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Uwe Nagel · Karim Adiprasito · Roberta Di Gennaro · Sara Faridi · Satoshi Murai *Editors*

Lefschetz Properties

Current and New Directions

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THE SERIES IS INDEXED IN SCOPUS

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Editors

Lefschetz Properties

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Foreword

The INdAM meeting “The Strong and Weak Lefschetz Properties” took place in Cortona from 12 to 16 September 2022 in the beautiful Villa Il Palazzone. The study of Lefschetz properties for Artinian algebras, initiated by Richard Stanley, began in the 1980s. Introduced as algebraic generalizations of the Hard Lefschetz property of the cohomology ring of a complex manifold, these properties soon proved to be present and interesting in many branches of mathematics, in particular in algebraic geometry, commutative algebra, and algebraic and combinatorial topology, and constitute a fertile meeting place for researchers with different backgrounds. Thirty-eight mathematicians from 14 countries and three continents participated in the Cortona meeting. Care was taken to invite expert researchers who have been working in this sector for years, as well as young post-docs and Ph.D. students. Furthermore, a significant percentage of female researchers were invited. Adopting a formula already tested in previous conferences (Banff 2016; Mittag Leffler 2017; Levico 2018; Luminy 2019; Oberwolfach 2020), an important part of the meeting was dedicated to group work on specific problems, proposed before the conference by some of the participants and selected by the organizers, with the aim of creating new collaborations, including interdisciplinary ones, and developing new techniques. The Cortona meeting was a success, not only because of the high level of the scientific talks, but also due to the climate of great friendliness and collaboration that was created among the participants, enthusiastic to meet again after the long and difficult interval due to the COVID-19 pandemic. This book is a testimony to that. The organizing and scientific committee was composed of Karim Adiprasito (Hebrew University of Jerusalem, Israel, and University of Copenhagen, Denmark), Roberta Di Gennaro (Università degli Studi di Napoli “Federico II”, Italy), Sara Faridi (Dalhousie University, Canada), Satoshi Murai (Waseda University, Japan), Uwe Nagel (University of Kentucky, USA), and myself. Many thanks to all of them for the nice collaboration and for taking care of this volume. The meeting was made

possible, thanks to the PRIN project "Moduli Theory and Birational Classification" of the MIUR and to the generous support of the Italian "Istituto Nazionale di Alta Matematica Francesco Severi"—INdAM—that provided not only funding but also logistical support. Many thanks for making this possible.

Trieste, Italy
December 2023

Emilia Mezzetti

Preface

This book grew out of a conference focusing on the Lefschetz properties that took place in Cortona, Italy, in September 2022. It was part of a series of meetings that was initiated by Junzo Watanabe who organized a first workshop on Lefschetz properties at Tokai International College, Hawaii, in 2012. Subsequent meetings took place at the University of Göttingen, Germany, 2015; Banff International Research Station, Canada, 2016; Mittag-Leffler Institute, Sweden, 2017; CIRM, Levico, Italy, 2018; CIRM, Luminy, France 2019; MFO, Oberwolfach, Germany, 2020.

The Lefschetz properties formalize the algebraic properties of the cohomology ring of a complex manifold as guaranteed by the Hard Lefschetz Theorem. Their investigation revealed connections to many other parts of mathematics. The present volume attempts to give an idea of the state of the art and the scope of the current research related to the Lefschetz properties. It consists of two survey articles, followed by nine research papers and a collection of open problems.

The first survey discusses the Jordan type of an Artinian algebra over a field, which gives more information than the strong Lefschetz property. The second survey describes connections to algebraic geometry in the form of hypersurfaces whose presence is unexpected based on an intuitive dimension count.

Some of the following research articles present new results related to topics discussed in the surveys. These concern, for example, the presence of the Lefschetz properties for certain Gorenstein algebras and for algebras related to graphs or flag complexes, results toward a characterization of sets of points whose general projection is a complete intersection as well as a method to detect the failure of the weak Lefschetz property. Two articles explore combinatorial aspects, a connection to nonnegative Toeplitz matrices via the Hodge-Riemann relations and an occurrence of the Kostka numbers, respectively. Another paper studies geometric properties of permutahedral varieties. This volume ends with an extensive collection of open problems covering many aspects of the investigations of the Lefschetz properties.

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Acknowledgements We are grateful to “Istituto Nazionale di Alta Matematica ‘Francesco Severi’—INdAM” and the PRIN project “Moduli Theory and Birational Classification” for their generous financial support.

Emilia Mezzetti deserves the lion’s share of the credit for the successful 2022 workshop in Cortona, which led to this volume.

Finally, we acknowledge the hard work and enthusiasm of the members of the Lefschetz community. While this volume is named after the Cortona conference, it showcases the ideas shared and developed over the past decade, during the many conferences this community has organized.

Contents

Jordan Type of an Artinian Algebra, a Survey

Nasrin Altafi, Anthony Iarrobino, and Pedro Macias Marques

Abstract We consider Artinian algebras A over a field k, both graded and local algebras. The Lefschetz properties of graded Artinian algebras have been long studied, but more recently the Jordan type invariant of a pair (ℓ, A) where ℓ is an element of the maximal ideal of A, has been introduced. The Jordan type gives the sizes of the Jordan blocks for multiplication by ℓ on A, and it is a finer invariant than the pair (ℓ, A) being strong or weak Lefschetz. The Jordan degree type for a graded Artinian algebra adds to the Jordan type the initial degree of "strings" in the decomposition of A as a $\mathsf{k}[\ell]$ module. We here give a brief survey of Jordan type for Artinian algebras, Jordan degree type for graded Artinian algebras, and related invariants for local Artinian algebras, with a focus on recent work and open problems.

