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Allaberen Ashyralyev
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Editors

Analysis and Applied Mathematics

Extended Abstracts of the 2022 Joint Seminar

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Preface

This book offers proceedings of the in-person and online "Analysis and Applied Mathematics" seminars organized jointly by the Bahcesehir University (Istanbul, Türkiye), Ghent Analysis and PDE Center (Ghent University, Ghent, Belgium), and the Institute of Mathematics and Mathematical Modeling (Almaty, Kazakhstan). This book of extended abstracts is part of our series Research Perspectives Ghent Analysis and PDE Center, devoted to the publication of abstracts, in an extended form, of talks presented at the events associated with the Ghent Analysis and PDE Center. We hope that this volume will be of value to professional mathematicians as well as advanced students in the fields of analysis and applied mathematics, providing an overview of some research topics in the wide area of analysis and their relevance to applied mathematics.

The goal of the joint seminar "Analysis and Applied Mathematics" is to provide a forum for researchers and scientists from different regions to communicate their recent developments and to present their original results in various fields of analysis and applied mathematics. The seminar originated in 2022, after the pandemic, mostly in the online format, to bring together mathematicians working in different institutions for discussions of joint topics of interest, fuelled by the work of the international community on these subjects. The website of the seminar can be found at https://sites.google.com/view/aam-seminars.

Many of the lectures given at the seminar have been recorded and are available on the YouTube Channel of the Institute of Mathematics of the University of Georgia. This includes many papers included in this volume, as well as other talks given at the seminar but do not appear here. The volume contains extended abstracts of these and a few related talks which were given at the seminar during the 2022–2023 period.

This book presents 23 papers by authors from different countries: Turkey, Kazakhstan, USA, Italy, Portugal, Spain, Serbia, Azerbaijan, Jordan, Lithuania, India, Iraq, Russian Federation, Uzbekistan, Tajikistan, and Turkmenistan. We are especially pleased with the fact that many articles are written by co-authors who work at different universities in the world. We are confident that such international integration provides an opportunity for a significant increase in the quality and quantity of scientific publications.

Publications in this book contain new results or overviews of some relevant mathematical areas. The volume reflects the latest developments in the area of analysis and applied mathematics and their interdisciplinary applications. This volume is organised in four parts. Part I contains the contributed papers focusing on various aspects of the analysis and its applications. Part II is devoted to the research on the theory of applied mathematics. Part III contains the results of studies on ordinary and partial differential equations and their applications. Finally, Part IV is focused on the simulation of problems arising in real-world applications of applied sciences.

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Istanbul, Turkey
Ghent, Belgium
Almaty, Kazakhstan
January 2024

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Part I
Analysis

Chapter 1
Some Measures of Noncompactness and Their Applications

Eberhard Malkowsky

Abstract This is the extended abstract of the author's talk in the *Analysis and Applied Mathematics Weekly Online Seminar* on important results on measures of noncompactness, and some recent applications on the characterisations of compact operators between certain BK spaces, and in fixed point theorems.

1.1 Introduction

Measures of noncompactness are very useful tools in functional analysis, for instance, in metric fixed point theory, the characterisations of compact operators between Banach spaces, and the study of differential and integral equations.

We present an axiomatic introduction to measures of noncompactness on the class of bounded subsets of complete metric spaces, the definition and most important properties of the Kuratowski and Hausdorff measures of noncompactness, a study of measures of noncompactness of operators between Banach spaces, and some applications to the characterisations of compact linear operators between certain BK spaces and the solvability of an infinite system of integral equations.

Compactness and *measures of noncompactness* play an important role in fixed point theory. There are, however, cases when the operators are not compact and the results have to be extended to noncompact operators. Perhaps the most important application of a measure of noncompactness is *Darbo's fixed point theorem* [4], which uses *Kuratowski's measure of noncompactness* α [8]. Darbo's theorem is a generalisation of Schauder's fixed point theorem [17].

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1.2 Measures of Noncompactness

Measures of noncompactness are studied in detail and their use is discussed, for instance, in the monographs [1, 2, 9, 10, 18].

First, we recall the *axiomatic introduction* of the concept of a *measure of noncompactness* in complete metric spaces.

Definition 1.2.1 Let (X, d) be a complete metric space, and $\mathcal{M}_X$ be the class of bounded subsets of X. A set function $\phi : \mathcal{M}_X \to [0, \infty)$ that satisfies the following conditions for all $Q, Q_1, Q_2 \in \mathcal{M}_X$

(MNC.1) $\phi(Q) = 0$ if and only if Q is relatively compact (*regularity*)

(MNC.2) $\phi(Q) = \phi(\overline{Q})$ (*invariance under closure*)

(MNC.3) $\phi(Q_1 \cup Q_2) = \max\{\phi(Q_1), \phi(Q_2)\}$ (*semi–additivity*)

is called a *measure of noncompactness on* $\mathcal{M}_X$ and $\phi(Q)$ is called the *measure of noncompactness of the set* Q.

