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Roland Duduchava
Eugene Shargorodsky
George Tephnadze
Editors

Tbilisi Analysis and PDE Seminar

Extended Abstracts of the 2020-2023 Seminar Talks

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Preface

The aim of this volume is to present new developments and ideas in various fields of *Analysis and Partial Differential Equations*, including harmonic analysis, operator theory, function spaces, inequalities, integral equations, and applications. It provides a comprehensive review of some latest results and ideas in these rapidly expanding fields. It also contains some open problems, which we hope will inspire further research. The book contains 21 research articles and covers a broad spectrum of topics and applications.

The contributing authors have given talks at, have participated in, or are related in some way to *Tbilisi Analysis & PDE seminar* and *Tbilisi Analysis & PDE workshop*. The seminars are held bi-weekly since 2020, and the workshop will be organised annually following the first workshop that was held at the University of Georgia, Tbilisi, on 30 August 30–2 September 2023. Participants of the workshop and the seminars have enjoyed inspiring talks by *Maria Esteban*, *Pavel Exner*, *Hans Feichtinger*, *Gerd Grubb*, *Ari Laptev*, *Volker Mehrmann*, *Lars-Erik Persson*, *Michael Ruzhansky*, *Kristian Seip*, *Mikhail Sodin*, and by many others.

The Seminars and Workshops advance mathematics research in Georgia, enhance its visibility, and foster collaboration between mathematicians from the region and from other countries. They aim to offer a general perspective of the current mathematics research in Georgia. Georgia has a strong tradition in Analysis, Integral and Partial Differential Equations (PDEs), and the seminar organisers continue efforts to inspire the young generation in Georgia to follow the traditions of the Georgian Mathematical School. We are particularly happy that a good proportion of the papers in this volume present research by young mathematicians in Georgia.

This series is supported by our partner—the Ghent Analysis and PDE Center, which has provided us with a framework to publish this book in the series: *Research Perspectives Ghent Analysis and PDE Center*. We thank the Editor in Chief of the series, Prof. Dr. Michael Ruzhansky, for his support, without which of our project would not have been possible.

We would like to extend our cordial thanks to our colleagues *Lars-Erik Persson* (UiT - The Arctic University of Norway), *Michael Ruzhansky* (Ghent Analysis and PDE Center), and *Duvan Cardona* (Ghent Analysis and PDE Center), who have

served as reviewers for this volume, for all the unconditional support they have given to this series. Also, we thank all our colleagues involved in the organisation of our past seminars and workshop, namely, *Medea Tsaava* (The University of Georgia), *Davit Baramidze* (The University of Georgia), *Giorgi Tutberidze* (The University of Georgia), and *Zurab Vashakidze* (The University of Georgia). We also thank *Kunda Kambaso* and *Daniel Jagadisan* from Birkhäuser-Springer for their help.

Tbilisi, Georgia — Roland Duduchava
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Tbilisi, Georgia — George Tephnadze
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Contents

Continuous Inequalities: Introduction, Examples and Related Topics

Joachim Jørgensen Ågotnes, Ludmila Nikolova, Lars-Erik Persson, and Sanja Varošanec

Abstract Classical inequalities in a measure space are usually described by involving finitely many functions $f_1, \ldots, f_N$, $N = 2, 3, \ldots$. In this paper we give an introduction and several examples when such inequalities can be given with infinitely many functions f_s involved, where index s can even be taken from another measure space. Such a development was also inspired when developing an interpolation between families (continuously many) of Banach spaces.

Keywords Inequalities · Continuous inequalities · Hölder's inequality · Minkowski's inequality · Bellman's inequality · Popoviciu's inequality · Refinement · Interpolation between families of Banach spaces

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1 Continuous/Family Hölder-Type Inequalities

The integral Hölder inequality reads: *if* p *and* q *are positive numbers such that* $\frac{1}{p}+\frac{1}{q}=1$ *and if* f *and* g *are non-negative functions on the measure space* (Y,ν), *such that* $f\in L_p(Y)$, $g\in L_q(Y)$, *then*

$$\int_Y f(t)g(t)\,d\nu(t)\le\left(\int_Y f^p(t)\,d\nu(t)\right)^{1/p}\left(\int_Y g^q(t)\,d\nu(t)\right)^{1/q}. \tag{1.1}$$