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1 Introduction

Notation

Let A be a graded or local Artinian algebra quotient of $R = \mathsf{k}[x_1, \dots, x_r]$ (polynomial ring) or of $\mathcal{R} = \mathsf{k}\{x_1, \dots, x_r\}$ (regular local ring) with maximal ideal $\mathfrak{m}$ and highest socle degree j: that is $A_j \neq 0$, but $A_i = 0$ for $i > j$. Here, for A local we take A_i to be the i-th graded piece of the associated graded algebra $A^* = \bigoplus \mathfrak{m}^i/\mathfrak{m}^{i+1}$ of A. For A graded we let $\mathfrak{m} = \oplus_1^j A_i$. The Hilbert function of A is the sequence $H(A) = (h_0, h_1, \dots, h_j)$ where $h_i = \dim_{\mathsf{k}} A_i$; the *Sperner number* of $H(A)$ is the maximum value of $H(A)$. The *Jordan type* $P_{\ell,A}$ of a nilpotent element $\ell \in \mathfrak{m}$ of A is the partition P giving the sizes of the Jordan blocks of the (nilpotent) multiplication map m_ℓ. The properties of (ℓ, A) being strong-Lefschetz ($P = H(A)^\vee$, the conjugate of the Hilbert function viewed as a partition) or weak-Lefschetz (the number of parts of P is the Sperner number) of a pair (ℓ, A), have been investigated as such since at least 1978—see [34, 66, 73]. Earlier, J. Briançon in 1972 showed the strong Lefschetz property $P_{\ell,A} = H(A)^\vee$ in characteristic zero for each codimension two Artinian algebra A and a generic $\ell \in R_1$ [17]. But Jordan type is a finer concept: there are in general many partitions that can occur for $P_{\ell,A}$ given just the Hilbert function $H = H(A)$. A basic introductory paper is the second two authors' joint paper with McDaniel [48]; other resources include [16, 23, 47, 49]. Our attention in this note will be to the more general notion of Jordan type, as opposed to merely the Lefschetz properties.

Let H be a sequence that occurs as the Hilbert function of an Artinian quotient of R or $\mathcal{R}$. First, take R to be the polynomial ring. We denote by $\mathrm{G}(H)$ and $\mathrm{GGor}(H)$ the family of graded or graded Gorenstein, respectively, quotients of R having Hilbert function H. Now take $\mathcal{R}$ to be the regular local ring, and denote by $\mathrm{Z}(H)$ or $\mathrm{ZGor}(H)$, respectively, the family of all (not necessarily graded) quotients of $\mathcal{R}$ having Hilbert function H, or, respectively, the Gorenstein quotients of $\mathcal{R}$ having Hilbert function H. We regard these in this survey as subvarieties (not necessarily irreducible) of the Grassmanian $\mathrm{Grass}(R/\mathfrak{m}^n)$, $n = |H|$; but some have also looked at the scheme structures, namely the Hilbert scheme $\mathrm{Hilb}^n(R)$ (see, for example [15, 19, 39, 52, 58] and [40, Appendix C]). We will write R for both R and $\mathcal{R}$, when considering both at the same time. There is a natural notion of dominance of Jordan types (see Definition 1.5). Our goals in this survey are

(a) Review the definitions and properties of Jordan type and Jordan degree type.
(b) Report on progress on the several major questions below, and
(c) Suggest some further problems.

1.1 Major Problems

The development of the subject has been related to some main questions:

(i) How does Jordan type behave as one deforms the element $\ell \in \mathfrak{m}$, or the algebra $A = R/I$ among algebras of a given Hilbert function? Two cases: graded A, and local $A = \mathcal{R}/I$. In particular, does the Hilbert function determine a bound (in the sense of domination) on the possible Jordan types?

(ii) For graded A, there is a refinement of Jordan type to a Jordan degree type [48]. Determine its properties and avatars (Sects. 2.3 and 2.4 below).
There is a natural generalization of Jordan type to "contiguous Jordan type" for graded algebras having non-unimodal Hilbert function. There are similar questions of deformation (see [48, Sect. 2F, Definition 2.28ii], not treated here).

(iii) Generalizations and refinements of Jordan type from graded algebras to local algebras [43] (see Sect. 3 below).

(iv) When A is local Gorenstein, what is the relation of these refinements of Jordan type to the symmetric decomposition of A (see [42, 43])?

(v) Using Jordan type and other invariants to show that various families $\mathrm{Z}(H)$ or $\mathrm{ZGor}(H)$ have several irreducible components [41] (Sect. 3.1 below).

(vi) Given the Artinian algebra A, and a fixed partition P of $|A|$, what is the locus $\mathfrak{Z}_P \subset \mathbb{P}(A_1) \cong \mathbb{P}^{r-1}$ of linear forms ℓ for which $P_{\ell,A} = P$? The non-Lefschetz locus [16]?

(vii) What is the relation between Jordan type and the Betti minimal resolution of A [4, 5]?

(viii) What pairs of Jordan type partitions $P_{\ell,A}$ and $P_{\ell',A}$ may occur together in an Artinian A? OR, what Jordan types P_M, P_N may occur for a pair (M, N) of $n \times n$ commuting matrices (see [55]).

Some of these questions are now partially answered, ideas behind them have inspired other questions that remain open. We discuss (i)–(v) in more detail below, and then pose some specific questions.