Proposition 1.2.2 *Let (X, d) be a complete metric space. Any measure of noncompactness ϕ on $\mathcal{M}_X$ satisfies the following conditions for all $Q, Q_1, Q_2 \in \mathcal{M}_X$*

$$Q_1 \subset Q_2 \textit{ implies } \phi(Q_1) \le \phi(Q_2) \quad \text{(monotonicity)} \tag{1.1}$$

$$\phi(Q_1 \cap Q_2) \le \min\{\phi(Q_1), \phi(Q_2)\} \tag{1.2}$$

$$\phi(Q) = 0 \textit{ for every finite set } Q \quad \text{(non–singularity)}. \tag{1.3}$$

$$\left\{\begin{array}{l} \textit{If } (Q_n) \textit{ is a decreasing sequence of nonempty, closed sets in } \mathcal{M}_X \textit{ and} \\ \lim_{n\to\infty} \phi(Q_n) = 0, \ \textit{then} \\ \qquad\qquad Q_\infty = \bigcap_{n=1}^{\infty} Q_n \neq \emptyset \textit{ is compact} \\ \text{(Cantor's generalised intersection property [18, p. 19]);} \\ \text{([8, 1930] for } \phi = \alpha.) \end{array}\right\} \tag{1.4}$$

Now we recall the definitions of the *Kuratowski* and *Hausdorff measures of noncompactness* in complete metric spaces (X, d).

Definition 1.2.3

(a) ([8] or [18, Definition II.2.1]) The *Kuratowski measure of noncompactness* is the map $\alpha : \mathcal{M}_X \to [0, \infty)$ with

$$\alpha(Q) = \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{k=1}^{n} S_k, \ S_k \subset X, \right.$$
$$\left. \operatorname{diam}(S_k) < \varepsilon \ (k = 1, 2, \ldots, n \in \mathbb{N})\right\}.$$

(b) ([21] or [18, Definition II.2.1]) The *Hausdorff* or *ball measure of noncompactness* is the map $\chi : \mathcal{M}_X \to [0, \infty)$ with

$$\chi(Q) = \inf \left\{ \varepsilon > 0 : Q \subset \bigcup_{k=1}^{n} B_{r_k}(x_k),\ x_k \in X, \right.$$
$$\left. r_k < \varepsilon \ (k = 1, 2, \ldots, n \in \mathbb{N}) \right\},$$

where $B_{r_k}(x_k)$, as usual, denotes the open ball of radius r_k and centre in x_k.

Remark 1.2.4 We note that the functions α and χ are measures of noncompactness in the sense of Definition 1.2.1. So they satisfy (1.1)–(1.4) ([9, Lemmas 2.6, 2.11, Theorem 2.7] and [18, Remark 3.2]). They are also equivalent ([18, Remark 3.2]), that is, $\chi(Q) \leq \alpha(Q) \leq 2 \cdot \chi(Q)$ for all $Q \in \mathcal{M}_X$. Studies on inequivalent measures of noncompactness can be found, for instance, in [12, 13].

Some measures of noncompactness such as α and χ satisfy several important conditions that are connected to the linear structure of Banach spaces; the statements for α in (1.5)–(1.8) of Proposition 1.2.5 are due to Darbo [4].

Proposition 1.2.5 ([10, Theorems 7.6.7, 7.7.6 (b)]) *Let X be a Banach space, $Q, Q_1, Q_2 \in \mathcal{M}_X$, ψ be any of the functions α or χ, and* co(Q) *denote the convex hull of Q. Then we have*

$$\psi(Q_1 + Q_2) \leq \psi(Q_1) + \psi(Q_2) \qquad \text{(sublinearity)}, \tag{1.5}$$

$$\psi(Q + x) = \psi(Q) \text{ for each } x \in X \qquad \text{(translation invariance)}, \tag{1.6}$$

$$\psi(\lambda Q) = |\lambda| \psi(Q) \text{ for each scalar } \lambda \qquad \text{(absolute homogeneity)} \tag{1.7}$$

$$\psi(Q) = \psi(co(Q)) \qquad \text{(invariance under passage to the convex hull)}. \tag{1.8}$$

If X is infinite dimensional, and B_X and S_X denote the open unit ball and the unit sphere in X, then $\alpha(B_X) = \alpha(S_X) = 2$ and $\chi(B_X) = \chi(S_X) = 1$ ([9, Theorems 2.9, 2.14]).

As an application of the results concerning measures of noncompactness we are going to state the famous theorem by *Goldenštein, Go'hberg and Markus*, which establishes an estimate for the Hausdorff measure of compactness of bounded sets in any Banach space with a Schauder basis.