The Hölder inequality (1.1) can obviously be formulated also for $n=3,4,5,\ldots$ functions involved. However, to get a hint how to generalize it with infinitely many functions involved, we rewrite (1.1) as an inequality between two geometric means:

$$\int_Y f^{1/p}(t)g^{1/q}(t)\,d\nu(t)\le\left(\int_Y f(t)\,d\nu(t)\right)^{1/p}\left(\int_Y g(t)\,d\nu(t)\right)^{1/q}.$$

The continuous version of the Hölder inequality (as an inequality between generalized geometric means) reads:

Theorem 1.1 *Let* u *and* v *be weight functions on the measure spaces* (X,μ) *and* (Y,ν), *respectively, such that* $\int_X u(x)\,d\mu(x)=1$. *Let* f *be a positive function on* $X\times Y$ *and measurable with respect to the measure* $\mu\times\nu$. *Then*

$$\int_Y \exp\left(\int_X \log f(x,y)u(x)\,d\mu(x)\right)v(y)\,d\nu(y)$$
$$\le\exp\left(\int_X\log\left(\int_Y f(x,y)v(y)\,d\nu(y)\right)u(x)\,d\mu(x)\right). \tag{1.2}$$

Corollary 1.2 *Let* $w_1,\ldots w_m\ge 0$ *be real numbers and let* $u\ge 0$, $p>0$, $a_i>0$ $(i=1,2,\ldots,m)$ *be functions on* X *such that* $\displaystyle\int_X\frac{u(x)}{p(x)}d\mu(x)=1$ *and* a_i^p *are measurable on* X. *Then*

$$\sum_{i=1}^m w_i\exp\left(\int_X\log a_i(x)u(x)\,d\mu(x)\right)$$
$$\le\exp\left(\int_X\log\left(\sum_{i=1}^m w_i a_i^{p(x)}(x)\right)\frac{u(x)}{p(x)}d\mu(x)\right).$$

Remark 1.3 Putting in Corollary 1.2: $m = 2$, $w_1 = w_2 = 1$, $p(x) = 1$ we get the following result, which can be described as superadditivity of geometric means:

$$\exp\left(\int_X \log a_1(x)u(x)\,d\mu(x)\right) + \exp\left(\int_X \log a_2(x)u(x)\,d\mu(x)\right)$$
$$\leq \exp\left(\int_X \log\Big(a_1(x) + a_2(x)\Big)u(x)\,d\mu(x)\right).$$

Example 1.4 Let $N = 2, 3, \ldots$, $X = [0, 1]$, $d\mu(x) = dx$, $u \equiv 1$, $p_i > 1$ for $i = 1, 2, \ldots, N$, with $\sum_{i=1}^{N} \frac{1}{p_i} = 1$, $f(x, y) = f_1^{p_1}(y)$, $0 \leq x \leq \frac{1}{p_1}$, $f(x, y) = f_2^{p_2}(y)$, $\frac{1}{p_1} < x \leq \frac{1}{p_1} + \frac{1}{p_2}, \ldots$, $f(x, y) = f_N^{p_N}$, $\sum_{i=1}^{N-1} \frac{1}{p_i} < x \leq 1$. Then (1.2) reads:

$$\int_Y \prod_{i=1}^{N} f_i(y)v(y)\,d\nu(y) \leq \prod_{i=1}^{N}\left(\int_Y f_i^{p_i}(y)v(y)\,d\nu(y)\right)^{1/p_i},$$

i.e. it is a standard form of the Hölder inequality involving N functions.

There exists some refinements of several of the classical inequalities. Such refinements can also be proved for several continuous inequalities. Our refinement of the continuous Hölder inequality in Theorem 1.1 reads:

Theorem 1.5 *Let $f(x, y)$ be a positive and measurable function on $(X \times Y, \mu \times \nu)$ and let $u(x)$ and $v(y)$ be weight functions on X and Y, respectively, such that $\int_X u(x)\,d\mu(x) = 1$ and $\int_Y f(x, y)v(y)d\nu(y) > 0$ μ-a.e. Moreover, let (Z, dz) be a measure space and $\alpha(z, y)$ be a non-negative integrable function on $Z \times Y$ such that $\int_Y \alpha(z, y) f(x, y)v(y)d\nu(y) > 0$ μ-a.e and $\int_Z \alpha(z, y)\,dz = 1$, for $y \in Y$. Then the following refinement of continuous form (1.2) of the Hölder inequality holds:*

$$\int_Y \exp\left(\int_X \log f(x, y)u(x)\,d\mu(x)\right) v(y)\,d\nu(y)$$
$$\leq \int_Z \left[\exp \int_X \log\left(\int_Y \alpha(z, y) f(x, y)v(y)\,d\nu(y)\right) u(x)\,d\mu(x)\right] dz$$
$$\leq \exp\left[\int_X \log\left(\int_Y f(x, y)v(y)d\nu(y)\right) u(x)\,d\mu(x)\right].$$

2 Continuous/Family Minkowski-Type Inequalities

The main result in this case reads:

Theorem 2.1 *Let $f(x, y)$ be non-negative and measurable on $(X \times Y, \mu \times \nu)$ and let $u(x)$ and $v(y)$ be weight functions.*

(a) If $p \geq 1$, then

$$\left(\int_Y \left(\int_X f(x, y)u(x)\, d\mu(x)\right)^p v(y)\, d\nu(y)\right)^{\frac{1}{p}} \leq \int_X \left(\int_Y f^p(x, y)v(y)\, d\nu(y)\right)^{\frac{1}{p}} u(x)\, d\mu(x). \tag{2.1}$$

(b) If $0 < p < 1$ and

(i) $\int_X \left(\int_Y f(x, y)v(y)\, d\nu(y)\right)^p u(x)\, d\mu(x) > 0$ *μ-a.e. and*
$\int_Y f(x, y)v(y)\, d\nu(y) > 0$ *ν-a.e.,*
then the reverse inequality in (2.1) holds.

If $p < 0$, the above-mentioned assumptions (i) and the additional one

(ii) $\int_X f^p(x, y)u(x)\, d\mu(x) > 0$ *ν-a.e.*
hold, then the reverse inequality in (2.1) holds.

Example 2.2 Let $X_1, \ldots, X_N$ be a partition of X, $\int_{X_i} u(x)\, d\mu(x) = \alpha_i$, and $f(x, y) = \frac{f_i(y)}{\alpha_i}$, for $x \in X_i$, $(i = 1, 2, \ldots, N)$.

If $p \geq 1$, then (2.1) becomes:

$$\left(\int_Y \left(\sum_{i=1}^N f_i(y)\right)^p v(y)\, d\nu(y)\right)^{\frac{1}{p}} \leq \sum_{i=1}^N \left(\int_Y f_i^p(y)v(y)\, d\nu(y)\right)^{\frac{1}{p}}, \tag{2.2}$$

i.e. the usual Minkowski inequality for integrals with N non-negative functions involved.

If $0 < p < 1$, then, with the obvious restrictions on the integrals, (2.2) holds in the reversed direction.

A standard proof of the Minkowski inequality (2.1) is just to use the corresponding Hölder inequality. In the continuous family situation, we even have the following remarkable equivalence theorem:

Theorem 2.3 *Let $f(x, y)$ be positive and measurable on $(X \times Y, \mu \times \nu)$. Assume that $p \geq 1$ and that $u(x)$ and $v(y)$ are weight functions. Then the following statements are equivalent:*

(i) *The continuous Hölder inequality* (1.2) *holds for all* X, $u(x)$ *and* $\mu(x)$ *such that* $\int_X u(x)\,d\mu(x) = 1$.

(ii) *The continuous Minkowski inequality* (2.1) *holds for all* X, $u(x)$ *and* $\mu(x)$ *such that* $\int_X u(x)\,d\mu(x) < \infty$.

As in the case of the Hölder inequality, we can also state the refinement of the continuous Minkowski inequality.