1.2 What Is Jordan Type?

We first present the definitions and some properties of Jordan type, and then in Sect. 2.1 discuss the relationship to the weak and strong Lefschetz properties for graded algebras. Since the definition of Jordan type does not require grading, we start by stating it in the general setting, for a module over an algebra that may not be graded.

Definition 1.1 (*Jordan type*) (See also [48, Definition 2.1] and [34, Sect. 3.5].) Let M be a finitely generated module over the Artinian algebra A, and let $\ell \in \mathfrak{m}$. The Jordan type of ℓ in M is the partition of $\dim_{\mathsf{k}} M$, denoted $P_\ell = P_{\ell,M} = (p_1, \dots, p_s)$, where $p_1 \geq \cdots \geq p_s$, whose parts p_i are the block sizes in the Jordan canonical form matrix of the multiplication map $m_\ell : M \to M$, $x \mapsto \ell x$. The *generic Jordan type* of A, denoted P_A, is the Jordan type $P_{\ell,A}$ for a generic element ℓ of A_1 (when A is graded), or of $\mathfrak{m}_A$ (A local).

The Jordan block form for the similarity class of a matrix is sometimes called the Segre characteristic, in contrast to its conjugate, the Weyr characteristic (see Note 1.10 below).

Definition 1.2 (*Jordan basis, pre-Jordan basis*) With the notation of the previous definition, a *pre-Jordan basis* for ℓ is a basis of M as a vector space over k of the form

$$\mathcal{B} = \{\ell^i z_k \mid 1 \le k \le s,\ 0 \le i \le p_k - 1\}, \tag{1}$$

where $P_{\ell,M} = (p_1, \dots, p_s)$ is the Jordan type of ℓ. We call the subsets $S_k = \{z_k, \ell z_k, \dots, \ell^{p_k-1} z_k\}$ *strings* of the basis $\mathcal{B}$, and each element $\ell^i z_k$ a *bead* of the string. The Jordan blocks of the multiplication m_ℓ are determined by the strings S_k, and M is the direct sum

$$M = \langle S_1 \rangle \oplus \cdots \oplus \langle S_s \rangle. \tag{2}$$

If the elements $z_1, \dots, z_s \in M$ satisfy $\ell^{p_k} z_k = 0$ for each k, we call $\mathcal{B}$ a *Jordan basis* for ℓ, recovering the usual definition in linear algebra, since a matrix representing the multiplication by ℓ with respect to $\mathcal{B}$, ordering elements as $(\ell^{p_1-1} z_1, \dots, z_1, \ell^{p_2-1} z_2, \dots, z_2, \dots, \ell^{p_s-1} z_s, \dots, z_s)$, is a canonical Jordan form. In that case the $\langle S_k \rangle$ are cyclic $\mathsf{k}[\ell]$-submodules of M.

The following is well-known (see [11, Sect. 4.7], [75]).

Lemma 1.3 *If M has a pre-Jordan basis $\mathcal{B}$ as in (1), then for each k, we have*

$$\ell^{p_k} z_k \in \langle \ell^a z_i \mid a \ge p_k, i < k \rangle.$$

There is a Jordan basis of M derived from the pre-Jordan basis, and having the same partition invariant $P_{\ell,M}$ giving the lengths of strings.

Algorithm 1.4 *Often it is useful to consider a pre-Jordan basis (or a Jordan basis) to study the Jordan type of an element $\ell \in \mathfrak{m}$. However, to compute the Jordan type of an element in a module, we do not need to choose a basis. We can consider the sequence $(d_0, \dots, d_{j+1})$, where $d_i = \dim_{\mathsf{k}} M/\ell^i M$, and compute the sequence of differences $\Delta_\ell = (\delta_1, \dots, \delta_{j+1})$, where $\delta_i = d_i - d_{i-1}$. Then taking the conjugate partition of this sequence, we get the Jordan type of ℓ in M (see [48, Lemma 2.3]):*

$$P_{\ell,M} = \Delta_\ell^\vee.$$

This is the algorithm we used in Macaulay 2 computations [32] in this paper.

A key notion is specialization of Jordan types, which follows the dominance partial order on partitions (Lemma 1.6).

$$H = (1,2,3,2,1), \qquad P_H = \begin{array}{l} \bullet\bullet\bullet \\ \bullet\bullet \\ \bullet\bullet \\ \bullet \\ \bullet \end{array} \qquad H^\vee = (P_H)^\vee = (5,3,1) : \begin{array}{l} \bullet\bullet\bullet\bullet\bullet \\ \bullet\bullet\bullet \\ \bullet \end{array}$$

Fig. 1 Hilbert function, its partition (3, 2, 2, 1, 1), and conjugate (Example 1.8)

Definition 1.5 (*Dominance order*) Let $P = (p_1, \ldots, p_s)$, $p_1 \geq \cdots \geq p_s$, and $Q = (q_1, \ldots, q_r)$, $q_1 \geq \cdots \geq q_r$, be two partitions of $n = \sum p_i = \sum q_i$. We say that P dominates Q (written $P \geq Q$, if for each $k \in [1, \min\{s, r\}]$, we have

$$\sum_{i=1}^{k} p_i \geq \sum_{i=1}^{k} q_i.$$

For example, the partition $(5, 4, 2) \geq (5, 3, 2, 1)$, but $(5, 3, 3, 2)$ and $(4, 4, 4, 1)$ are incomparable.

Let P be a partition of n we denote by $P^\vee$ the conjugate partition of n: switch rows and columns in the Ferrers diagram of P. Let H be a sequence that occurs as the Hilbert function of an Artinian algebra, and denote by P_H the associated partition of $n = |A|$, $H^\vee$ its conjugate.

