Earth and Environmental Sciences Library

Farid El-Dossoki
Mohamed Hassan
Amer Shehata *Editors*

Proceedings of The First International Conference on Green Sciences

Environmental Science for Green and Sustainable Environment

Springer

Earth and Environmental Sciences Library

Series Editors

Abdelazim M. Negm, *Faculty of Engineering, Zagazig University, Zagazig, Egypt*
Tatiana Chaplina, *Antalya, Türkiye*

Earth and Environmental Sciences Library (EESL) is a multidisciplinary book series focusing on innovative approaches and solid reviews to strengthen the role of the Earth and Environmental Sciences communities, while also providing sound guidance for stakeholders, decision-makers, policymakers, international organizations, and NGOs.

Topics of interest include oceanography, the marine environment, atmospheric sciences, hydrology and soil sciences, geophysics and geology, agriculture, environmental pollution, remote sensing, climate change, water resources, and natural resources management. In pursuit of these topics, the Earth Sciences and Environmental Sciences communities are invited to share their knowledge and expertise in the form of edited books, monographs, and conference proceedings.

Farid El-Dossoki · Mohamed Hassan ·
Amer Shehata
Editors

Proceedings of The First International Conference on Green Sciences

Environmental Science for Green and Sustainable Environment

Springer

Editors
Farid El-Dossoki
Faculty of Science
Port Said University
Port Said, Egypt

Mohamed Hassan
Faculty of Science
Port Said University
Port Said, Egypt

Amer Shehata
Faculty of Science
Port Said University
Port Said, Egypt

ISSN 2730-6674 ISSN 2730-6682 (electronic)
Earth and Environmental Sciences Library
ISBN 978-3-031-62671-5 ISBN 978-3-031-62672-2 (eBook)
https://doi.org/10.1007/978-3-031-62672-2

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

We are delighted to introduce this conference publication, highlighting the research papers presented at the "The First International Conference on Green Science". The primary objectives of the conference encompass a multifaceted approach towards promoting sustainable development, inspired by Egypt's pivotal role as the host for the prestigious UN conference, COP27. Through the integration of basic scientific principles with sustainability goals, the conference endeavors to foster a deeper understanding of environmental challenges and some potential scientific maneuvering to overcome. The conference paves the way for interdisciplinary scientific collaboration between academia research institutions and civil society to bridge knowledge gaps. Moreover, the conference seeks to strengthen the nexus between scientific research and industries, fostering partnerships to face environmental concerns. This has ultimately encouraged researchers to disseminate their findings at scientific forums as pivotal in fostering dialogue and collective action toward mitigating environmental issues. With a broad scope covering different topics of basic sciences including environmental sciences, chemistry, physics, microbiology and plants, biotechnology, marine sciences, geology, mathematics, and computer sciences, the conference serves as a platform to explore and exchange diverse scientific research aiming at sustainable development goals.

The initial section showcased in this book delves into the realms of "**Mathematics and Computer Science**". In the realm of mathematical modeling, the integration of innovative methods such as the Shifted Chebyshev Polynomials with Residual Power Series stands out as a promising approach for tackling diverse problem sets. Concurrently, in the domain of information systems, the application of Neighborhood Morphological Operators paired with precise Accuracy Measures proves invaluable for enhancing system performance. Meanwhile, the Fractional Relaxation Oscillation Equations find a series of approximate solution, offering a nuanced understanding of complex dynamics. In the educational sphere, the advent of Deep Learning Methods enables the discernment of intricate student behaviors, heralding advancements in pedagogical strategies. In agricultural technology, the Optimization of Convolutional Neural Network Models facilitates the efficient classification of multi-plant leaf diseases, bolstering crop management practices. Furthermore, the utilization of Virtual Machine technology for Adaptive Service Level Agreement adaptation showcases dynamic resource allocation in real-time scenarios. Lastly, in medical diagnostics, the employment of CNN models for lung diseases detection demonstrates the potential for leveraging cutting-edge technology in healthcare for early intervention and treatment. Many of the research papers introduced valuable output with potential applicability to save effort, time, and cost as one step toward sustainable development.

The papers reported in "**Part II: Environmental Science and Oceanography**" encompass a diverse range of environmental research topics. They delve into various aspects of environmental management and sustainability. One paper explores air pollutant loads linked to petroleum production, highlighting its relevance in assessing operational efficiency. Another introduces an innovative regression model utilizing environmental compliance data to predict risk scores for industrial facilities. Optimization of environmental inspection operations through data-driven risk classification is explored in a case study, emphasizing efficiency improvements. Additionally, there is research on nano-composites for heavy metal sorption from industrial wastewater, and the hydrographic changes in the Red Sea over decades are also included. Furthermore, the role of environmental engineering and sustainable agriculture in development is discussed, bridging reality and aspiration. Studies also assess the spatial distribution and pollution levels of physicochemical properties and heavy metals in water and sediments of specific drainage systems in the Nile Delta, Egypt. Section two ends by the assessment of carbon footprints that serves as a proactive response to climate change concerns, emphasizing the importance of environmental consciousness in modern-day practices.

Part III focuses on research efforts in diverse fields of biotechnology and plant biology. "**Microbiology and Biotechnology** are highlighted in a collection of papers addressing various environmental and medical concerns. The exploration of lipase-producing bacteria from oil-contaminated sites in Port-Said, Egypt, underscores the significance of bioprospecting for solutions to environmental challenges. Meanwhile, the study on auxin and cytokinin-mediated regeneration techniques in Paulownia tomentosa propagules sheds light on innovative methods for plant propagation and regeneration. Another paper delves into the synthesis of zinc oxide and copper oxide nanoparticles mediated by Sesuvium sesuvioides, revealing potential applications in cytotoxicity and apoptosis studies. Additionally, the role of Staphylococcus aureus biofilm formation in wound environments is examined, providing insights into antimicrobial resistance mechanisms crucial for medical advancements. These studies collectively demonstrate the multifaceted approaches researchers employ to address pressing issues in environmental sustainability and human health.

In addition a diverse range of scientific studies focusing on various aspects of research encountered in "**Part IV: Physical Chemistry and Physics**". The establishment of diagnostic reference levels for patients with renal colic using non-contrast computed tomography is explored in a pilot study, indicating advancements in medical imaging techniques. Additionally, thermodynamic, kinetics, and adsorption mechanism studies investigate the behavior of methyl orange on surfactant-modified activated carbon, shedding light on potential applications in wastewater treatment. Furthermore, research on Solenostemma Argel as a sustainable and eco-friendly corrosion inhibitor for aluminum in acidic environments underscores the importance of natural alternatives in mitigating industrial corrosion, aligning with efforts toward environmental sustainability and green chemistry initiatives.

The summarized research papers endeavors in "**Part V: Biochemistry**" cover a wide array of scientific investigations spanning diverse fields. The studies focused on the biosynthesis and cytotoxicity of curcumin loaded on CuO NPs against hepatocellular and colorectal carcinoma, emphasizing potential therapeutic applications. Additionally, green synthesis methods are employed to characterize zinc oxide and copper oxide nanoparticles conjugated with curcumin, reflecting their antioxidant and anti-tumor properties both in vitro and in vivo. Further research explores the anti-cancer activities of basil loaded with ZnO NPs against hepatocellular and colorectal carcinoma.

Genetic associations with male infertility risk in the Egyptian population are investigated, highlighting the involvement of specific gene polymorphisms.