Theorem 2.4 *Let* $f(x, y)$ *be a non-negative and measurable function on* $(X \times Y, \mu \times \nu)$, *let* $u(x)$ *and* $v(y)$ *be weight functions on* X *and* Y, *respectively. Moreover, let* $\alpha(z, y)$ *be a non-negative function such that* $\int_Z \alpha(z, y)\,dz = 1$ *for* $y \in Y$. *If* $p \geq 1$, *then*

$$\begin{aligned}
&\int_Y \left(\int_X f(x, y) u(x)\, d\mu(x) \right)^p v(y)\, d\nu(y) \\
&\qquad \leq \int_Z \left[\int_X \left(\int_Y \alpha(z, y) f^p(x, y) v(y)\, d\nu(y) \right)^{1/p} u(x)\, d\mu(x) \right]^p dz \\
&\qquad \leq \left[\int_X \left(\int_Y f^p(x, y) v(y)\, d\nu(y) \right)^{1/p} u(x)\, d\mu(x) \right]^p .
\end{aligned}$$

3 Continuous/Family Popoviciu and Bellman-Type Inequalities

Popoviciu-type inequalities are some type of reversed Hölder-type inequalities. We just state one such result involving two functions:

Proposition 3.1 *Let* p, q *be positive real numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* f *and* g *be non-negative measurable functions on the measure space* (Y, ν). *Then the following inequality*

$$\begin{aligned}
&v_0 c_1 c_2 - \int_Y v(y) f(y) g(y)\, d\nu(y) \\
&\qquad \geq \left(v_0 c_1^p - \int_Y v(y) f^p(y)\, d\nu(y) \right)^{\frac{1}{p}} \left(v_0 c_2^q - \int_Y v(y) g^q(y)\, d\nu(y) \right)^{\frac{1}{q}}
\end{aligned} \tag{3.1}$$

holds, where $v(y)$ *is a weight,*

$$v_0 c_1^p - \int_Y v(y) f^p(y)\, d\nu(y) \geq 0, \quad \textit{and} \quad v_0 c_2^q - \int_Y v(y) g^q(y)\, d\nu(y) \geq 0.$$

Our continuous version of Proposition 3.1 reads:

Theorem 3.2 *Let $u(x)$ and $v(y)$ be weight functions on the measure spaces (X, μ) and (Y, ν), respectively, such that $\int_X u(x)\,d\mu(x) = 1$, let $f(x, y)$ be a positive measurable function on $X \times Y$, $v_0 \in (0, \infty)$, and assume that $f_0(x)$ is a function on X such that $v_0 f_0(x) > \int_Y f(x, y)v(y)\,d\nu(y)$, for all $x \in X$. Then the following continuous form of the Popoviciu inequality holds:*

$$
\begin{aligned}
&\exp\left(\int_X \log(v_0 f_0(x))u(x)\,d\mu(x)\right)\\
&\quad - \int_Y \exp\left(\int_X \log f(x, y)u(x)\,d\mu(x)\right) v(y)\,d\nu(y)\\
&\quad \geq \exp\left[\int_X \log\left(v_0 f_0(x) - \int_Y f(x, y)v(y)\,d\nu(y)\right) u(x)\,d\mu(x)\right].
\end{aligned}
$$

Example 3.3 Let $u(x) = 1$, $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ with $\displaystyle\int_{X_1} d\mu(x) = \frac{1}{p}$, $\displaystyle\int_{X_2} d\mu(x) = \frac{1}{q}$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

$$
f_0(x) = \begin{cases} c_1^p, & x \in X_1 \\ c_2^q, & x \in X_2 \end{cases} \quad \text{and} \quad f(x, y) = \begin{cases} f^p(y), & x \in X_1 \\ g^q(y), & x \in X_2 \end{cases}.
$$

Then we rediscover the Popoviciu inequality (3.1) in the finite form with $v_0 = 1$.

Theorem 3.2 also implies the following result of independent interest:

Corollary 3.4 *Let $w_1 > 0$, $w_2, \ldots, w_m \geq 0$ be reals, $p, a_i, i = 1, 2, \ldots, m$, be positive functions on X such that $\displaystyle\int_X \frac{d\mu(x)}{p(x)} = 1$ and a_i^p are measurable on X. Then*

$$
\begin{aligned}
&w_1 \exp\left(\int_X \log a_1(x)\,d\mu(x)\right) - \sum_{i=2}^{m} w_i \exp\left(\int_X \log a_i(x)\,d\mu(x)\right)\\
&\quad \geq \exp\left\{\int_X \log\left[w_1(a_1(x))^{p(x)} - \sum_{i=2}^{m} w_i(a_i(x))^{p(x)}\right]\frac{d\mu(x)}{p(x)}\right\},
\end{aligned}
$$

provided that all integrals exist.