Theorem 1.2.6 (Goldenštein, Go'hberg, Markus) (R-BIB.GGM1 *or* [18, Theorem II.4.2] *or* [9, Theorem 2.23])
Let X be a Banach space with a Schauder basis (b_k). Then the function $\mu : \mathcal{M}_X \to [0, \infty)$ defined by

$$\mu(Q) = \limsup_{n \to \infty} \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \text{ with } \mathcal{R}_n(x) = \sum_{k=n+1}^{\infty} \lambda_k b_k \tag{1.9}$$

for all $x = \sum_{k=0}^{\infty} \lambda_k b_k \in X$ *satisfies the following inequality for every* $Q \in \mathcal{M}_X$

$$\frac{1}{a} \cdot \mu(Q) \leq \chi(Q) \leq \mu(Q),$$

where $a = \limsup_{n\to\infty} \|\mathcal{R}_n\|$ *is the basis constant.*

So far, we measured the noncompactness of bounded subsets of complete metric spaces and Banach spaces. Now we introduce the concept of measures of noncompactness of operators between Banach spaces.

Definition 1.2.7 ([9, Definition 2.24]) Let ϕ_1 and ϕ_2 be measures of noncompactness on the Banach spaces X and Y, respectively. An operator $T : X \to Y$ is said to be (ϕ_1, ϕ_2)*–bounded* if $T(Q) \in \mathcal{M}_Y$ for each $Q \in \mathcal{M}_X$, and there exists a real number $k > 0$ such that $\phi_2(T(Q)) \leq k\phi_1(Q)$ for each $Q \in \mathcal{M}_X$.
If an operator T is (ϕ_1, ϕ_2)–bounded, then $\|T\|_{\phi_1,\phi_2}$ defined by

$$\|T\|_{\phi_1,\phi_2} = \inf\{k \geq 0 : \phi_2(T(Q)) \leq k\phi_1(Q) \text{ for each } Q \in \mathcal{M}_X\}$$

is called (ϕ_1, ϕ_2)*–operator norm of* T, or (ϕ_1, ϕ_2)*–measure of noncompactness of* T, or simply *measure of noncompactness of* T.
If $\phi_1 = \phi_2 = \phi$, then we write $\|T\|_\phi$ instead of $\|T\|_{\phi,\phi}$.

Theorem 1.2.8 *Let* X *and* Y *be Banach spaces,* $L \in \mathcal{B}(X, Y)$*,* S_X *and* $\overline{B}_X$ *be the unit sphere and the closed unit ball in* X*.*

(a) ([9, Theorem 2.25]) *Then we have* $\|L\|_\chi = \chi(L(S_X)) = \chi(L(\overline{B}_X))$*.*
(b) ([9, Corollary 2.26]) *Let* $\mathcal{C}(X, Y)$ *be the set of all compact operators in* $\mathcal{B}(X, Y)$*. Then* $\|\cdot\|_\chi$ *is a seminorm on* $\mathcal{B}(X, Y)$*,*

$$\|L\|_\chi = 0 \text{ if and only if } L \in \mathcal{C}(X, Y), \tag{1.10}$$

and $\|L\|_\chi \leq \|L\|$.

Important applications of the theory of measures of noncompactness are Darbo's fixed point theorem and its generalisation, the *Darbo–Sadovskiĭ theorem*. The important hypotheses are the condensing property (1.11), the invariance of the passage to the convex hull (1.8) of the measures of noncompactness involved, and Cantor's generalised intersection property (1.4).

Theorem 1.2.9 (Darbo's Fixed Point Theorem) ([4]) *Let* C *be a non–empty bounded, closed and convex subset of a Banach space* X *and* α *be the Kuratowski measure of noncompactness on* X*. If* $f : C \to C$ *is continuous such that there exists a constant* $c \in [0, 1)$ *with*

$$\alpha(f(Q)) \leq c \cdot \alpha(Q) \text{ for every } Q \subset C, \tag{1.11}$$

then f *has a fixed point in* C*.*

Theorem 1.2.10 (Darbo–Sadovskiĭ) (R-BIB.Sad2, [18, Theorem 5.4, p. 40] or [10, Theorem 7.10.3])
Let X be a Banach space, ϕ be a measure of noncompactness which is invariant under passage to the convex hull, $C \neq \emptyset$ be a bounded, closed and convex subset of X and $f : C \to C$ be an operator that satisfies the condensing property (1.11), with ϕ in place of α. Then f has a fixed point in C.

1.3 Some Applications

Here we apply the results of Sect. 1.2 to the characterisations of some classes of bounded linear and compact operators on the *generalised Hahn space* h_d, and give a generalisation of Darbo's fixed point theorem and its application to the solution of an integral equation. We recommend [2] and [3] for further comprehensive studies of applications of measures of noncompactness to the solvability of infinite systems of differential and integral equations.