The following result is well known.

Lemma 1.6 ([48, Theorem 2.5]) *Let A be a standard graded Artinian algebra, and let $\ell \in A_1$ be a linear form. Then $P_{\ell,A} \leq H(A)^\vee$ in the dominance partial order on partitions.*

There is an analogous statement for local algebras A (ibid.).

Corollary 1.7 *Let A be an Artinian quotient of R and let $\ell \in \mathfrak{m}_A$. Then $P_{\ell,A}$ has at least as many parts as the Sperner number of $H(A)$.*

Proof That $H(A)^\vee \geq P_{\ell,A}$ and are partitions of $n = \dim_{\mathsf{k}} A$ is equivalent to $H(A) = \left(H(A)^\vee\right)^\vee \leq P_{\ell,A}^\vee$ [21, Lemma 6.3.1]. So the largest part of $H(A)$ (viewed as a partition) is less or equal the largest part in $P_{\ell,A}^\vee$, which is just the number of parts of $P_{\ell,A}$. $\square$

Example 1.8 (*Comparison of Jordan type for algebra B and associated graded algebra $A = B^*$*)

(a) Consider the graded complete intersection (CI) algebra $A = \mathsf{k}[x, y]/I$, $I = (x^3, y^3) = \mathrm{Ann}(X^2Y^2)$, with $H(A) = (1, 2, 3, 2, 1)$ and $H^\vee = (5, 3, 1)$ (Fig. 1). Here

$$P_{\ell,A} = (5, 3, 1) \text{ for } \ell = ax + by \text{ when } ab \neq 0, \text{ but } P_{x,A} = P_{y,A} = (3, 3, 3).$$

The strings for $\ell = x$ are $\{1, x, x^2\}, \{y, xy, x^2y\}, \{y^2, xy^2, x^2y^2\}$, and $(5, 3, 1) > (3, 3, 3)$.

(b) Consider the non-homogeneous CI algebra $B = \mathcal{R}/J$, with $\mathcal{R} = \mathsf{k}\{x, y\}$, and ideal $J = (x^3, y^3 - x^2y^2) = \mathrm{Ann}(X^2Y^2 + Y^3)$ satisfying $B^* = A$. We have for char $\mathsf{k} = 0$, again $P_{\ell,B} = (5, 3, 1)$ for $\ell = ax + by$ when $ab \neq 0$, and $P_{x,B} = (3, 3, 3)$. But now $P_{y,B} = (4, 3, 2)$, as the multiplication m_y has pre-Jordan basis strings $\{1, y, y^2, y^3 = x^2y^2\}$, $\{x, xy, xy^2\}$, and $\{x^2, x^2y\}$. Applying Algorithm 1.4, a Jordan basis for m_y has the strings $\{1, y, y^2, y^3 = x^2y^2\}$, $\{x, xy, xy^2\}$, and $\{x^2 - y, x^2y - y^2\}$, as y^4, xy^3, and $(x^2 - y)y^2$ are zero. The algebra B is a deformation of A, and $P_{y,B} = (4, 3, 2) > P_{y,A} = (3, 3, 3)$ in the dominance partial order, consistent with Corollary 3.8.

The following example illustrates some of the methods of determining Jordan type for a non-homogeneous Artinian Gorenstein (AG) algebra. See also [41, Sect. 2.4].

Example 1.9 (*Determining Jordan type, C non-homogeneous*) Let $\mathcal{R} = \mathsf{k}\{x, y, z\}$ and $C = \mathcal{R}/\mathrm{Ann}G$, where $G = X^3Y + Y^2Z$. Then C is a non-homogeneous AG algebra, not CI, defined by $\mathrm{Ann}G = (xz,\ yz - x^3,\ z^2,\ xy^2,\ y^3)$, with Hilbert function $H(C) = (1, 3, 3, 2, 1)$ and $H(C)^\vee = (5, 3, 2)$.

i. **Generic Jordan type of** C. Assume char $\mathsf{k} \notin \{2, 3\}$ and consider a general element $\ell \in \mathfrak{m}_C$. We write $\ell = ax + by + cz + h$, with $h \in \mathfrak{m}_C^2$. Suppose $ab \neq 0$. Then $\ell^4 = 4a^3bx^3y \neq 0$. Also, $\ell^3 = a^3x^3 + 3a^2bx^2y + h'$ and $\ell^2x = a^2x^3 + 2abx^2y + h''$, with $h', h'' \in \mathfrak{m}_C^4$ (note that $yz = x^3$ in A, so $y^2z = x^3y \in \mathfrak{m}_C^4$). We can easily check that ℓ^3 and ℓ^2x are linearly independent, so we have already two strings in a pre-Jordan basis for ℓ, namely $\{1, \ell, \ell^2, \ell^3, \ell^4\}$ and $\{x, \ell x, \ell^2x\}$. According to Lemma 3.7 the Jordan type of ℓ in C is at most $(5, 3, 2)$, and we already have two string of lengths 5 and 3, so we will check if we can get a new string of length 2. Note that $\mathfrak{m}_C^3 = \langle \ell^3, \ell^2x, \ell^4\rangle$, so if there is a further string of length two, there must be an order-one element $\alpha \in \mathfrak{m}_C \setminus \mathfrak{m}_C^2$ such that $\ell\alpha \notin \langle \ell^2, \ell^3, \ell^4, \ell x, \ell^2x\rangle$. Using ℓ and x to cancel terms in α if necessary, we can assume that $\alpha = z + g$, with $g \in \mathfrak{m}_C^2$. Now $\ell\alpha = bx^3 + \ell g \in \mathfrak{m}_C^3$, meaning there is no new length-two string. Therefore the Jordan type of ℓ is

$$P_{\ell,C} = (5, 3, 1, 1),$$

and since the set $\{ax + by + cz + h \in \mathfrak{m}_C : ab \neq 0, h \in \mathfrak{m}_C^2\}$ is an open dense subset of $\mathfrak{m}_C$, this is the generic Jordan type of C (Definition 1.1). We can consider $\{z\}$ and $\{y^2\}$ as new strings to complete the pre-Jordan basis.