Moreover, in vivo investigations focus on the hypocholesterolemic effects of synthetic cholesterol congeners and predictive biomarkers for COVID-19 severity, including neutrophil/lymphocyte and lactate dehydrogenase/lymphocyte ratios. Furthermore, the ameliorative effects of oleoylchalcones on oxidative damage and hypertrophy in the liver and cardiac tissues induced by a high-fat diet are examined, demonstrating potential therapeutic interventions. Lastly, prospective assessment of urinary biomarkers for diagnosing recurrence and progression in patients with non-muscle-invasive bladder cancer, along with the effect of heavy metals on bladder cancer and its relation to the Toll-like receptor signaling pathway, offer insights into diagnostic and therapeutic strategies for this malignancy.

We gratefully acknowledge the continuous help and support of the editor of the "Earth and Environmental Sciences Library book series", Prof. Abdelazim M. Negm for his invaluable contributions, including the precise review of articles of the conference proceedings and unwavering support throughout the entire lifecycle of the conference proceeding publication. Thanks are also extended to include Springer's team, starting from the evaluation of the proposal till the end of the publication processes.

Farid El-Dossoki
Mohamed Hassan
Amer Shehata

Contents

Mathematics and Computer Science

Shifted Chebyshev Polynomials with Residual Power Series Method for Solving Various Types of Models

Saad Z. Rida[1], Anas A. M. Arafa[2,3], Hussein S. Hussein[1], Ismail Gad Ameen[1], and Marwa M. M. Mostafa[1(✉)]

[1] Department of Mathematics, Faculty of Science, South Valley University, Qena 83523, Egypt
{hshafei,marwa.masoud}@sci.svu.edu.eg, ismailgad@svu.edu.eg
[2] Department of Mathematics, College of Science, Qassim University, Buraydah, Saudi Arabia
a.arafa@qu.edu.sa
[3] Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said, Egypt
anas_arafa@sci.psu.edu.eg

Abstract. Recently, Chebyshev's polynomials have attracted much attention in finding solutions for fractional order differential equations (FODEs).

In this research study, we represent shifted second-kind Chebyshev orthonormal polynomials (SSKCOP) with residual power series (RPS) technique to solve various types of models such as diffusion model, Backward Kolmogorov model, homogeneous and nonhomogeneous advection models taking into account with time fractional derivative in Caputo manner. By utilization of SSKCOP and their orthogonality properties, these models will be reduce into system of FODEs which can be solved by using RPS technique. The numerical simulations and outcomes are presented through various graphs and tables demonstrating that present method is accurate and powerful to obtain approximate solutions of nonlinear models that arise in engineering and physics.

Keywords: Collocation Method · Residual Power Series Technique · Fractional Derivative · Fractional Diffusion Model · Backward Kolmogorov Model · Advection Partial Differential Equation

1 Introduction

Liouville, Riemann, Grunwald, Leibniz, Letnikov and others developed the fractional calculus theory (FC) [1–5]. The FC is one of the branches of mathematical analysis which interested with studying non integer order of integrals and derivatives. In last few decades, it has played a significant role in different applications, for instance fluid mechanics, entropy, physics, economic, engineering and biological applications [6–9]. As a result, all modeling of problems in life depend on

F. El-Dossoki et al. (Eds.): ICGS 2023, EESL, pp. 3–20, 2024.
https://doi.org/10.1007/978-3-031-62672-2_1

the current and previous time that can be done by FC (for details, see [10–13]). Fractional partial differential equations (FPDEs) have become a powerful instrument to model scientific phenomena (more details, see [14–17]). The solutions of FODEs have an important role to describe the characteristics of non-linear models which appear in nature. Its complex to get an actual solutions for FODEs to describe non-linear phenomena, therefore there exist many various analytical and numerical techniques. Such techniques are: tanh method [18], variational iteration method (VIM) [19], finite difference method [20], Homotopy analysis method (HAM) [21–24], spectral collocation method [25], Adomian decomposition method(ADM) [26,27], diferential quadrature method [28] and RPS technique [29–32].

In recent years, orthonormal Chebyshev polynomials have a huge significance in numerical analysis, from the points of view both of theoretical and practical. These polynomials have been used widely for approximations due to their accuracy to solve many problems [33,34].

The target of this research study is to construct the approximate solution of different types of problems by using second-kind shifted Chebyshev collocation method with RPS technique.

The rest of the research is structured as follows: In Sect. 2, we review some mathematical definitions and theorems of FC and fractional power series (FPS). In Sect. 3, we give the basic features of SSKCOP. In Sect. 4, we provide our algorithm to solve various types of problems such as diffusion problem, Backward Kolmogorov equation, homogeneous and nonhomogeneous advection partial differential equation by using the second-kind shifted Chebyshev collocation with RPS technique. In Sect. 5, we represent numerical outcomes and simulations which shown the accuracy and the high efficiency for present method. Section 6 contains concluding remarks.

2 Fundamental Concepts and Theories

This section includes the definitions of Riemann-Liouville (RL), Caputo's fractional derivative (CFD) [2,35] and FPS-related theorems [36,37].

Definition 1. RL fractional integral I^{μ} of order μ for a function $\mathbb{V}(t) \in \mathbb{C}_{\kappa}$, $\kappa \geqslant -1$ is given by:

$$I^{\mu}\mathbb{V}(t) = \begin{cases} \frac{1}{\Gamma(\mu)} \displaystyle\int_0^t (t-s)^{\mu-1}\ \mathbb{V}(s)ds, & t>0,\ \mu>0, \\ \mathbb{V}(t), & \mu = 0. \end{cases} \tag{1}$$

Definition 2. The CFD of order μ is defined as:

$$D^{\mu}\mathbb{V}(t) = I^{n-\mu}D^{n}\mathbb{V}(t) = \frac{1}{\Gamma(n-\mu)}\int_0^t (t-s)^{n-\mu-1}\mathbb{V}^{(n)}(s)ds, \tag{2}$$

$t > 0,\ n-1 < \mu < n,\ n \in \mathbb{N}$, where D^n refers to n order classic differential operator.

Definition 3. Caputo fractional partial derivative (CFPD) of order μ for a function $\mathbb{V}(x,t) \in \mathbb{C}_\kappa,\ \kappa \geqslant -1$ is given as below:

$$\mathbb{D}_t^{\mu}\mathbb{V}(x,t) = \begin{cases} \frac{1}{\Gamma(n-\mu)} \displaystyle\int_0^t (t-s)^{n-\mu-1} \frac{\partial^n \mathbb{V}(x,s)}{\partial s^n} ds, & n-1<\mu<n, \\ \frac{\partial^n \mathbb{V}(x,t)}{\partial t^n}, & \mu = n \in \mathbb{N}. \end{cases} \tag{3}$$

Definition 4. A power series expansion of the formula

$$\sum_{n=0}^{\infty} A_n(t-t_0)^{n\mu} = A_0 + A_1(t-t_0)^{\mu} + A_2(t-t_0)^{2\mu} + ...,$$

where $0 \le n-1 < \mu \le n,\ n \in \mathbb{N}$ and $t \ge t_0$ is called FPS about t_0.