Our refinement of the continuous Popoviciu inequality reads:

Theorem 3.5 *Let $u(x)$ and $v(y)$ be weight functions on the measure spaces (X, μ) and (Y, ν), respectively, where $\int_X u(x)\,d\mu(x) = 1$. Let $f(x, y)$ be a positive measurable function on $X \times Y$, $v_0 \geq 0$, and assume that $f_0(x)$ is a function on X such that $v_0 f_0(x) > \int_Y f(x, y)v(y)\,d\nu(y)$, for all $x \in X$. Moreover, let $\alpha(z, y)$*

be a non-negative integrable function on $Z \times Y$ *such that* $\int_Z \alpha(z, y)\, dz = 1$ *for* $y \in Y$, *where* (Z, dz) *is a* $\sigma-$*finite measure space. Then the following refinement of the continuous form of the Popoviciu inequality holds:*

$$\begin{aligned}
&\exp\left(\int_X \log\left(v_0 f_0(x)\right) u(x)\, d\mu(x)\right) \\
&\quad - \int_Y \exp\left(\int_X \log(f(x, y)) u(x)\, d\mu(x)\right) v(y)\, d\nu(y) \\
&\quad \geq \exp\left(\int_X \log\left(v_0 f_0(x)\right) u(x)\, d\mu(x)\right) \\
&\quad - \int_Z \left[\exp \int_X \log\left(\int_Y f(x, y)\alpha(z, y) v(y)\, d\nu(y)\right) u(x)\, d\mu(x)\right] dz \\
&\quad \geq \exp\left[\int_X \log\left(v_0 f_0(x) - \int_Y f(x, y) v(y)\, d\nu(y)\right) u(x)\, d\mu(x)\right] \geq 0.
\end{aligned}$$

Example 3.6 Using a suitable choice for the measure μ and functions u and f, we get the following refinement of the integral Popoviciu inequality (3.1) for two functions with $v_0 = 1$.

$$\begin{aligned}
&c_1 c_2 - \int_Y v(y) f(y) g(y)\, d\nu(y) \\
&\quad \geq c_1 c_2 - \int_Z \left(\int_Y \alpha(z, y) f^p(y) v(y)\, d\nu(y)\right)^{1/p} \\
&\quad \left(\int_Y \alpha(z, y) g^q(y) v(y)\, d\nu(y)\right)^{1/q} dz \\
&\quad \geq \left(c_1^p - \int_Y v(y) f^p(y)\, d\nu(y)\right)^{1/p} \left(c_2^q - \int_Y v(y) g^q(y)\, d\nu(y)\right)^{1/q}.
\end{aligned}$$

Bellman-type inequalities are just a type of reversed Minkowski-type inequalities in a similar way as the Popoviciu inequality was related to the Hölder inequality. Our continuous form of this inequality reads:

Theorem 3.7 *Let* $f_0(x)$, $f(x, y)$, v_0, $u(x)$, $v(y)$, X, Y, μ, ν *be defined as in Theorem 3.2. Then, for* $p \geq 1$,

$$\begin{aligned}
&\left(\int_X \left[v_0 f_0^p(x) - \int_Y f^p(x, y) v(y)\, d\nu(y)\right]^{\frac{1}{p}} u(x)\, d\mu(x)\right)^p \\
&\quad \leq v_0 \left[\int_X f_0(x) u(x)\, d\mu(x)\right]^p - \int_Y \left[\int_X f(x, y) u(x)\, d\mu(x)\right]^p v(y)\, d\nu(y),
\end{aligned}$$

whenever $v_0 f_0^p(x) \geq \int_Y f^p(x, y)v(y)\, d\nu(y)$, *for all* $x \in X$.

Example 3.8 By applying Theorem 3.7 with $u(x) = v(y) = 1$, $v_0 = 1$, $X = \bigcup_{i=1}^n X_i$, $X_i = [i-1, i)$ for $i = 1, 2, \ldots, n$, $d\mu(x) = dx$, $f(x, y) = f_i(y)$ and $f_0(x) = c_i$ for each $x \in X_i$, $i = 1, 2, \ldots, n$, we get the following version of Bellman's inequality:

$$\sum_{i=1}^n \left(c_i^p - \int_Y f_i^p(y) d\nu(y)\right)^{\frac{1}{p}} \leq \left(\left(\sum_{i=1}^n c_i\right)^p - \int_Y \left(\sum_{i=1}^n f_i(y)\right)^p d\nu(y)\right)^{\frac{1}{p}}$$

whenever

$$c_i \geq \left(\int_Y f_i^p(y)\, d\nu(y)\right)^{\frac{1}{p}}, \quad i = 1, 2, \ldots, n.$$