ii. **Why we cannot attain a last length-two string**. That a last two-length string is not attainable is related to a construction from [42, Proposition 1.33]. The module $Q_C(1)$ can be explained by the relations between the terms Y^2Z and X^3Y in G (we refer to [42] for details on the $Q(a)$ modules, introduced by the second author in [45]; see also Lemma 3.1 below). Here, $Q_C(1)$ has two homogeneous terms:

$$Q_C(1)_1 = \frac{(0:\mathfrak{m}_C^3)}{\mathfrak{m}_C^2 + (0:\mathfrak{m}_C^2)} \quad \text{and} \quad Q_C(1)_2 = \frac{\mathfrak{m}_C^2 \cap (0:\mathfrak{m}_C^2)}{\mathfrak{m}_C^3 + (0:\mathfrak{m}_C)}. \tag{3}$$

Note that Y^2 is not a partial of X^3Y, but all further partials of Y^2 belong to $\langle 1, Y\rangle$, and thus are also partials of X^3Y. So acting on G with z yields $z \circ G = Y^2$, and this means that the class of z is non-zero in $Q_C(1)_1$ (in fact, it generates this module). However, $\mathfrak{m}_{\mathcal{R}} z \circ G = \langle 1, Y\rangle$, so if $\ell' \in \mathfrak{m}_{\mathcal{R}}$ is a lifting ot ℓ, we have $\ell' z \circ G = bY + d = (bx^3 + dx^3y) \circ G$, for some $d \in \mathsf{k}$, which explains why $\ell z \in \mathfrak{m}_C^3$ and its class is zero in $Q_C(1)_2$, so the module $Q_C(1)$ is acyclic. Coincidently, $Q_C(1) = \langle z, y^2\rangle$, so its generators are the elements we chose for the last two strings of the pre-Jordan basis.[1]

iii. **Special Jordan types of** C. When $\ell = ax + by + cz + h$, $h \in \mathfrak{m}_C^2$ and $ab = 0$, we find lower Jordan types in the dominance order. For instance,

$$\begin{aligned} &P_{x,C} = (4^2, 1^2), \quad P_{y+z,C} = (4, 2^3), \quad P_{y,C} = (3^2, 2^2), \\ &P_{x^2,C} = (2^4, 1^2), \qquad P_{z,C} = (2^3, 1^4). \end{aligned} \tag{4}$$

The strings for a pre-Jordan basis for z are particularly interesting, and illustrate the issues of the non-graded case: since $yz = x^3$ a possible choice is $\{1, z\}$, $\{y, x^3\}$, $\{y^2, x^3y\}$, $\{x\}$, $\{x^2\}$, $\{xy\}$, $\{x^2y\}$. Note that in the strings $\{y, x^3\}$ and $\{y^2, x^3y\}$ there is a jump in order: the orders of y and y^2 are 1 and 2, but multiplying by z makes these orders jump to 3 and 4, respectively.

iv. **Deformation** $C(t)$. Consider the family of Artinian Gorenstein algebras $\big(C(t)\big)_{t\in\mathsf{k}}$, where $C(t) = \mathcal{R}/\mathrm{Ann}G(t)$ is defined by the dual generator

$$G(t) = X^3Y + Y^2Z + tYZ^2.$$

Then $C(0) = C$, and for $t \neq 0$, $C(t)$ is a CI algebra, as

$$\mathrm{Ann}G(t) = (xz,\ ty^2 - yz + x^3,\ z^2 - tx^3).$$

We have $H\big(C(t)\big) = H(C) = (1, 3, 3, 2, 1)$ for all t. We can check that for $t \neq 0$ the Jordan type of $\ell = ax + by + cz + h$, with $h \in \mathfrak{m}_{C(t)}^2$ and $ab \neq 0$, is

$$P_{\ell,C(t)} = (5, 3, 2) = H\big(C(t)\big)^{\vee},$$

admitting strings $\{1, \ell, \ell^2, \ell^3, \ell^4\}$, $\{x, \ell x, \ell^2 x\}$, and $\{z, \ell z\}$. So the generic Jordan type of $C(t)$, for $t \neq 0$, strictly dominates that of $C = C(0)$ which is $(5, 3, 1, 1)$ (only domination is required by Lemma 3.7). For x, $y + z$, y, and x^2, we find the same Jordan types in $C(t)$ as in (4), but $P_{z,C(t)} = (3^2, 1^4) > P_{z,C}$.

The associated graded algebra is $C(t)^* = R/(xz,\ ty^2 - yz,\ z^2,\ x^4)$, with $R = \mathsf{k}[x, y, z]$, for $t \neq 0$, and $C(0)^* = C^* = R/(xz,\ yz,\ z^2,\ xy^2,\ y^3,\ x^4)$. The generic

[1] Further examples and discussion of these points are found in [41, Sect. 2.4, Remark 2.11ff.].