We use the standard notations ω, ℓ_∞ and c_0 for the sets of all complex, bounded and null sequences $x = (x_k)_{k=1}^\infty$; bs and $bv = \{x \in \omega : \sum_{k=1}^\infty |x_k - x_{k+1}| < \infty\}$, for the sets of all bounded series, and of all series of bounded variation. We also write $bv_0 = bv \cap c_0$. If $m \in \mathbb{N}$ and $x = (x_k)_{k=1}^\infty \in \omega$, then we write $x^{[m]} = (x_k^{[m]})_{k=1}^\infty$ for the *m–section of* x, where $x_k^{[m]} = x_k$ for $1 \leq k \leq m$ and $x_k^{[m]} = 0$ for $k > m$.

We refer the reader to [10, Definitions 9.2.1 and 9.2.12] for the concepts and fundamental properties of BK and AK spaces.

Let $d = (d_k)_{k=1}^\infty$ be a given monotone increasing unbounded sequence of positive real numbers. For every sequence $x = (x_k)_{k=1}^\infty \in \omega$, let $\Delta x = (\Delta x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty$ be the sequence of the forward differences of the sequence x. The *generalised Hahn space* is defined as [6]

$$h_d = \left\{ x = (x_k)_{k=1}^\infty \in \omega : \sum_{k=1}^\infty d_k |\Delta x_k| < \infty \right\} \cap c_0.$$

If $d_k = k$ for all k, then $h_d = h$, the original Hahn space h [7, 1922], and if $d = e = (1, 1, \dots)$, then $h_e = bv_0$.

Since h_d is a BK space with AK by Malkowsky et al. [11, Proposition 2.1], every $L \in \mathcal{B}(h_d) = \mathcal{B}(h_d, h_d)$ is given by an infinte matrix $A = (a_{nk})_{n,k=1}^\infty$ such that $L(x) = Ax = (A_n(x))_{n=1}^\infty$ for all sequences $x = (x_k)_{k=1}^\infty$, where $A_n x = \sum_{k=1}^\infty a_{nk} x_k$ for all $n \in \mathbb{N}$, and conversely, if $Ax \in h_d$ for all $x \in h_d$, then $L_A \in \mathcal{B}(h_d)$, where $L_A x = Ax$ for all $x \in h_d$ ([10, Theorem 9.3.3]).

First, we need to characterise the class $\mathcal{B}(h_d)$ and deternmine the operator norm of $L \in \mathcal{B}(h_d)$.

Theorem 1.3.1 ([11, Theorem 3.9 and Corollary 3.15 (a)]) *We have $L \in \mathcal{B}(h_d)$ if and only if $Ax = L(x) \in h_d$ for all $x \in h_d$ and this is the case if and only if*

$$\lim_{n\to\infty} a_{nk} = 0, \text{ for all } k, \tag{1.12}$$

and

$$\|A\|_{(h_d,h_d)} = \sup_m \left(\frac{1}{d_m} \sum_{n=1}^{\infty} d_n \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) \right| \right) < \infty. \tag{1.13}$$

If $L \in \mathcal{B}(h_d)$, then

$$\|L\| = \|A\|_{(h_d,h_d)}. \tag{1.14}$$

Proof (Outline) The proof uses the concept of *determining sets for BK spaces* ([19, Definition 7.4.2]), [11, Propositions 3.2 and 2.3] and [19, Theorem 8.3.4].

(i) First we note that, by Malkowsky et al. [11, Proposition 3.2], $E = \left\{(1/d_k) \cdot e^{[k]} : k \in \mathbb{N}\right\}$ is a determining set for h_d. Also, by Malkowsky et al. [11, Proposition 2.3], the continuous dual h_d^* of h_d is normisomorphic to $bs_d = \{a \in \omega : \sup_n (1/d_n) |\sum_{k=1}^n a_k| < \infty\}$ with the natural norm $\|a\|_{bs_d} = \sup_n (1/d_n) |\sum_{k=1}^n a_k|$ for all $a \in bs_d$.

(ii) Writing $y^{[m]} = (1/d_m) \cdot e^{[m]}$ for all $m \in \mathbb{N}$, we show $\sup_m \|Ay^{[m]}\|_{bs} < \infty$ and $Ay^{[m]} \in c_0$ for all $y^{[m]} \in E$. We note that the first condition is (1.12) and the seond condition is equivalent to (1.13). Hence we have obtained Condition (ii) in [19, Theorem 8.3.4]. Also condition (i) in [19, Theorem 8.3.4] is redundant, since the columns $A^k = (a_{nk})_{n=1}^{\infty}$ of A are in c_0 for each k by (1.12), and

$$\|A^k\|_{h_d} \le d_k \|Ay^{[k]}\|_{h_d} + d_{k-1} \|Ay^{[k-1]}\|_{h_d} < \infty$$

for all k. Thus we obtain the characterization of $\mathcal{B}(h_d)$.

(iii) We obtain $\|L(x)\|_{h_d} \le \|A\|_{(h_d,h_d)} \|x\|_{h_d}$ for all $x \in h_d$, so $\|L\| \le \|A\|_{(h_d,h_d)}$. Conversely $\|L(y^{[m]})\|_{h_d} \le \|L\|$ for all m yields $\|A\|_{(h_d,h_d)} \le \|L\|$. This yields (1.14).