Our refinement of the continuous Bellman inequality reads:

Theorem 3.9 *Let the assumptions of Theorem 2.4 hold. Assume that* $v_0 \geq 0$ *and* $f_0(x)$ *is a function on* X *such that* $v_0 f_0^p(x) - \int_Y f^p(x, y)\ v(y)\, d\nu(y) \geq 0$. *Let* $\alpha(z, y)$ *satisfies the assumptions of Theorem 3.5.*

Then the following refinement of the continuous Bellman inequality holds for $p \geq 1$*:*

$$\begin{aligned} v_0 \left[\int_X f_0(x)u(x)\, d\mu(x)\right]^p &- \int_Y \left(\int_X f(x, y)u(x)\, d\mu(x)\right)^p v(y)\, d\nu(y) \\ &\geq v_0 \left[\int_X f_0(x)u(x)\, d\mu(x)\right]^p \\ &- \int_Z \left[\int_X \left(\int_Y \alpha(z, y) f^p(x, y)v(y)\, d\nu(y)\right)^{1/p} u(x)\, d\mu(x)\right]^p dz \\ &\geq \left(\int_X \left[v_0 f_0^p(x) - \int_Y f^p(x, y)v(y)\, d\nu(y)\right]^{\frac{1}{p}} u(x)\, d\mu(x)\right)^p. \end{aligned}$$

4 Continuous Beckenbach-Dresher-Type Inequalities

The Beckenbach-Dresher inequality in its original form looks like a Minkowski-type inequality with the Gini means involved. Inspired by an extension of this result by Peetre-Persson, the continuous version reads:

Theorem 4.1 *Let (X,μ), (Y,ν) and (Y,λ) be measure spaces. Let f,g be non-negative functions on $X\times Y$ such that f is integrable with respect to the measure $\mu\times\nu$ and g is integrable with respect to $\mu\times\lambda$.*

(a) If

(i) $u\geq 1$ and $q\leq 1\leq p$ $(q\neq 0)$, or
(ii) $u<0$ and $p\leq 1\leq q$ $(p\neq 0)$, and all terms exist, then

$$\frac{\left(\int_Y\left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y\left(\int_X g(x,y)d\mu(x)\right)^q d\lambda(y)\right)^{\frac{u-1}{q}}} \leq \int_X \frac{\left(\int_Y f^p(x,y)d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y g^q(x,y)d\lambda(y)\right)^{\frac{u-1}{q}}}d\mu(x). \tag{4.1}$$

If (iii) $0<u\leq 1$, $p\leq 1$ and $q\leq 1$, $p,q\neq 0$, then inequality (4.1) is reversed.

(b) If $u\geq 1$ and $p\geq 1$, then

$$\begin{aligned}&\left(\int_Y\left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y)\right)^{\frac{u}{p}}\\ &\quad\times\exp\left(\frac{1-u}{\int_Y d\lambda}\int_Y \log\left(\int_X g(x,y)d\mu(x)\right)d\lambda(y)\right)\\ &\leq \int_X\left(\int_Y f^p(x,y)d\nu(y)\right)^{\frac{u}{p}}\exp\left(\frac{1-u}{\int_Y d\lambda}\int_Y \log g(x,y)d\lambda(y)\right)d\mu(x).\end{aligned}$$

Remark 4.2 Using a discrete measure μ in (4.1), we can formulate the following Beckenbach-Dresher inequality (of Peetre-Persson type) for integrals: If u,p,q satisfy the assumptions of Theorem 4.1 (a), then

$$\frac{\left(\int_Y\left(\sum_{i=1}^n f_i(y)\right)^p d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y\left(\sum_{i=1}^n g_i(y)\right)^q d\lambda(y)\right)^{\frac{u-1}{q}}} \leq \sum_{i=1}^n \frac{\left(\int_Y f_i^p(y)d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y g_i^q(y)d\lambda(y)\right)^{\frac{u-1}{q}}}.$$