Jordan type of $C(t)^*$ is the same as that of $C(t)$ likewise for the special Jordan types of x and x^2, but $P_{y+z,C(t)^*} = (3, 2^3, 1)$, $P_{y,C(t)^*} = (3, 2^3, 1)$, for any $t \in \mathsf{k}$, and $P_{z,C(t)^*} = (2^2, 1^6)$, for $t \neq 0$, $P_{z,C(0)^*} = (2, 1^8)$. All these are dominated by the respective Jordan types in $C(t)$, as expected from Corollary 3.8.

Note 1.10 *The* Weyr characteristic *is an invariant of the similarity class of a matrix introduced by Eduard Weyr in 1885; for our nilpotent maps m_ℓ on A, it is just the conjugate (switch rows and columns of the Ferrers diagram) of the Jordan partition $P_{\ell,A}$ ([68, Sect. 2.4]). See [72] for an excellent introduction; K. O'Meara and J. Watanabe point out that for some problems the Weyr form may be more useful than the Jordan type [69]; see also [48, Note p. 371] for further references.*

1.3 Historical Note

Lefschetz properties of the cohomology rings of algebraic varieties had been long studied before the algebraists adapted it. R. Stanley showed that graded Artinian complete intersection algebras $A = R/(x_1^{a_1}, \dots, x_r^{a_r})$ satisfy a strong Lefschetz property [73]: he proved this using the hard Lefschetz theorem for the cohomology of the product $\mathbb{P} = \mathbb{P}^{a_1-1} \times \cdots \times \mathbb{P}^{a_r-1}$ of projective spaces. This inspired many to explore the Lefschetz properties of Artinian algebras. Results and open problems at the time concerning Lefschetz properties of graded Artinian algebras were well set out in the 2013 foundational opus by Harima et al. [34] and also surveyed by Migliore and Nagel [66]. Other articles on the Lefschetz properties include those by Harima [35, 36] in 1995 and 1999, one by Harbourne, Schenck, and Seceleanu on Gelfand-Tsetlin patterns and the weak Lefschetz property in 2011 [33], and that on singular hypersurfaces and Lefschetz properties by Di Gennaro, Ilardi, and Vallès in 2014 [24]. The latter direction was extended by others as E. Mezzetti, R.M. Miró-Roig, G. Ottaviani on Laplace Equations and weak Lefschetz in 2013 [65], and Miró-Roig and Salat on Togliatti equations in 2018 [67].

Despite advocacy by the second author beginning in 2012 at the Lefschetz conference organized by Junzo Watanabe at Tokai University, for using the finer Jordan type invariant of a pair (ℓ, A), it was not until 2022 that an introduction [48] to the topic appeared. This was at the instigation of Yong-Su Shin, who asked prior to coauthoring [70], where one could find an introduction to Jordan type! There was none. The authors of [48] attempted to give a comprehensive introduction, including new results, doing for Jordan type what Migliore and Nagel had done earlier in the same journal in "Tour of the strong and weak Lefschetz Properties" [66]. For some topics, such as modular tensor products, they were able to exhibit several threads of work by different communities who seemed unaware of each other's work on the same subject [48, Sect. 3B]. Several other articles by the same group treated Jordan type for certain free extensions, which are deformations of tensor products: see [47, Theorem 2.1], and also [64] which gives a connection of free extensions to invariant theory.

A main advance in the study of Lefschetz properties of Artinian Gorenstein (or AG) algebras was the 2009 article of T. Maeno and J. Watanabe, showing that the ranks of multiplication by powers of a linear form ℓ on the degree components A_i of a graded Gorenstein algebra A was given by the ranks of certain higher Hessians formed from the Macaulay dual generator of A, at a point p_ℓ [63]. This result was extended and used by many, including Gondim [28], Gondim and Zappalà [29]; and it was generalized by Gondim and Zappalà in 2019 to the mixed Hessians [30]. These have been used to prove that some Nagata idealization examples of graded AG algebras in embedding dimension at least four, are not strong Lefschetz (as [20]). The Hessian tools have been used recently by a growing cohort to study Jordan types for pairs (A, ℓ) where A is a graded AG algebra and $\ell \in A_1$: see, for example [6, 8, 10] and Sect. 2.2 below).

Recent Articles on Jordan Type and Artinian Algebras

We here mention several recent articles and research areas, with emphasis on those that mention Jordan type. Fixing codimension two, and a Hilbert function H, we can study the "Jordan cells" $\mathbb{V}(E_P)$ of the family G_H, comprised of ideals having initial monomial ideal E_P in a direction given by a linear form ℓ, determined by the partition P, which must have "diagonal lengths" T: see [6, Theorem 2.8]. The cell $\mathbb{V}(E_P)$ is comprised of all graded Artinian algebras $A = \mathsf{k}[x, y]/I$ such that $P_{\ell,A} = P$. The dimension of these cells, and some of their geometric properties were known by Yaméogo's [46, 76, 77]; the article [6] determines the generic number of generators of ideals in each cell [6, Theorems 3.11, 5.15] using a decomposition of cells into a product of simpler components. See Question 4.3.

There has been the beginning of tying the Jordan type with the Betti resolution of A, see Abdallah and Schenck [4] and Abdallah's [5], and as well Jelisiejew et al. [51], where they investigate local complete intersections of codimension three, also the book-length [54] has some Betti number vs. Jordan type calculations.