Theorem 1. *Assume that Y has FPS representation at $t = t_0$ of the form*

$$Y(t) = \sum_{n=0}^{\infty} A_n(t-t_0)^{n\mu} = A_0 + A_1(t-t_0)^{\mu} + A_2(t-t_0)^{2\mu} + ...,$$

where $0 \le n-1 < \mu \le n,\ n \in \mathbb{N}$ and $t_0 \le t \le t_0 + \mathbb{R}$, where $\mathbb{R}$ is a radius of convergence for multiple FPS. If $D^{n\mu}\ Y(t)$ are continuous on $(t_0, t_0+\mathbb{R})$, $n = 0, 1, 2, ...$, then

$$A_n = \frac{D^{n\mu}Y(t_0)}{\Gamma(n\mu+1)}, \quad n\mu > -1.$$

Definition 5. A power series expansion is defined as:

$$\sum_{n=0}^{\infty} Y_n\ (t-t_0)^{n\mu} = Y_0 + Y_1\ (t-t_0)^{\mu} + Y_2\ (t-t_0)^{2\mu} + ...,$$

is called multiple FPS about $t = t_0$.

Theorem 2. *Assume that $\mathbb{V}(x,t)$ has a multiple FPS representation at t_0 is given by:*

$$\mathbb{V}(x,t) = \sum_{n=0}^{\infty} Y_n(x)(t-t_0)^{n\mu},$$

where $x \in I,\ 0 \le n-1 < \mu \le n,\ n \in \mathbb{N}$ and $t_0 \le t \le t_0 + \mathbb{R}$.

If $\mathbb{D}_t^{n\mu}\mathbb{V}(x,t))$ are continuous on $I \times (t_0, t_0+\mathbb{R})$, $n = 0, 1, 2, ...$, then coefcients is given by:

$$Y_n(x) = \frac{\mathbb{D}_t^{n\mu}\ \mathbb{V}(x,t_0)}{\Gamma(n\mu+1)}.$$

3 Second-Kind Chebyshev Orthonormal Polynomials (SKCOP)

The SKCOP $\mathcal{U}_n(x)$ of degree n are defined on $[-1,1]$ as [38]:

$$\mathcal{U}_n(x) = \frac{sin(n+1)\alpha}{sin(\alpha)},$$

where $x = cos(\alpha),\ \alpha \in [0,\pi]$. The orthogonal property of $\mathcal{U}_n(x)$ on $[-1,1]$ is given below:

$$<\mathcal{U}_n(x), \mathcal{U}_m(x)> = \int_{-1}^{1} w(x)\mathcal{U}_n(x)\mathcal{U}_m(x)dx = \begin{cases} 0 & \text{if}\, n \neq m, \\ \frac{\pi}{2} & \text{if}\, n = m, \end{cases}$$

with respect to weight function $w(x) = \sqrt{1-x^2}$.

The polynomials $\mathcal{U}_n(x)$ may be created by utilizing recurrence relation as:

$$\mathcal{U}_n(x) = 2x\mathcal{U}_{n-1}(x) - \mathcal{U}_{n-2}(x),\ n = 2,3,4,...,$$

where

$$\mathcal{U}_0(x) = 1,\ \mathcal{U}_1(x) = 2x.$$

The SSKCOP on the interval $x \in [0,1]$ are given as:

$$\mathcal{U}^*_n(x) = \mathcal{U}_n(2x-1).$$

The orthogonality relation of SSKCOP according to weight function $w^*(x) = \sqrt{x-x^2}$ is:

$$<\mathcal{U}^*_n(x), \mathcal{U}^*_m(x)> = \int_{0}^{1} w^*(x)\ \mathcal{U}^*_n(x)\mathcal{U}^*_m(x)dx = \begin{cases} 0 & \text{if}\, n \neq m, \\ \frac{\pi}{8} & \text{if}\, n = m. \end{cases}$$

The recurrence relation of SSKCOP:

$$\mathcal{U}^*_n(x) = 2(2x-1)\mathcal{U}^*_{n-1}(x) - \mathcal{U}^*_{n-2}(x),\ n = 2,3,4,...,$$

where

$$\mathcal{U}^*_0(x) = 1,\ \mathcal{U}^*_1(x) = 4x-2.$$

Analytical formula of SSKCOP $\mathcal{U}^*_n(x)$ of degree n is expressed as:

$$\mathcal{U}^*_n(x) = \sum_{i=0}^{n} (-1)^i \frac{2^{2n-2i}\Gamma(2n-i+2)}{\Gamma(2n-2i+1)\Gamma(i+1)} x^{n-i},\ n > 0. \tag{4}$$

Let $\mathbb{V}(x,t)$ the expansion series of SSKCOP as:

$$\mathbb{V}(x,t) = \sum_{i=0}^{\infty} A_i(t)\mathcal{U}^*_i(x), \tag{5}$$

where the coefficients $A_i(t)$ are given by:

$$A_i(t) = \frac{8}{\pi} \int_0^1 \sqrt{x - x^2}\ \mathbb{V}(x,t) \text{Ų}_i^*(x) dx, i = 0, 1, 2, ... \tag{6}$$

In practice, we truncate the infinite series up to $(m+1)$ terms of SSKCOP as follows:

$$\mathbb{V}_m(x,t) = \sum_{i=0}^{m} A_i(t) \text{Ų}_i^*(x). \tag{7}$$

Theorem 3. *The μ^{th} fractional-order derivative of the function approximation $\mathbb{V}(x,t)$ which is defined in Eq. (7) is calculated as*

$$\mathbb{D}_x^{\mu} \mathbb{V}_m(x,t) = \sum_{i=\lceil \mu \rceil}^{m} \sum_{k=0}^{i-\lceil \mu \rceil} A_i(t)\ \varUpsilon_{i,k}^{(\mu)}\ x^{i-k-\mu}, \tag{8}$$

where $\lceil \mu \rceil$ the smallest integer number greater than or equal to μ and $\varUpsilon_{i,k}^{(\mu)}$ is defined as:

$$\varUpsilon_{i,k}^{(\mu)} = (-1)^k \frac{2^{2i-2k} \varGamma(2i-k+2)\ \varGamma(i-k+1)}{\varGamma(k+1)\varGamma(2i-2k+2)\varGamma(i-k-\mu+1)}, \quad \forall\ i = \lceil \mu \rceil, \lceil \mu \rceil + 1, ..., m.$$

4 Technique of Proposed Method

Here, we need to construct an analytic solution for diffusion model, Backward Kolmogorov model and advection partial differential equation.

4.1 Fractional Diffusion Model

$$\mathbb{D}_t^{\mu} \mathbb{V}(x,t) = g(x)\ \mathbb{D}_x^{2} \mathbb{V}(x,t) + H(x,t),\ 0 < \mu \leq 1,\ (x,t) \in [0,1] \times [0,1], \tag{9}$$

with initial conditions (ICs)

$$\mathbb{V}(x,0) = \lambda(x), \tag{10}$$

and boundary conditions (BCs)

$$\begin{cases} \mathbb{V}(0,t) = \varPsi_1(t), \\ \mathbb{V}(1,t) = \varPsi_2(t), \end{cases} \tag{11}$$

where $g(x)$ and $H(x,t)$ are called diffusion coefficient and source function, respectively.