5 Final Remarks

Proofs of all results in this paper can be found in papers [1–5] and in the book [6]. In this book, we also pointed out that the close connection between convexity and interpolation theory is partly kept also in this general continuous/family situation. In particular, in the Appendix in [6], it is given a short introduction to the theory of interpolation between infinite many Banach spaces, even between families of Banach spaces (in the classical case only two or finite many Banach spaces are involved). It was in this connection two of the present authors first met the need to prove a classical inequality of Hölder-type in continuous/family form. We also mention that this new book manuscript contains more than 100 references on the subject, several other examples of classical inequalities in continuous/family form (e.g. those by Jensen, Jensen-Mercer and Hardy) and a view on some inequalities in Banach lattice norm settings. Finally, we pronounce that the research presented in this paper implies, in our opinion, an interesting direction of research, namely to prove some corresponding continuous/family forms of other inequalities than those covered in the book [6].

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Approximation by Vilenkin-Nörlund Means in Lebesgue Spaces

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Abstract In this paper we improve and complement a result by Móricz and Siddiqi (J Approx Theory 70(3):375–389, 1992). In particular, we prove that their estimate of the Nörlund means with respect to the Vilenkin system holds also without their additional condition. Moreover, we prove a similar approximation result in Lebesgue spaces for any $1 \leq p < \infty$.

Keywords Vilenkin group · Vilenkin system · Fejér means · Nörlund means · Approximation

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1 Introduction

Concerning some definitions and notations used in this introduction we refer to Sect. 2.

It is well-known (see e.g. [20, 39, 56]) that, for any $1 \leq p \leq \infty$ and $f \in L_p(G_m)$, there exists an absolute constant C_p, depending only on p such that

$$\|\sigma_n f\|_p \leq C_p \|f\|_p .$$

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Moreover, (for details see [39]) if $1 \leq p \leq \infty$, $M_N \leq n < M_{N+1}$, $f \in L^p(G_m)$ and $n \in \mathbb{N}$, then

$$\|\sigma_n f - f\|_p \leq 2R^5 \sum_{s=0}^{N} \frac{M_s}{M_N} \omega_p \left(1/M_s, f\right), \tag{1.1}$$

where $R := \sup_{k\in\mathbb{N}} m_k$ and $\omega_p(\delta, f)$ is the modulus of continuity of L^p, $1 \leq p \leq \infty$ functions defined by

$$\omega_p(\delta, f) = \sup_{|t|<\delta} \|f(x+t) - f(x)\|, \qquad \delta > 0.$$

It follows that if $f \in lip\,(\alpha, p)$, i.e.,

$$lip(\alpha, p) := \{f \in L^p : \omega_p(\delta, f) = O(\delta^\alpha) \quad \text{as} \quad \delta \to 0\},$$

then

$$\|\sigma_n f - f\|_p = \begin{cases} O\left(1/M_N\right), & \text{if} \quad \alpha > 1, \\ O\left(N/M_N\right), & \text{if} \quad \alpha = 1, \\ O\left(1/M_n^\alpha\right), & \text{if}\ \alpha < 1. \end{cases}$$

Moreover, (for details see [39]) if $1 \leq p < \infty$, $f \in L^p(G)$ and

$$\left\|\sigma_{M_n} f - f\right\|_p = o\left(1/M_n\right), \text{ as } n \to \infty,$$

then f is a constant function.

The weak-(1, 1) type inequality for the maximal operators of Vilenkin-Fejer means σ^*, defined by

$$\sigma^* f = \sup_{n\in\mathbb{N}} |\sigma_n f|$$

can be found in Schipp [40] for Walsh series and in Pĺ, Simon [33] and Weisz [54] for bounded Vilenkin series. Boundedness of the maximal operators of Vilenkin-Féjer means of the one- and two-dimensional cases can be found in Fridli [15], Gát [17], Goginava [18], Nagy and Tephnadze [29–32], Simon [42, 43], Tephnadze [44–49], Tutberidze [37, 50], Weisz [25, 38, 55].

Convergence and summability of Nörlund means with respect to Vilenkin systems were studied by Areshidze and Tephnadze [3] (see also [2]), Baramidze, Persson and Tephnadze [4–8], Blahota et al. [13] (see also [10–12, 34–36]), Fridli et al. [16], Goginava [19], Nagy [26–28] (see also [9, 14]), Memic [21, 22].