Not totally unrelated, the preprint [2] studies Jordan types for codimension three graded Gorenstein algebras of Sperner number at most 6 and all linear forms. This is facilitated by the D. Buchsbaum–D. Eisenbud Pfaffian structure theorem and related work [18, 22, 25] which specifies the Betti resolutions possible given $H(A)$. The results are still complex with 26 Jordan types for $H = (1, 3, 4^k, 3, 1)$ when $k \geq 3$ and 47 for $H = (1, 3, 5^k, 3, 1)$ with $k \geq 4$.

In [1] the weak Lefschetz property and Jordan types for linear forms of a class of graded AG algebras, called Perazzo algebras, of codimension five were studied. For Perazzo algebras, the multiplication map ℓ^{j-2} from degree 1 to degree $j - 1$ does not have maximal rank, where j is the socle degree. Thus, the strong Lefschetz property is not satisfied for this family. In [1] all Jordan types for linear forms of Perazzo algebras of codimension five with the smallest possible Hilbert function were determined.

2 Properties of Jordan Type, and of Jordan Degree Type

2.1 Lefschetz Properties and Jordan Type

Definition 2.1 (*Lefschetz properties*) Let A be a graded Artinian algebra of highest socle degree j and let $\ell \in A_1$. We say that the pair (A, ℓ) is *weak Lefschetz* (WL) if for each $i \geq 0$ the multiplication map $\times \ell : A_i \to A_{i+1}$ has maximal rank. The algebra A satisfies the *weak Lefschetz property* (WLP) if it has a WL element. We say that the pair (ℓ, A) is *strong Lefschetz* (SL) if for each $i, d \geq 0$ the multiplication map $\times \ell^d : A_i \to A_{i+d}$ has maximal rank. The algebra A satisfies the *strong Lefschetz property* (SLP) if it has a SL element.

The following result part A is a portion of [48, Proposition 2.10]; part B is essentially [48, Lemma 2.11], shown when $H(A)$ is also symmetric in [34, Proposition 3.5]. We say that a Hilbert function $H(A) = (h_0, h_1, \dots, h_j)$ is *unimodal* if there is an integer k such that $h_0 \leq \cdots \leq h_k$ and $h_k \geq h_{k+1} \geq \cdots \geq h_j$. Recall that the Sperner number $\mathrm{Sperner}(A)$ is the maximum value of $H(A)$.

Lemma 2.2 *A. Let A be a graded Artinian algebra (possibly non-standard), and $\ell \in A_1$. Then the following are equivalent*

(i) The pair (A, ℓ) is strong Lefschetz;
(ii) The Jordan type $P_{A,\ell} = H(A)^\vee$, the conjugate of the Hilbert function viewed as a partition.

B. Assume further that $H(A)$ is unimodal. Then the following are equivalent

(i) The pair (A, ℓ) is weak Lefschetz.
(ii) The dimension $\dim_k A/\ell A = \mathrm{Sperner}(A)$.
(iii) The number of parts of the Jordan partition $P_{A,\ell}$ is $\mathrm{Sperner}(A)$*, the minimum possible given $H(A)$ (Corollary 1.7).*

Proof The proof of Lemma 2.2(A) under the hypothesis is a bit subtle see [48, Proposition 2.10]. For Lemma 2.2(B), the proof of B(i) $\Leftrightarrow$ B(ii) is straightforward from the definitions; the proof of B(ii) $\Leftrightarrow$ B(iii) follows from decomposing A as a direct sum of *strings* (Lemma 1.3). □

2.2 Higher Hessians and Mixed Hessians

Graded Artinian Gorenstein algebras are determined by a single polynomial in the Macaulay dual ring, by a result of Macualay [62]. Let $A = R/\mathrm{Ann}F$ be an Artinian Gorenstein algebra with dual generator $F \in \mathcal{E}_j = \mathsf{k}_{DP}[X_1, \dots, X_r]_j$, where $\mathrm{Ann}F$

is the ideal generated by all the forms $g \in R$ such that $g \circ F = 0$. Maeno and Watanabe [63] introduced higher Hessians associated to the dual generator F and provided a criterion for Artinian Gorenstein algebras having the SLP.

We first briefly recall the Macaulay duality [60], see [26, Sect. 21.2], [44]; the recent emendation by Kleiman and Kleppe gives a geometric view consistent with studying deformation [57]. We let $R = \mathsf{k}[x_1, \ldots, x_r]$ act on $\mathcal{E}$ by contraction[2] where for $u \geq k$, $x_i^k \circ X_j^u = \delta_{i,j} X_i^{u-k}$ (we will call this $\partial^k X_j^u / \partial X_i^k$) and extending this multilinearly to an action of $h \in R$ on $F \in \mathcal{E}$.

$$h \circ F = h(\partial/\partial X_1, \ldots, \partial/\partial X_r) \circ F, \tag{1}$$

so taking $F = X_1^3 X_2^2 + X_1 X_2^4$ we have $x_1 x_2^2 \circ F = X_1^2 + X_2^2$.

Definition 2.3 ([63, Definition 3.1]) Let F be a polynomial in $\mathcal{E}$ and $A = R/\mathrm{Ann} F$ be its associated Gorenstein algebra. Let $\mathcal{B}_k = \{\alpha_i^{(k)}\}_i$ be an ordered k-basis of A_k. The entries of the k-th Hessian matrix of F with respect to $\mathcal{B}_k$ are

$$\mathrm{Hess}^k(F) = (\alpha_u^{(k)} \alpha_v^{(k)} F)_{u,v}.$$

Note that when $k = 1$, $\mathrm{Hess}^1(F)$ coincides with the usual Hessian. P. Gordan and M. Noether proved that the (first) Hessian of every homogeneous form F in at most 4 variables has non-zero determinant unless F defines a cone [31]. This is no longer the case in polynomial rings with at least 5 variables: a family of forms that do not define a cone and for which the Hessian has zero determinant was provided by [31, 71], they are called Perazzo forms.