► By substituting from Eqs. (7) and (8) into Eq. (9), we get

$$\sum_{i=0}^{m} \mathbb{D}_t^{\mu} A_i(t) Ꝗ_i^*(x) = g(x) \sum_{i=\lceil 2\rceil}^{m} \sum_{k=0}^{i-\lceil 2\rceil} A_i(t)\, \Upsilon_{i,k}^{(2)}\, x^{i-k-2} + H(x,t). \tag{12}$$

► We collocate Eq. (12) at the roots x_p, $p = 0, 1, 2, ..., m - \lceil\mu\rceil$ are collocation points of SSKCOP $Ꝗ_{m+1-\lceil\mu\rceil}^*(x)$, we obtain a system of FODEs as:

$$\sum_{i=0}^{m} \mathbb{D}_t^{\mu} A_i(t) Ꝗ_i^*(x_p) = g(x_p) \sum_{i=\lceil 2\rceil}^{m} \sum_{k=0}^{i-\lceil 2\rceil} A_i(t)\, \Upsilon_{i,k}^{(2)}\, x_p^{i-k-2} + H(x_p,t). \tag{13}$$

► Appling Eq. (7) into the ICs and BCs in Eqs. (10) and (11) leads a system of algebraic equations:

$$\sum_{i=0}^{m} A_i(0) Ꝗ_i^*(x_p) = \lambda(x_p),\ x_p,\ p = 0, 1, 2, ..., m - \lceil\mu\rceil,\ m-1 < \mu \le m, \tag{14}$$

$$\begin{cases} \sum_{i=0}^{m} A_i(t) Ꝗ_i^*(0) = \Psi_1(t), \\ \sum_{i=0}^{m} A_i(t) Ꝗ_i^*(1) = \Psi_2(t). \end{cases} \tag{15}$$

Hence, we get $(m+1)$ system of algebraic equations and FODEs in $A_i(t)$, $\forall\ i = 0, 1, 2, ..., m$. These system of FODEs solved by using RPS technique.

4.2 Backward Kolmogorov Model

$$\mathbb{D}_t^{\alpha}\mathbb{V}(x,t) = -B_1(x,t)\ \mathbb{D}_x\mathbb{V}(x,t) + B_2(x,t)\ \mathbb{D}_x^2\mathbb{V}(x,t),\ 0 < \mu \le 1,\ (x,t) \in [0,1]\times\mathbb{R}, \tag{16}$$

with ICs

$$\mathbb{V}(x,0) = F(x), \tag{17}$$

and BCs

$$\begin{cases} \mathbb{V}(0,t) = \Phi_1(t), \\ \mathbb{V}(1,t) = \Phi_2(t), \end{cases} \tag{18}$$

where $B_1(x,t)$ is diffusion coefficients and $B_2(x,t)$ is drift coefficients.

♦ By substituting from Eqs. (7) and (8) into Eq. (16), we obtain

$$\begin{aligned} \sum_{i=0}^{m} \mathbb{D}_t^{\mu} A_i(t) Ꝗ_i^*(x) = &-B_1(x,t) \sum_{i=\lceil 1\rceil}^{m} \sum_{k=0}^{i-\lceil 1\rceil} A_i(t)\, \Upsilon_{i,k}^{(1)}\, x^{i-k-1} \\ &+ B_2(x,t) \sum_{i=\lceil 2\rceil}^{m} \sum_{k=0}^{i-\lceil 2\rceil} A_i(t)\, \Upsilon_{i,k}^{(2)}\, x^{i-k-2}. \end{aligned} \tag{19}$$

♦ By collocating Eq. (19) at the roots x_p, $p = 0, 1, 2, ..., m - \lceil\mu\rceil$ are collocation points of SSKCOP $\mathcal{U}^*_{m+1-\lceil\mu\rceil}(x)$, we get system of FODEs as:

$$\sum_{i=0}^{m} \mathbb{D}_t^{\mu} A_i(t)\mathcal{U}_i^*(x_p) = -B_1(x_p,t) \sum_{i=\lceil 1\rceil}^{m} \sum_{k=0}^{i-\lceil 1\rceil} A_i(t)\ \Upsilon_{i,k}^{(1)}\ x_p^{i-k-1} + B_2(x_p,t) \sum_{i=\lceil 2\rceil}^{m} \sum_{k=0}^{i-\lceil 2\rceil} A_i(t)\ \Upsilon_{i,k}^{(2)}\ x_p^{i-k-2}. \quad (20)$$

♦ By appling Eq. (7) into the ICs and BCs in Eqs. (17) and (18) leads a system of algebraic equations:

$$\sum_{i=0}^{m} A_i(0)\mathcal{U}_i^*(x_p) = F(x_p),\ x_p,\ p = 0, 1, 2, ..., m - \lceil\mu\rceil,\ m-1 < \mu \leq m, \quad (21)$$

$$\begin{cases} \sum_{i=0}^{m} A_i(t)\mathcal{U}_i^*(0) = \Phi_1(t), \\ \sum_{i=0}^{m} A_i(t)\mathcal{U}_i^*(1) = \Phi_2(t). \end{cases} \quad (22)$$

Therefore, we obtain $(m+1)$ system of algebraic equations and FODEs in $A_i(t)$, $\forall\ i = 0, 1, 2, ..., m$. These system of FODEs solved by using RPS technique.

4.3 Advection Partial Differential Equations (PDEs)

$$\mathbb{D}_t^{\alpha}\mathbb{V}(x,t) = f\Big(\mathbb{V}(x,t), \mathbb{D}_x\mathbb{V}(x,t)\Big) + K(x,t),\ 0 < \mu \leq 1,\ (x,t) \in [0,1]\times\mathbb{R}, \quad (23)$$

with ICs

$$\mathbb{V}(x,0) = Z(x), \quad (24)$$

and BCs

$$\begin{cases} \mathbb{V}(0,t) = \Theta_1(t), \\ \mathbb{V}(1,t) = \Theta_2(t), \end{cases} \quad (25)$$

where f is a nonlinear function and $K(x,t)$ is the source function.

• By substituting from Eqs. (7) and (8) into Eq. (23), we get

$$\sum_{i=0}^{m} \mathbb{D}_t^{\mu} A_i(t)\mathcal{U}_i^*(x) = f\Big(\mathbb{V}_m(x,t), \mathbb{D}_x\mathbb{V}_m(x,t)\Big) + K(x,t). \quad (26)$$

• Now, we collocate Eq. (26) at the roots x_p, $p = 0, 1, 2, ..., m-\lceil\mu\rceil$ are collocation points of SSKCOP $\mathcal{U}^*_{m+1-\lceil\mu\rceil}(x)$, we obtain a system of FODEs as:

$$\sum_{i=0}^{m} \mathbb{D}_t^{\mu} A_i(t)\mathcal{U}_i^*(x_p) = f\Big(\mathbb{V}_m(x_p,t), \mathbb{D}_x\mathbb{V}_m(x,t)|x = x_p\Big) + K(x_p,t). \quad (27)$$

Table 1. Absolute errors at $m = 2$ and $t = 1$ at various values of μ for Problem 1.

x	$\mu = 1$			$\mu = 0.9$			$\mu = 0.7$		
	A	B	C	A	B	C	A	C	B
0	0	0	0	0	0	0	0	0	0
0.1	1.61516e−05	2.69e−03	5.16e−04	5.09416e−05	6.53e−03	9.89e−04	4.45161e−04	3.52e−02	3.43e−03
0.2	1.09251e−09	4.78e−03	2.06e−03	1.57668e−08	1.16e−02	3.96e−03	2.40266e−06	6.25e−02	1.37e−02
0.3	4.52416e−15	6.27e−03	4.65e−03	3.85025e−13	1.52e−02	8.90e−03	1.67122e−09	8.20e−02	3.09e−02
0.4	0	7.16e−03	8.26e−03	0	1.74e−02	1.58e−03	3.24851e−13	9.38e−02	5.49e−02
0.5	0	7.46e−03	1.29e−02	0	1.81−02	2.47e−02	0	9.77e−02	8.58e−02
0.6	1.11022e−16	7.16e−03	1.86e−02	2.22045e−16	1.74e−02	3.56e−02	0	9.38e−02	1.24e−01
0.7	2.22045e−16	6.27e−03	2.53e−02	0	1.52e−02	4.85e−02	2.22045e−16	8.20e−02	1.68e−01
0.8	0	4.78e−03	3.30e−02	0	1.16e−02	6.33e−02	0	6.25e−02	2.20e−01
0.9	0	2.69e−03	4.18e−02	4.44089e−16	6.53e−03	8.01e−02	4.44089e−16	3.52e−02	2.78e−01
1	0	0	5.16e−02	0	0	9.89e−02	0	0	3.43e−01