□

An application of Theorem 1.3.1 yields the multiplier $M(h_d, h_d)$, and the value a of the basis constant for h_d. We recall that the multiplier of $X \subset \omega$ in $Y \subset \omega$ is the set

$$M(X, Y) = \left\{ z \in \omega : z \cdot x = (z_k x_k)_{k=1}^{\infty} \in Y \text{ for all } x = (x_k)_{k=1}^{\infty} \in X \right\}.$$

We also obtain the value of the basis constant a of h_d.

Example 1.3.2 ([11, Remark 4.6])

(a) It follows from Theorem 1.3.3 that

$$M(h_d, h_d) = \left\{ z \in \omega : \left(\frac{1}{d_m} \cdot \left\| z^{[m-1]} \right\|_{h_d} \right)_{m=1}^{\infty} \in \ell_\infty \right\}.$$

(b) Let $l \in \mathbb{N}$ be given, $(c_m^{(l)})_{m=1}^{\infty}$ be the sequence with $c_m^{(l)} = 0$ for $1 \leq m \leq l$ and $c_m^{(l)} = 1 + d_l/d_m$ for $m \geq l+1$, then

$$a = \limsup_{l \to \infty} \|\mathcal{R}_l\| = \limsup_{l \to \infty} \left(\sup_{m \geq l} c_m^{(l)} \right) = \limsup_{l \to \infty} \left(\sup_{m \geq l} \left(1 + \frac{d_l}{d_m} \right) \right) = 2.$$

Now we use Theorem 1.3.1 to establish an estimate for $\|L\|_\chi$ for every $L \in \mathcal{B}(h_d)$.

Theorem 1.3.3

(a) ([11, Theorem 4.8 (a)]) *Let* $L \in \mathcal{B}(h_d)$. *We write*

$$\gamma_m^{<l>} = \frac{1}{d_m} \left(d_l \left| \sum_{k=1}^{m} a_{l+1,k} \right| + \sum_{n=l+1}^{\infty} d_n \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) \right| \right)$$

for all m and l. Then we have

$$\frac{1}{2} \cdot \limsup_{l \to \infty} \left(\sup_m \gamma_m^{<l>} \right) \leq \|L\|_\chi \leq \limsup_{l \to \infty} \left(\sup_m \gamma_m^{<l>} \right). \tag{1.15}$$

(b) ([11, Theorem 4.10 (d)]) *We have* $L \in C(h_d) = C(h_d, h_d)$ *if and only if*

$$\lim_{l \to \infty} \left(\sup_m \gamma_m^{<l>} \right) = 0.$$

Proof (Outline) Let A be an infinite matrix with the rows A_n ($n \in \mathbb{N}$). For each $m \in \mathbb{N}$, we write $A^{<m>}$ for the matrix with the rows $A_n^{<m>} = 0$ for $n \leq m$ and $A_n^{<m>} = A_n$ for $n \geq m+1$. Also let $L^{<m>}$ denote the operator represented by $A^{<m>}$. Obviously $L^{<m>} = \mathcal{R}_m \circ L$ ($m \in \mathbb{N}$) for $L \in \mathcal{B}(h_d)$. First we have by Theorem 1.3.1 for all l

$$\|L^{<l>}\| = \sup_m \left(\frac{1}{d_m} \sum_{n=1}^{\infty} d_n \left| \sum_{k=1}^{m} (a_{nk}^{<l>} - a_{n+1,k}^{<l+1>}) \right| \right) = \sup_m \gamma_m^{<l>}.$$

Since $a = 2$ by Example 1.3.2 (b), (1.9) yields the inequalities in (1.15). Finally, Part (b) follows from (1.15) and (1.10). □

We apply Theorem 1.3.3 and Example 1.3.2 (a) to obtain two results by *Sawano* and *El–Shabrawy* [16, Corollary 5.1 and Lemma 5.1].

Rhaly [14] introduced the generalised Cesàro operator C_t on ω for $t \in [0, 1)$ by the matrix $C_t = (a_{nk}(t))_{n,k=0}^{\infty}$ with $a_{nk} = t^{n-k}/(n+1)$ for $(0 \le k \le n)$ and $a_{nk} = 0$ for $k > n$ $(n = 0, 1, \dots)$.

Example 1.3.4 ([16, Corollary 5.1]) Let $0 \le t < 1$. Then $L_{C_t} \in \mathcal{B}(h)$.

The special case of $d_k = k$ for all k of the next example yields [16, Lemma 5.1].

Example 1.3.5 ([5, Example 10]) Let $(\lambda_k)_{k=1}^{\infty}$ be a decreasing sequence of positive real numbers which converges to 0 and $D(\lambda) = diag(\lambda_1, \lambda_2, \dots)$ denote the diagonal matrix with the sequence λ on its diagonal. Then $L_{D(\lambda)} \in C(h_d)$.

