := (\sum_i \|U_i\|^2_{H_i})^{1/2} < \infty$ and the operator $\bigotimes_i T_i$ acts componentwise on its domain $\bigotimes_i \mathsf{D}(T_i) \subseteq \bigotimes_i H_i$.

Exponents If $p \in [0, \infty]$ then its *Hölder conjugate* p' is determined by

$$\frac{1}{p'} = 1 - \frac{1}{p},$$

where we use the convention $1/\infty = 0$. The *upper* and *lower Sobolev conjugates* are defined through

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d},$$
$$\frac{1}{p_*} = \frac{1}{p} + \frac{1}{d}.$$

If $p > d$, then p^* is not a number in $[1, \infty]$.

Chapter 2
Locally Uniform Domains

In this chapter, we will introduce and explore the underlying geometry for this monograph: locally uniform domains near the Neumann boundary. The precise definition will be given in Definition 2.1. The original idea of uniform domains and their relation with the Sobolev extension problem originates from Jones' work [47]. Variations in the case of mixed boundary conditions were first given in [21]. We discuss their framework in Sect. 2.2. The geometric framework presented in this chapter is a special case of [14] that is better suited to conquer the Kato square root property later on.

2.1 Introduction of Locally Uniform Domains

Definition 2.1 Let $\varepsilon \in (0, 1]$ and $\delta \in (0, \infty]$. Let $O \subseteq \mathbb{R}^d$ be open and $N \subseteq \partial O$. Set $N_\delta := \{z \in \mathbb{R}^d : \mathrm{d}(z, N) < \delta\}$. Then O is called *locally an* (ε, δ)*-domain near* N if the following properties hold.

(i) All points $x, y \in O \cap N_\delta$ with $|x - y| < \delta$ can be joined in O by an ε*-cigar with respect to* $\partial O \cap N_\delta$, that is to say, a rectifiable curve $\gamma \subseteq O$ of length

$$\ell(\gamma) \leq \varepsilon^{-1}|x - y| \tag{LC}$$

such that

$$\mathrm{d}(z, \partial O \cap N_\delta) \geq \frac{\varepsilon|z - x|\,|z - y|}{|x - y|} \qquad (z \in \gamma). \tag{CC}$$

(ii) O has *positive radius near* N, that is, there exists $c > 0$ such that all connected components O' of O with $\partial O' \cap N \neq \emptyset$ satisfy $\operatorname{diam}(O') \geq c$.

S. Bechtel, *Square Roots of Elliptic Systems in Locally Uniform Domains*,
Linear Operators and Linear Systems 303,
https://doi.org/10.1007/978-3-031-63768-1_2

If the values of ε and δ need not be specified, then O is simply called *locally uniform near* N.

Remark 2.2 Definition 2.1 describes a quantitative local connectivity property of O near N. For an illustration of ε-cigars with respect to ∂O the reader can refer for instance to [71, Fig. 3.1]. Having positive radius is of course only a restriction if O has infinitely many connected components.

As a starter, we show a corkscrew condition near N. This condition implies, for instance, that N is a null set (see Lemma 4.8). Moreover, the corkscrew condition has a close connection with the notions of a porous set (which we will study in more detail in Chap. 7) and the interior thickness condition (see Definition 8.15).

Proposition 2.3 (Corkscrew Condition) *Suppose that* $O \subseteq \mathbb{R}^d$ *is open and locally an* (ε, δ)*-domain near* $N \subseteq \partial O$*. Then there is some* $\kappa \in (0, 1]$ *such that*

$$\forall x \in \overline{N_{\delta/2} \cap O},\ r \leq 1 \quad \exists z \in \mathrm{B}(x, r)\ :\ \mathrm{B}(z, \kappa r) \subseteq O \cap \mathrm{B}(x, r).$$

Proof By assumption, O is locally an (ε, δ)-domain near N. Moreover, recall the positive radius constant c from Definition 2.1. With these numbers in hand, put $C := \min(\delta/2, c, 1)$.

First, observe that it suffices to show the claim for $r \leq C$ and $x \in N_{\delta/2} \cap O$. Indeed, for such x and any $r \leq 1$ we find $z \in \mathrm{B}(x, Cr) \subseteq \mathrm{B}(x, r)$ with $\mathrm{B}(z, (\kappa C)r) \subseteq O \cap \mathrm{B}(x, r)$. That is to say, we just have to replace κ by κC for a general r. Finally, with a constant *strictly* smaller than κC we can allow all $x \in \overline{N_{\delta/2} \cap O}$ in virtue of a limiting argument.

That being said, let $r \leq C$ and $x \in N_{\delta/2} \cap O$. We claim that there is some $y \in O$ satisfying $r/2 \leq |x - y| \leq 3r/4$. Suppose that this was not true and let O' be the connected component of O that contains x. First, by choice of x there is $x_0 \in N$ with $|x - x_0| < \delta/2$. Any ball of radius at most $\delta/2$ around x_0 intersects O in a point y that can be joined to x by an ε-cigar in O. Thus, $y \in O'$. Hence, $x_0 \in \partial O'$, so that $\partial O' \cap N \neq \emptyset$. Second, $O' \subseteq \mathrm{B}(x, r/2)$, since otherwise connectedness would yield some $y \in O'$ with $r/2 \leq |x - y| \leq 3r/4$. Consequently, $\mathrm{diam}(O') \leq r/2 < c$, a contradiction to the positive radius property.

Fix any $y \in O$ as above. Then $|x - y| \leq 3r/4 < \delta/2$, and in particular $y \in N_\delta \cap O$. Let γ be a joining ε-cigar. By continuity we pick $z \in \gamma \subseteq O$ with $|x - z| = \frac{1}{2}|x - y|$. Let us verify the required property for $\kappa := \varepsilon/8$. First, we have $\mathrm{B}(z, \kappa r) \subseteq \mathrm{B}(x, r)$ by construction. Second, $|z - y| \geq \frac{1}{2}|x - y|$ and $|x - y| \geq \frac{r}{2}$ plugged into (CC) give $\mathrm{d}(z, \partial O \cap N_\delta) \geq \kappa r$. Suppose that $\mathrm{B}(z, \kappa r)$ intersects ∂O in a point w. It follows that $w \notin N_\delta$. Again by choice of x there is $x_0 \in N$ with $|x - x_0| < \delta/2$. Now

$$\delta \leq |w - x_0| \leq |w - z| + |z - x| + |x - x_0| \leq \kappa r + \frac{1}{2}|x - y| + \delta/2 < \delta,$$

a contradiction. Thus, $\mathrm{B}(z, \kappa r) \subseteq \mathbb{R}^d \setminus \partial O$. Third, as $z \in O$ we must have $\mathrm{B}(z, \kappa r) \subseteq O$. □