[A] present method
[B] Chebyshev collocation method and RPSM
[C] HAM

- By appling Eq. (7) into the ICs and BCs in Eqs. (24) and (25) leads a system of algebraic equations:

$$\sum_{i=0}^{m} A_i(0)\text{Ụ}_i^*(x_p) = Z(x_p),\ x_p,\ p = 0, 1, 2, ..., m - \lceil\mu\rceil,\ m - 1 < \mu \leq m, \quad (28)$$

$$\begin{cases} \sum_{i=0}^{m} A_i(t)\text{Ụ}_i^*(0) = \Theta_1(t), \\ \sum_{i=0}^{m} A_i(t)\text{Ụ}_i^*(1) = \Theta_2(t). \end{cases} \quad (29)$$

Then, we get $(m + 1)$ system of algebraic equations and FODEs in $A_i(t)$, $\forall\ i = 0, 1, 2, ..., m$. These system of FODEs solved by using RPS technique.

5 Numerical Simulation and Discussions

Problem 1. Consider the time-fractional diffusion problem [39]:

$$\mathbb{D}_t^{\mu}\mathbb{V}(x, t) = \frac{x^2}{2}\ \mathbb{D}_x^2\mathbb{V}(x, t),\ 0 < \mu \leq 1,\ 0 \leq x, t \leq 1, \quad (30)$$

with ICs

$$\mathbb{V}(x, 0) = x^2, \quad (31)$$

and BCs

$$\begin{cases} \mathbb{V}(0, t) = 0, \\ \mathbb{V}(1, t) = E_{\mu}(t^{\mu}), \end{cases} \quad (32)$$

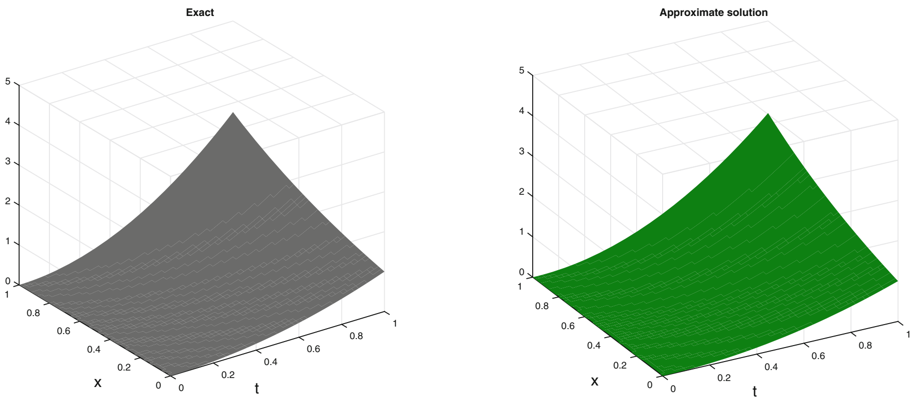

Fig. 1. Exact and approximate solution figures at $\mu = 1$ for Problem 1.

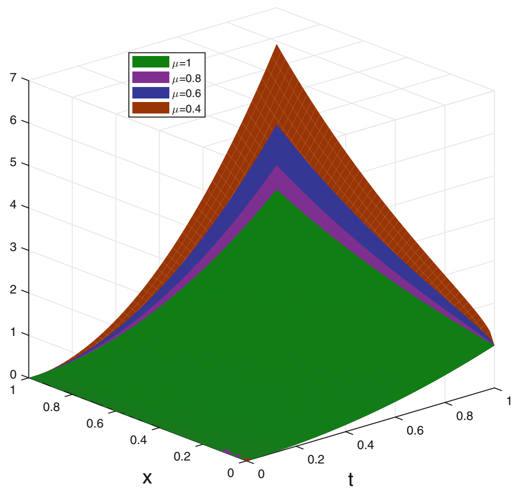

Fig. 2. The different fractional order graph of approximate solution for Problem 1.

which has an exact solution at $\mu = 1$ is $\mathbb{V}(x,t) = x^2 E_\mu(t^\mu)$, where $E_\mu(t^\mu)$ the Mittag-Leffler function.

In Table 1, we have been compared the outcomes obtained by present method with Chebyshev collocation method with RPSM and HAM [39]. Figure 1 expressed the comparison of an actual and computational results of present method for Problem 1. Figure 2 displays the three dimensions graphs for approximate solution of the present method when $\mu = \{1, 0.8, 0.6, 0.4\}$ for Problem 1.

Problem 2. Consider the Backward Kolmogorov equation [40,41]:

$$\mathbb{D}_t^\mu \mathbb{V}(x,t) = (x+1)\ \mathbb{D}_x \mathbb{V}(x,t) + x^2\ e^t\ \mathbb{D}_x^2 \mathbb{V}(x,t),\ 0 < \mu \le 1,\ (x,t) \in [0,1] \times \mathbb{R}, \tag{33}$$

with ICs

$$\mathbb{V}(x,0) = x + 1, \tag{34}$$

and BCs

$$\begin{cases} \mathbb{V}(0,t) = e^{t}, \\ \mathbb{V}(1,t) = 2e^{t}, \end{cases} \tag{35}$$

which has an exact solution at $\mu = 1$ is $\mathbb{V}(x,t) = (x+1)\ e^{t}$.

In Tables 2 and 3, we have been compared the outcomes obtained by the present method with Q-HATM [40] and new iterative method (NIM) [41]. In Table 4, we have been presented the approximate solution at different value of μ. Figure 3 expressed the comparison of an actual and computational results of present method for Problem 2. Figure 4 shows the three dimensions graphs for approximate solution of the present method when $\mu = \{1, 0.8, 0.6, 0.4\}$ for Problem 2.