We also give an application of our results to Fredholm operators. We recall the definition of Fredholm operators ([10, Definition 8.4.1]). Let X and Y be Banach spaces, $L \in \mathcal{B}(X, Y)$, and $N(L)$ and $R(L)$ denote the null space and the range of L, respectively. Then L is said to be a *Fredholm operator*, if $R(L)$ is closed, and both dimensions $\dim N(T)$ and $\dim X/R(L)$ are finite. The *index* of a Fredholm operator L is defined as $i(L) = \dim N(L) - \dim X/R(L)$. Let us recall that if $L \in \mathcal{B}(X)$ and $\|L\|_{\chi} < 1$, then $I - L$ is a Fredholm operator and $i(I - L) = 0$ ([20] or [10, Section 7.13]).

Corollary 1.3.6 ([11, Corollary 4.11]) *Let $\alpha = (\alpha_n)_{n=1}^{\infty}$, $\beta = (\beta_n)_{n=1}^{\infty}$ and $\gamma = (\gamma_n)_{n=1}^{\infty}$ be given sequences of complex numbers, and $A(\alpha, \beta, \gamma)$ denote the tridiagonal matrix with α on the main diagonal, γ on the subdiagonal and β on the diagonal above the main diagonal.*
Then the operator $L \in \mathcal{B}(h_d)$ represented by the matrix $A(\gamma, \alpha, \beta) = A(0, \alpha, 0) + A(\gamma, 0, 0) + A(0, 0, \beta)$ is Fredholm with index $i(A(\alpha, \beta, \gamma)) = 0$ if $A(0, \alpha, 0)$ is Fredholm with index $i(A(0, \alpha, 0)) = 0$ and $A(\gamma, 0, 0)$ and $A(0, 0, \beta)$ are compact.

Example 1.3.7 ([11, Example 4.12]) Let $d_k = k$, $\alpha_k = 1 - 1/k$ and $\beta_k = \gamma_k = 1/k$ for all k. Then the operator $L \in \mathcal{B}(h_d)$ represented by the matrix $A(\gamma, \alpha, \beta)$ is Fredholm.

Finally, we consider a generalisation of Darbos's fixed point theorem, Theorem 1.2.9, and its application to the existence of solutions of a functional integral equation of Volterra type [15, Theorem 3.1]. We need the following definition.

Definition 1.3.8 ([15, Definition 2.1]) Let X be a Banach space and ϕ be a measure of noncompactness on $\mathcal{M}_X$ which is invariant under the passage to the convex hull (1.8), and homogeneous (1.7). Furthermore, let $H : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing map such that, for each sequence (a_n) of positive real numbers, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} H(a_n) = 0$. A map $T : X \to X$ is said to be a *countable H–set contraction* if there exists a $\tau > 0$ such that, for all countable $Q \in \mathcal{M}_X$, $\phi(T(Q)) > 0$ implies $\tau + H(\phi(T(Q))) \le H(\phi(Q))$.

The next result generalises Darbo's fixed point theorem.

Theorem 1.3.9 ([15, Theorem 2.8]) *Let C be a non–empty, bounded, closed and convex subset of a Banach space X, ϕ be a measure of noncompactness (as above) and $T : X \to X$ be a continuous H–contraction. Then T has a fixed point.*

An application of Theorem 1.3.9 yields a result on the solvability of the nonlinear integral equation

$$x(t) = f(t, x(t)) + \int_0^t g(t, s, x(s))\, ds \ \ (t \in \mathbb{R}^+) \tag{1.16}$$

in the space $\mathcal{BC}(\mathbb{R}^+)$ which consists of all real functions defined continuous and bounded on $\mathbb{R}^+$; the norm on $\mathcal{BC}(\mathbb{R}^+)$ is defined by $\|x\| = \sup_{t\in\mathbb{R}^+}\{|x(t)|\}$.

Theorem 1.3.10 ([15, Theorem 3.1]) *We consider the following conditions:*

(i) *The function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous, but, for any nonempty bounded subset X of $\mathcal{BC}(\mathbb{R}^+)$, the family $\{f(t, x) : x \in X\}$ is equi–continuous for all $t \in \mathbb{R}^+$, and the function $t \mapsto f(t, 0)$ is a member of the space $\mathcal{BC}(\mathbb{R}^+)$. Moreover, there exists $\tau > 0$ such that*

$$|f(t, x)-f(t, y)| \neq 0 \text{ implies } \tau+H(|f(t, x)-f(t, y)|) \leq H(|x-y|).$$

(ii) *The function $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist continuous functions $a, b : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $|g(t, s, x)| \leq a(t)b(s)$ for all $t, s \in \mathbb{R}^+$ with $s \leq t$ and $x \in \mathbb{R}$, where $\lim_{t\to\infty} a(t) \int_0^t b(s)\, ds = 0$.*