Up to non-zero constant multiple, $\det \mathrm{Hess}^k(F)$ is independent of the choice of basis $\mathcal{B}_k$. For every $0 \leq k \leq \lfloor \frac{j}{2} \rfloor$ and a linear form $\ell = a_1 x_1 + \cdots + a_r x_r$ the rank of $\times \ell^{j-2k} : A_k \to A_{j-k}$ is equal to the rank of $\mathrm{Hess}_\ell^k(F)$; i.e. the Hessian matrix evaluated at the point $P_\ell = (a_1, \ldots, a_r)$—see Theorem 2.2 below. For now we state,

Theorem 2.1 ([63, Theorem 3.1], [74]) *An AG algebra $A = R/\mathrm{Ann} F$ with socle degree j has the SLP if and only if there exists linear form $\ell \in R_1$ such that*

$$\det \mathrm{Hess}_\ell^k(F) \neq 0,$$

for every $k = 0, \ldots, \lfloor \frac{j}{2} \rfloor$.

As mentioned above, for Perazzo forms F the determinant of the first order Hessian, $\mathrm{Hess}^1(F)$, is identically zero. So by the above theorem the associated AG algebra of a Perazzo form fails to have the SLP. The WLP and Jordan types of Perazzo algebras in 5 variables have been studied in [1, 27].

For an AG algebra for which all higher Hessians have non-vanishing determinants, the above theorem shows that for a general enough linear form ℓ all the multiplication maps $\ell^{j-2k} : A_k \to A_{j-k}$ have maximal rank. It is natural to ask: if an AG algebra

[2] When char $\mathsf{k} = 0$ or char $\mathsf{k} > j$ we may use the usual differentiation action, see [40, Appendix A].

A has at least one Hessian with vanishing determinant, which multiplication maps have maximal rank and which ones do not? Gondim and Zappalà [30] introduced mixed Hessians that generalize the higher Hessians.

Definition 2.4 ([30, Definition 2.1]) Let $A = R/\mathrm{Ann}F$ be the AG algebra associated to $F \in \mathcal{E}_j$. Let $\mathcal{B}_k = \{\alpha_i^{(k)}\}_i$ and $\mathcal{B}_u = \{\beta_i^{(u)}\}_i$ be k-basis of A_k and A_u respectively. The entries of the *mixed Hessian* matrix of order (k, u) for F with respect to $\mathcal{B}_k$ and $\mathcal{B}_u$ is given by

$$\mathrm{Hess}^{(k,u)}(F) = (\alpha_u^{(k)}\beta_v^{(u)}F)_{u,v}.$$

Notice that this generalizes the definition of higher Hessians and we have $\mathrm{Hess}^k(F) = \mathrm{Hess}^{(k,k)}(F)$.

Theorem 2.2 ([30, Theorem 2.4]) *Let A be a standard graded AG algebra. Then the rank of the mixed Hessian matrix of order (k, u) evaluated at the point $P_\ell = (a_1, \dots, a_r)$ is the same as the rank of the multiplication map $\ell^{u-k} : A_k \to A_u$ for $\ell = a_1x_1 + \cdots + a_rx_r$.*

The method of higher Hessians and mixed Hessians has been used to study the Lefschetz properties for graded AG algebras, for instance see [8, 27, 28]. The ranks of higher and mixed Hessians together at the point P_ℓ completely determine the ranks of multiplication maps by different powers of the linear form ℓ in all degrees, and hence, when the Hilbert function $H(A)$ is unimodal, also the Jordan degree type of (ℓ, A) (Proposition 2.16). Costa and Gondim in [23] determined the Jordan types for general linear forms of AG algebras having low codimension and low socle degree in terms of the ranks of the associated mixed Hessians.

The first and second authors with Khatami classified [10] all partitions that can occur as Jordan types of linear forms for AG algebras in codimension two (these are exactly complete intersection algebras by [61]) having a fixed Hilbert function. It has been shown that in codimension two, the Jordan types of linear forms of AG algebras are completely determined by the rank of higher Hessians. In fact, they are uniquely determined by the sets of higher Hessians that have vanishing determinants.

Theorem 2.3 ([10, Theorem 3.8]) *Assume that $H = \big(1, 2, 3, \dots, d^k, \dots, 3, 2, 1\big)$, is a Hilbert function of some complete intersection algebra for $d \geq 2$ and $k \geq 2$ ($k = 1$, respectively). Let P be a partition that can occur as the Jordan type of a linear form and an Artinian complete intersection algebra having Hilbert function H. Then the following are equivalent.*

(i) *$P = P_{\ell,A}$ for a linear form $\ell \in R_1$ and an Artinian complete intersection algebra $A = R/\mathrm{Ann}F$, and there is an ordered partition $n = n_1 + \cdots + n_c$ of an integer n satisfying $0 \leq n \leq d$ (or $0 \leq n \leq d-1$, respectively) such that $\det \mathrm{Hess}_\ell^{n_1+\cdots+n_i-1}(F) \neq 0$, for each $1 \leq i \leq c$, and the remaining Hessians are zero;*

(ii) *P satisfies*

$$P = \big(p_1^{n_1}, \dots, p_c^{n_c}, (d-n)^{d-n+k-1}\big), \tag{2}$$

where $p_i = k - 1 + 2d - n_i - 2(n_1 + \cdots + n_{i-1})$, for $1 \leq i \leq c$.