Problem 3. Consider the nonlinear time-fractional advection PDE [44]

$$\mathbb{D}_t^{\mu}\mathbb{V}(x,t) + \mathbb{V}(x,t)\ \mathbb{D}_x\mathbb{V}(x,t) = x + xt^2,\ t > 0,\ 0 \le x \le 1, \quad 0 < \mu \le 1, \tag{36}$$

with ICs

$$\mathbb{V}(x,0) = 0, \tag{37}$$

Table 2. Numerical outcomes at various t and x for $\mu = 1$ for Problem 2.

t	x	Exact	Approximate solutions		Absolute error	
			present method	Q-HATM	present method	Q-HATM
0.1	0	1.105171	1.105171	1.105166	0	4.251e−6
0.1	0.2	1.326205	1.326205	1.326200	3.99680e−15	5.102e−6
0.1	0.4	1.547239	1.547239	1.547233	8.88178e−16	5.952e−6
0.1	0.6	1.768273	1.768273	1.547239	8.88178e−16	6.802e−6
0.1	0.8	1.989308	1.989308	1.989300	6.66134e−16	7.652e−6
0.1	1	2.210342	2.210342	2.210333	4.44089e−16	8.503e−6
0.2	0	1.221403	1.221400	1.221333	2.22045e−16	6.9425e−5
0.2	0.2	1.465683	1.465683	1.465600	1.84275e−12	8.3310e−5
0.2	0.4	1.709964	1.709964	1.709866	4.44089e−16	9.7194e−5
0.2	0.6	1.954244	1.954244	1.954133	4.44089e−16	1.11080e−4
0.2	0.8	2.198525	2.198525	2.198400	0	1.24964e−4
0.2	1	2.442806	2.442806	2.442666	4.44089e−16	1.38849e−4

Table 3. Numerical outcomes at $x = 0.1$ for $\mu = 1$ for Problem 2.

t	Exact	Approximate solutions		Absolute error	
		present method	NIM	present method	NIM
0.2	1.343543	1.343543	1.343466	2.08722e−14	7e−5
0.3	1.484845	1.484845	1.484450	1.20459e−12	3e−4
0.4	1.641007	1.641007	1.639733	2.15881e−11	1e−3
0.5	1.813593	1.813593	1.810416	2.02952e−10	3e−3
0.6	2.004331	2.004331	1.997600	1.26860e−09	6e−3
0.7	2.215128	2.215128	2.202383	5.98331e−09	1e−2
0.8	2.448095	2.448095	2.425866	2.29638e−08	2e−2
0.9	2.705563	2.705563	2.669150	7.52985e−08	3e−2

Table 4. Numerical solutions at $x = 0.1$ for $\mu = 0.99$ and $\mu = 0.85$ for Problem 2.

t	$\mu = 0.99$		$\mu = 0.85$	
	present method	NIM	present method	NIM
0.2	1.347334	1.349248	1.411917	1.447511
0.3	1.490063	1.492376	1.577614	1.492376
0.4	1.647637	1.649728	1.757935	1.649728
0.5	1.821670	1.822452	1.955363	1.822452
0.6	2.013924	2.011662	2.172205	2.011662
0.7	2.226332	2.218455	2.410817	2.218455
0.8	2.461028	2.4439194	2.673696	2.443919
0.9	2.720365	2.705563	2.963541	2.689133

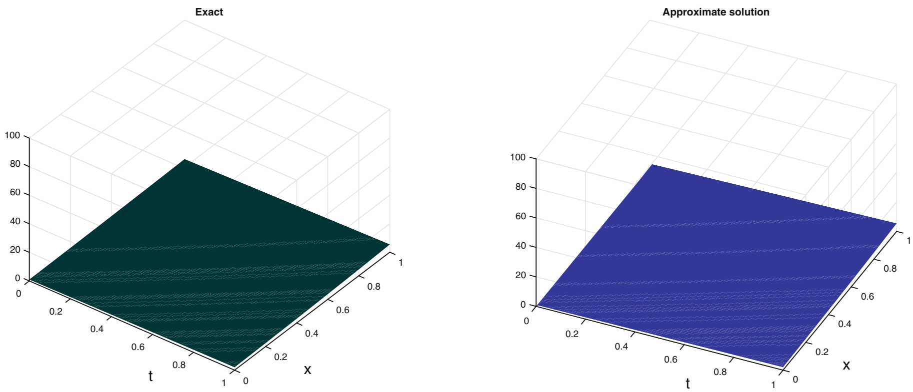

Fig. 3. Exact and approximate solution figures at $\mu = 1$ for Problem 2.

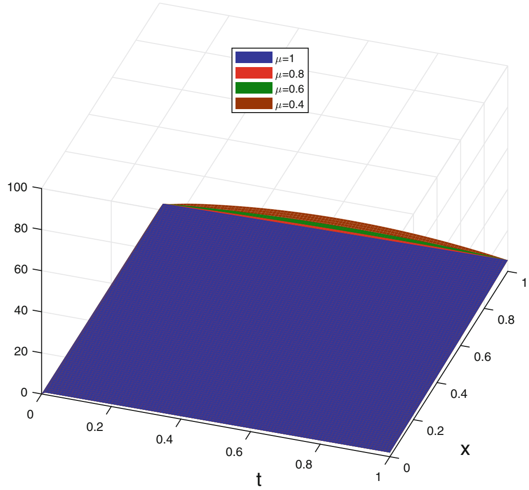

Fig. 4. The different fractional order graph of approximate solution for Problem 2.

Table 5. Numerical outcomes of present method with other existing techniques for Problem 3.

x	t	Exact	present method	VIM [42]	ADM [42]	HPM [43]	VHPIM [43]	PIA [44]
0.2	0.25	0.050000	0.050000	0.0503090	0.0500000	0.0499876	0.0499876	0.0500001
0.2	0.5	0.100000	0.100000	0.1006190	0.1000000	0.0999780	0.0999746	0.1000002
0.2	0.75	0.150000	0.150000	0.1509280	0.1500010	0.1499680	0.1499620	0.1500004
0.2	1	0.200000	0.200000	0.2012370	0.2000010	0.1999570	0.1999510	0.2000005
0.4	0.25	0.100000	0.100000	0.1018940	0.1000230	0.0995290	0.0996450	0.1000158
0.4	0.5	0.200000	0.200000	0.2037870	0.2000460	0.1990590	0.1992900	0.2000316
0.4	0.75	0.300000	0.300000	0.3056810	0.3000690	0.2985880	0.2989350	0.3000475
0.4	1	0.400000	0.400000	0.4075750	0.4000920	0.3981180	0.3985800	0.4000633
0.6	0.25	0.150000	0.150000	0.1530940	0.1504110	0.1471580	0.1456900	0.1502739
0.6	0.5	0.300000	0.300000	0.3061880	0.3008230	0.2943170	0.2913800	0.3005478
0.6	0.75	0.450000	0.450000	0.4592820	0.4512340	0.4414750	0.4370700	0.4508218
0.6	1	0.600000	0.600000	0.6123760	0.6016460	0.5886340	0.5827590	0.6010957

and the BCs

$$\begin{cases} \mathbb{V}(0,t) = 0, \\ \mathbb{V}(1,t) = t, \end{cases} \tag{38}$$

which has an exact solution at $\mu = 1$ is $\mathbb{V}(x,t) = xt$.

In Table 5, we have been compared the outcomes obtained by the present method with VIM, ADM [42], variational homotopy perturbation iteration method (VHPIM), HPM [43] and Perturbation Iteration Algorithm (PIA) [44]. Figure 5 expressed the comparison of an actual and computational results of present method for Problem 3. Figure 6 represents the three dimensions graphs

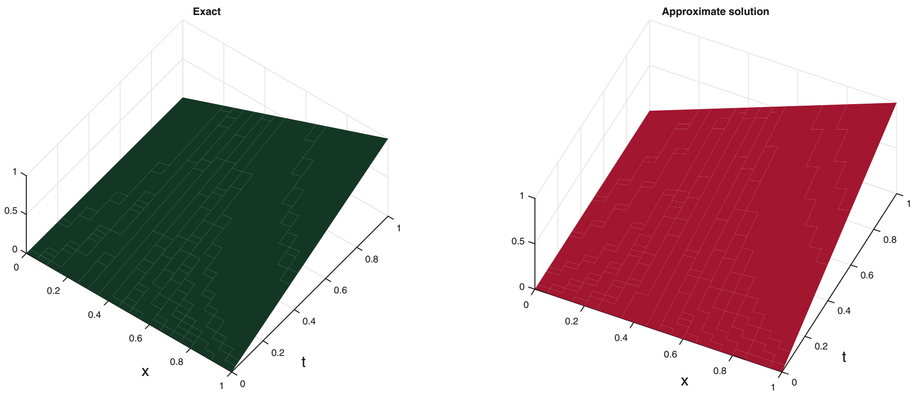

Fig. 5. Exact and approximate solution figures at $\mu = 1$ for Problem 3.