(iii) *There exists a positive solution r_0 of the inequality $H^{-1}(H(r_0) - \tau) + q \leq r_0$, where q is the constant defined by $q = \sup_{t\geq 0} \left\{|f(t, 0)| + a(t) \int_0^t b(s)\, ds\right\}$.*

Let (i), (ii) *and* (iii) *be satisfied. Then the nonlinear integral equation (1.16) has at least one solution in the space $\mathcal{BC}(\mathbb{R}^+)$.*

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Chapter 2
Definition of Hessians for $m-$Convex Functions as Borel Measures

Azimbay Sadullaev

Abstract In this work, $m-$convex functions are defined in the class of bounded upper semi-continuous functions of real arguments and a connection is established between $m-$convex and well-known violent $m-$subharmonic functions. As a consequence, we define in the class of $m-$convex functions, the Hessians H^k, $k = 1, 2, \dots, n - m + 1$, as Borel measures.

2.1 Introduction

$m-$convex functions in $\mathbb{R}^n$ are a real analogue of violent $m-$subharmonic (sh_m) functions in complex space $\mathbb{C}^n$. Let us recall the definition of a class of sh_m-functions, which has become the subject of research by many authors (Blocki [1], Dinev and Kolodziej [2], Li [3], Lu [4, 5], Abdullaev and Sadullaev [6, 7], etc.).

A twice smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is called violent subharmonic $u \in sh_m(D)$, if at each point of the domain D

$$sh_m(D) = \left\{u \in C^2 : \ (dd^c u)^k \wedge \beta^{n-k} \geq 0, \ \ k = 1, 2, \dots, n - m + 1\right\}$$
$$= \left\{u \in C^2 : \ dd^c u \wedge \beta^{n-1} \geq 0, (dd^c u)^2 \wedge \beta^{n-2} \geq 0, \dots, \ (dd^c u)^{n-m+1} \wedge \beta^{m-1} \geq 0\right\}, \tag{2.1}$$

where $\beta = dd^c \, \|z\|^2$ —is the standard volume form in $\mathbb{C}^n$.

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Operators $(dd^c u)^k \wedge \beta^{n-k}$ are closely related to Hessians. For a doubly smooth function $u \in C^2(D)$, the second order differential

$$dd^c u = \frac{i}{2} \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$$

(at a fixed point $o \in D$) is Hermitian quadratic form. After a suitable unitary coordinate transformation, it is reduced to diagonal form

$$dd^c u = \frac{i}{2} \left[\lambda_1 dz_1 \wedge d\bar{z}_1 + \ldots + \lambda_n dz_n \wedge d\bar{z}_n\right],$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hermitian matrix $(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k})$, which are real: $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. Note that the unitary transformation does not change the differential form $\beta = dd^c \|z\|^2$. Therefore, it is easy to see that

$$(dd^c u)^k \wedge \beta^{n-k} = k!(n-k)! H_o^k(u) \beta^n,$$

where $H_o^k(u) = \sum_{1 \le j_1 < \ldots < j_k \le n} \lambda_{j_1} \ldots \lambda_{j_k}$ is the Hessian of the vector $\lambda = \lambda(u) \in \mathbb{R}^n$ of dimension k.

Consequently, a doubly smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is violent $m-$subharmonic if at each point $o \in D$ we have

$$H^k(u) = H_o^k(u) \ge 0, \quad k = 1, 2, \ldots, n-m+1. \tag{2.2}$$

Note that the concept of a violent m-subharmonic function in the generalized sense is determined in the general case.

Definition 2.1 A function $u \in L^1_{loc}(D)$ is called sh_m in the domain $D \subset \mathbb{C}^n$, if it is upper semi-continuous, $u(z) \ge \overline{\lim_{w \to z}} u(w)$ $\forall z \in D$ and for any doubly smooth sh_m functions $v_1, \ldots, v_{n-m} \in C^2(D) \bigcap sh_m(D)$ the following

$$dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1},$$

defined as

$$\begin{aligned} &\left[dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1}\right](\omega) \\ &= \int u \, dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \qquad \omega \in F^{0,0} \end{aligned} \tag{2.3}$$

is positive.

Blocki in the work [1] proved that this definition is correct, that for $u \in C^2(D)$ functions this definition coincides with the original definition of sh_m-functions. Moreover, in the class of bounded sh_m-functions, the operators

$$(dd^c u)^k \wedge \beta^{n-k} \geq 0, \quad k = 1, 2, \ldots, n-m+1$$

are defined as Borel measures in the domain D (see [1, 6]).