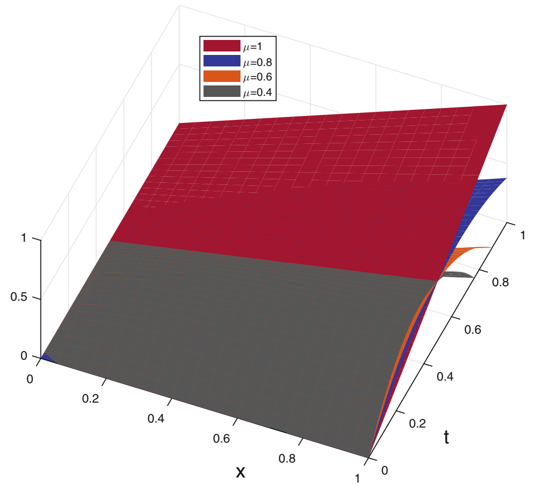

Fig. 6. The different fractional order graph of approximate solution for Problem 3.

for approximate solution of the present method when $\mu = \{1, 0.8, 0.6, 0.4\}$ for Problem 3.

Problem 4. Consider the nonlinear time-fractional homogeneous advection PDE [45]

$$\mathbb{D}_t^{\mu}\mathbb{V}(x,t) + \mathbb{V}(x,t)\ \mathbb{D}_x\mathbb{V}(x,t) = 0,\ t > 0,\ 0 \leq x \leq 1, \quad 0 < \mu \leq 1, \tag{39}$$

with ICs

$$\mathbb{V}(x,0) = -x, \tag{40}$$

Table 6. Numerical solutions of present method with LWM at $\mu = 1$ for Problem 4.

x	t	Exact solution	Approximate solution		Absolute error	
			present method	LWM [45]	present method	LWM [45]
0.25	0.2	−0.312500	−0.312500	−0.312	2.56000e−09	7e−04
0.25	0.4	−0.416667	−0.416653	−0.415	1.39810e−05	1e−03
0.25	0.6	−0.625000	−0.622279	−0.607	2.72098e−03	1.8e−02
0.25	0.8	−1.250000	−1.078201	−0.987	1.71799e−01	2.63e−1
0.5	0.2	−0.625000	−0.625000	−0.624	3.41333e−09	1e−03
0.5	0.4	−0.833333	−0.833315	−0.831	1.86414e−05	2e−03
0.5	0.6	−1.250000	−1.246372	−1.215	3.62797e−03	3.5e−02
0.5	0.8	−2.500000	−2.270935	−1.975	2.29065e−01	5.25e−01
0.75	0.2	−0.937500	−0.937500	−0.937	2.56000e−09	2e−04
0.75	0.4	−1.250000	−1.249986	−1.247	1.39810e−05	3e−03
0.75	0.6	−1.875000	−1.872279	−1.822	2.72098e−03	5.3e−2
0.75	0.8	−3.750000	−3.578201	−2.963	1.71799e−01	7.87e−1

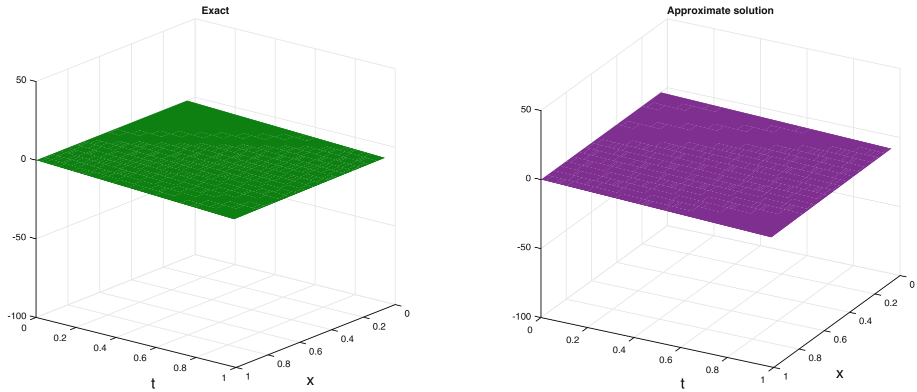

Fig. 7. Exact and approximate solution figures at $\mu = 1$ for Problem 4.

and the BCs

$$\begin{cases} \mathbb{V}(0,t) = 0, \\ \mathbb{V}(1,t) = \dfrac{1}{t-1}, \end{cases} \tag{41}$$

which has an exact solution at $\mu = 1$ is $\mathbb{V}(x,t) = \dfrac{x}{t-1}$.

In Table 6, we have been compared the outcomes obtained by present method with Legendre wavelets method (LWM) [45] with exact solution. Figure 7 expressed the comparison of an actual and computational results of present method for Problem 4. Figure 8 shows the three dimensions graphs for approximate solution of the present method when $\mu = \{1, 0.8, 0.6, 0.4\}$ for Problem 4.

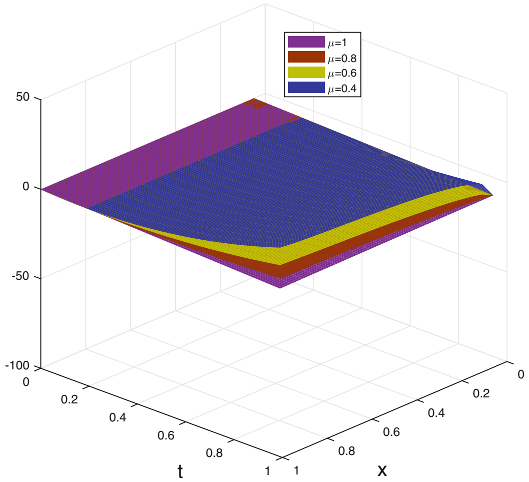

Fig. 8. The different fractional order graph of approximate solution for Problem 4.

6 Concluding Remarks

In this article, we determined second-kind shifted Chebyshev collocation method with RPS technique for solving different models such as fractional diffusion model, Backward Kolmogorov model, homogeneous and nonhomogeneous advection partial differential equation. By using shifted Chebyshev collocation method, these models are turned to a system of FODEs. Hence, we obtain system of algebraic equations with the help ICs and BCs. Resulting, we get system contains FODEs and algebraic equations. This system solved by utilization RPS technique. Tables and graphs illustrations the high accuracy and efficiency of the present method. All numerical results introduced in this paper were obtained by MATLAB software package.

7 Recommendation

According to the abovementioned numerical and results simulations, it is recommended to employ the present method for solving complex non-linear models that appear in engineering and mathematical physics due to its accurate and efficience.

Acknowledgement. The authors would like to express their sincere thanks to the reviewers for their valuable suggestions which have been improved the manuscript.

References

1. Oldham, K., Spanier, J.: The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order. Elsevier (1974). https://doi.org/10.1016/s0076-5392(09)x6012-1
2. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, vol. 204. Elsevier (2006).https://doi.org/10.1016/s0304-0208(06)x8001-5
3. Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity. World Scientific (2010). https://doi.org/10.1142/p614
4. Almeida, R., Tavares, D., Torres, D.F.: The Variable-Order Fractional Calculus of Variations. Springer, Cham (2019). https://doi.org/10.1007/978-3-319-94006-9
5. Miller, K.S., Ross, B.: An introduction to the fractional calculus and fractional differential equations (1993). http://ci.nii.ac.jp/ncid/BA20696637
6. Magin, R.L., Ingo, C., Colon-Perez, L., Triplett, W., Mareci, T.H.: Characterization of anomalous diffusion in porous biological tissues using fractional order derivatives and entropy. Microporous Mesoporous Mater. **178**, 39–43 (2013). https://doi.org/10.1016/j.micromeso.2013.02.054
7. Yang, X.J.: Fractional derivatives of constant and variable orders applied to anomalous relaxation models in heat transfer problems. Therm. Sci. **21**, 1161–1171 (2016). https://doi.org/10.2298/tsci161216326y
8. Yang, X.J., Machado, J.T., Cattani, C., Gao, F.: On a fractal LC-electric circuit modeled by local fractional calculus. Commun. Nonlinear Sci. Numer. Simul. **47**, 200–206 (2017). https://doi.org/10.1016/j.cnsns.2016.11.017
9. Lopes, A.M., Machado, J.T., Pinto, C.M., Galhano, A.M.: Fractional dynamics and MDS visualization of earthquake phenomena. Comput. Math. Appl. **66**, 647–658 (2013). https://doi.org/10.1016/j.camwa.2013.02.003
10. Baleanu, D., Golmankhaneh, A.K., Golmankhaneh, A.K., Baleanu, M.C.: Fractional electromagnetic equations using fractional forms. Int. J. Theor. Phys. **48**, 3114–3123 (2009). https://doi.org/10.1007/s10773-009-0109-8
11. Al-Smadi, M.H., Gumah, G.N.: On the homotopy analysis method for fractional SEIR epidemic model. Res. J. Appl. Sci. Eng. Technol. **7**, 3809–3820 (2014). https://doi.org/10.19026/rjaset.7.738 https://doi.org/10.19026/rjaset.7.738
12. Kumar, S.: A new fractional modeling arising in engineering sciences and its analytical approximate solution. Alex. Eng. J. **50**, 813–819 (2013). https://doi.org/10.1016/j.aej.2013.09.005
13. Al-Smadi, M., Freihat, A., Arqub, O.A., Shawagfeh, N.: A novel multistep generalized differential transform method for solving fractional-order Lu Chaotic and hyperchaotic systems. J. Comput. Anal. Appl. **19**, 713 (2015)
14. Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J.: Fractional Calculus: Models and Numerical Methods, 3. World Scientific (2012)
15. Zaslavsky, G.M.: Chaos, fractional kinetics, and anomalous transport. Phys. Rep. **371**, 461–580 (2002). https://doi.org/10.1016/s0370-1573(02)00331-9
16. Hirota, R.: Exact envelope-soliton solutions of a nonlinear wave equation. J. Math. Phys. **14**, 805–809 (1973). https://doi.org/10.1063/1.1666399
17. Kumar, S., Kumar, D., Singh, J.: Numerical computation of fractional Black-Scholes equation arising in financial market. Egypt. J. Basic Appl. Sci. **1**, 177–183 (2014). https://doi.org/10.1016/j.ejbas.2014.10.003
18. Malfliet, W.: The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations. J. Comput. Appl. Math. **164**, 529–541 (2004). https://doi.org/10.1016/s0377-0427(03)00645-9

19. Das, S.: Analytical solution of a fractional diffusion equation by variational iteration method. Comput. Math. Appl. **57**, 483–487 (2009). https://doi.org/10.1016/j.camwa.2008.09.045
20. Albadarneh, R.B., Batiha, I.M., Zurigat, M.: Numerical solutions for linear fractional differential equations of order $1<\alpha<2$ Using Finite Difference Method (FFDM). Int. J. Math. Comput. Sci. **16**, 103–111 (2016). https://doi.org/10.22436/jmcs.016.01.11
21. Arafa, A.A.M., Rida, S.Z., Mohamed, H.: Approximate analytical solutions of Schnakenberg systems by homotopy analysis method. Appl. Math. Model. **36**, 4789–4796 (2012). https://doi.org/10.1016/j.apm.2011.12.014
22. Arafa, A.A.M., Rida, S.Z., Mohamed, H.: Homotopy analysis method for solving biological population model. Commun. Theor. Phys. **56**, 797–800 (2011). https://doi.org/10.1088/0253-6102/56/5/01
23. Dehghan, M., Manafian, J., Saadatmandi, A.: Solving nonlinear fractional partial differential equations using the homotopy analysis method. Numer. Methods Partial Differ. Equ. **26**, 448–479 (2010). https://doi.org/10.1002/num.20460
24. Ganjiani, M.: Solution of nonlinear fractional differential equations using homotopy analysis method. Appl. Math. Model. **34**, 1634–1641 (2010). https://doi.org/10.1016/j.apm.2009.09.011
25. Javidi, M.: A numerical solution of the generalized Burger's-Huxley equation by spectral collocation method. Appl. Math. Comput. **178**, 338–344 (2006). https://doi.org/10.1016/j.amc.2005.11.051
26. Wang, Q.: Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method. Appl. Math. Comput. **182**, 1048–1055 (2006). https://doi.org/10.1016/j.amc.2006.05.004
27. Hu, Y., Luo, Y., Lu, Z.: Analytical solution of the linear fractional differential equation by Adomian decomposition method. J. Comput. Appl. Math. **215**, 220–229 (2008). https://doi.org/10.1016/j.cam.2007.04.005
28. Sari, M., Gürarslan, G.: Numerical solutions of the generalized Burgers-Huxley equation by a differential quadrature method. Math. Probl. Eng. (2009). https://doi.org/10.1155/2009/370765
29. Arafa, A.A.M.: A new algorithm of residual power series (RPS) technique. Int. J. Appl. Comput. Math. **6**, 1–13 (2020). https://doi.org/10.1007/s40819-020-00812-7
30. Arafa, A., Elmahdy, G.: Application of residual power series method to fractional coupled physical equations arising in fluids flow. Int. J. Differ. Equ. **2018**, 1–10 (2018). https://doi.org/10.1155/2018/7692849
31. Jaradat, H.M., Al-Shara, S., Khan, Q.J., Alquran, M., Al-Khaled, K.: Analytical solution of time-fractional Drinfeld-Sokolov-Wilson system using residual power series method. Int. J. Appl. Math. **46**, 64–70 (2016). https://squ.pure.elsevier.com/en/publications/analytical-solution-of-time-fractional-drinfeld-sokolov-wilson-sy
32. Shqair, M., El-Ajou, A., Nairat, M.: Analytical solution for multi-energy groups of neutron diffusion equations by a residual power series method. Mathematics **7**, 633 (2019). https://doi.org/10.3390/math7070633
33. Sweilam, N.H., Nagy, A.M., El-Sayed, A.A.: On the numerical solution of space fractional order diffusion equation via shifted Chebyshev polynomials of the third kind. J. King Saud Univ. Sci. **28**, 41–47 (2016). https://doi.org/10.1016/j.jksus.2015.05.002
34. Mahdy, A., Mukhtar, N.A.H.: Second kind shifted Chebyshev polynomials for solving the model nonlinear ODEs. Am. J. Comput. Math. **7**, 391–401 (2017). https://doi.org/10.4236/ajcm.2017.74028