2.2 $m-$ Convex Functions

Now let $D \subset \mathbb{R}^n$ and $u(x) \in C^2(D)$. Similar to (2.2), we want to define $m-$convex functions in the domain $D \subset \mathbb{R}^n$. The matrix $(\frac{\partial^2 u}{\partial x_j \partial x_k})$ is orthogonal, i.e., $\frac{\partial^2 u}{\partial x_j \partial x_k} = \frac{\partial^2 u}{\partial x_k \partial x_j}$. Therefore, after a suitable orthonormal transformation, it is transformed into a diagonal form,

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right) \rightarrow \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix},$$

where $\lambda_j = \lambda_j(x) \in \mathbb{R}$ are the eigenvalues of the matrix $(\frac{\partial^2 u}{\partial x_j \partial x_k})$. Let $H_k(u) = H_k(\lambda) = \sum_{1 \leq j_1 < \ldots < j_k \leq n} \lambda_{j\,1} \ldots \lambda_{j_k}$ be the Hessians of $k-$ dimensional of the eigenvalue vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Definition 2.2 A twice smooth function $u \in C^2(D)$ is called $m-$convex in $D \subset \mathbb{R}^n$, $u \in m-cv(D)$, if its eigenvalue vectors $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x))$ satisfy the conditions

$$m - cv \bigcap C^2(D) = \{H_k(u) = H_k(\lambda(x)) \geq 0, \ \forall x \in D, \ \ k = 1, \ldots, n-m+1\}$$

at each point $o \in D$.

Function theory of $m-cv$ is not studied much and is a new direction in the theory of real geometry. However, when $m = 1$ this class

$$1 - cv \bigcap C^2(D) = \{H_1(\lambda) \geq 0\} = \{\lambda_1 \geq 0, \ldots, \lambda_n \geq 0\}$$

coincides with the class of convex functions in $\mathbb{R}^n$, and when $m = n$ the class $n - cv \bigcap C^2(D) = \{\lambda_1 +, \ldots, \lambda_n \geq 0\}$ coincides with the class of subharmonic, (sh) functions. The class of convex functions has been well studied (Aleksandrov [8, 9], Bakelman [10, 11], Pogorelov [12], Artykbaev [20], etc.). For $m > 1$ this class was studied in a series of works by N. Ivochkina, N. Trudinger, X. Wang, S. Li, H. Lu et al. (see [3–5, 13–19].)

The principle difficulty in the theory of $m - cv$ functions is the introduction of class $m - cv \bigcap L^1_{loc}$, i.e. the definition of functions $m - cv(D)$ in the class of upper semi-continuous, locally integrable or bounded functions. So, for $m = n$ (the case of subharmonic functions) in the class of upper semi-continuous, locally integrable function $u(x) \in n - cv(D)$ is defined as a generalized function, and the Laplace operator Δu is a Borel measure.

2.3 Definition of Hessians for $m - cv$ Functions

In this work, we establish a connection between $m - cv$ functions and violent subharmonic (sh_m) functions and using the well-known and rich properties of sh_m functions we give the definitions of Hessians $H_k(u),\ \ k = 1, \dots, n - m + 1$ for $m-$convex functions, like Borel measures.
To do this, we embed $\mathbb{R}^n_x$ in $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y (z = x + iy)$, as a real $n-$dimensional subspace of the complex space $\mathbb{C}^n_z$.

Theorem 2.1 *A twice smooth function $u(x) \in C^2(D),\ \ D \subset \mathbb{R}^n_x$, is $m - cv$ in D if and only if the function $u^c(z) = u^c(x + iy) = u(x)$, that does not depend on variable $y \in \mathbb{R}^n_y$, is sh_m in the domain $D \times \mathbb{R}^n_y$.*

Proof Let us establish a connection between the Hessians $H_k(u)$ and $H^k(u^c)$. We have

$$\frac{\partial u^c}{\partial z_j} = \frac{1}{2}\left[\frac{\partial u^c}{\partial x_j} - \frac{\partial u^c}{\partial y_j}\right] = \frac{1}{2}\frac{\partial u^c}{\partial x_j},$$

$$\frac{\partial^2 u^c}{\partial z_j \bar{\partial}\, z_k} = \frac{1}{2}\frac{\partial}{\bar{\partial}\, z_k}\left[\frac{\partial u^c}{\partial x_j}\right] = \frac{1}{4}\left[\frac{\partial^2 u^c}{\partial x_k \partial x_j} + \frac{\partial^2 u^c}{\partial x_k \partial y_j}\right] = \frac{1}{4}\frac{\partial^2 u^c}{\partial x_k \partial x_j}.$$

Thus,

$$\frac{\partial^2 u^c}{\partial z_j \partial\, \bar{z}_k} = \frac{1}{4}\frac{\partial^2 u}{\partial x_j \partial\, x_k}$$

and, therefore, $H_k(u) = H^k(u^c)$ and $H^k(u) \geq 0,\ \ k = 1, \dots, n - m + 1$, if and only if $H^k(u^c) \geq 0,\ \ k = 1, \dots, n - m + 1$. □

Now, let $u(x)$ be an upper semi-continuous function in the domain $D \subset \mathbb{R}^n_x$. Then, $u^c(z)$ will also be an upper semi-continuous function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$.