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Contents

The Yang Problem for Complete Bounded Complex Submanifolds: a Survey

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Abstract. We survey the history as well as recent progress in the Yang problem concerning the existence of complete bounded complex submanifolds of the complex Euclidean spaces. We also point out some open questions on the topic.

1 The Problem and the First Solution

In 1977, P. Yang asked the following (see [66, p. 135, Question II]):

Problem 1. Do there exist complete immersed complex submanifolds $\varphi\colon M^k \to \mathbb{C}^n$ $(1 \leq k < n)$ with bounded image?

Completeness is a very natural condition to impose on a Riemannian manifold when one is interested in its global properties. Recall that an immersed submanifold $\varphi\colon M \to \mathbb{C}^n$ is said to be complete if the Riemannian metric $\varphi^* d\sigma^2$ induced on M by pulling back the Euclidean metric $d\sigma^2$ on $\mathbb{C}^n$ by the immersion φ is a complete metric on M: geodesics go on indefinitely. By the Hopf-Rinow theorem, φ is complete if and only if $\varphi \circ \gamma\colon [0,1) \to \mathbb{C}^n$ has infinite Euclidean length for every divergent path $\gamma\colon [0,1) \to M$. Every compact Riemannian manifold is complete, but compact complex manifolds cannot be found in $\mathbb{C}^n$ by the maximum principle for holomorphic functions, so the question in Problem 1 is in order.

The main original motivation for P. Yang to pose the aforementioned question is that a positive answer would prove the existence of complete immersed complex submanifolds $M^k \to \mathbb{C}^{2n}$ with strongly negative holomorphic sectional curvature [67]. On the other hand, since complex submanifolds of $\mathbb{C}^n$ are minimal (i.e., critical points for the volume functional; see e.g. [25, 55], among many others, for an introduction to the subject), the Yang problem is also related to the so-called Calabi-Yau problem, which dates back to E. Calabi's conjectures from 1965 [54, p. 170] and asked whether there are complete bounded minimal hypersurfaces in $\mathbb{R}^n$ $(n \geq 3)$; we refer to [12, Ch. 7] for a recent survey on this fascinating topic. Nevertheless, the Yang problem for complex submanifolds has become an active focus of interest in its own right, having received many important contributions in the last decade; see [38, §4.3] for a brief introduction to the topic.

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We shall denote by $\mathbb{D} = \{\zeta \in \mathbb{C} \colon |\zeta| < 1\}$ the open unit disc in $\mathbb{C}$ and by $\mathbb{B}_n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon |z| < 1\}$ the open unit ball in $\mathbb{C}^n$ for $n \geq 2$. The first affirmative answer to the question in Problem 1 was given only two years later, in 1979, by P.W. Jones. Recall that a map $f \colon X \to Y$ between topological spaces is said to be proper if $f^{-1}(K) \subset X$ is compact for every compact set $K \subset Y$.

Theorem 1 (Jones [51]). *There exist a complete bounded immersed complex disc $\mathbb{D} \to \mathbb{C}^2$, a complete bounded embedded complex disc $\mathbb{D} \hookrightarrow \mathbb{C}^3$, and a complete properly embedded complex disc $\mathbb{D} \hookrightarrow \mathbb{B}_4$.*

P.W. Jones' construction method is strongly complex analytic. It relies on using the BMO duality theorem in order to find a pair of bounded holomorphic functions f_1 and f_2 on $\mathbb{D}$ satisfying the property that

$$\int_\gamma \big(|f_1'(\zeta)| + |f_2'(\zeta)|\big)\, d\sigma(\zeta) = +\infty$$

for all paths $\gamma \subset \mathbb{D}$ terminating on $\mathbb{S}^1 = b\mathbb{D}$, where σ denotes Euclidean arc length. It follows that $\mathbb{D} \ni \zeta \mapsto (\zeta, f_1(\zeta), f_2(\zeta))$ is a complete bounded embedded complex disc in $\mathbb{C}^3$; the other two assertions in the theorem are obtained by slight modifications of this procedure.

2 Curves

Despite having to wait more than three decades for it, Theorem 1 has been generalized in several directions. The first extension of P.W. Jones' existence result was given by A. Alarcón and F.J. López in 2013 and concerns the topology of the examples.

Theorem 2 (Alarcón-López [15]). *Let $n \geq 2$. Every open orientable surface S admits a complex structure J such that the open Riemann surface $M = (S, J)$ carries a complete proper holomorphic immersion $M \to \mathbb{B}_n$ which is an embedding if $n \geq 3$. The same holds true if we replace the ball by any convex domain in $\mathbb{C}^n$.*

The embeddedness condition for $n \geq 3$ in this statement was not explicitly stated in [15]; nevertheless, it trivially follows from a standard transversality argument, as was later pointed out by A. Alarcón and F. Forstnerič in [6]. We emphasize that the examples in Theorem 2 may have any topological type, even infinite.

The proof in [15] is completely different from that in [51]. In particular, it is much more geometric, and is reminiscent of the method developed by N. Nadirashvili in his seminal paper [60] for constructing a complete bounded minimal disc in $\mathbb{R}^3$; see also [12, §7.1]. The construction goes by induction, the rough idea being the following. In the step $j \in \mathbb{N}$ we begin with a smoothly bounded compact complex curve, say X_{j-1}, whose boundary bX_{j-1} lies inside the ball

$R_{j-1}\mathbb{B}_n$ of some radius $R_{j-1} > 0$ but close to its boundary sphere. Then, we apply to X_{j-1} a deformation that is arbitrarily small outside a neighborhood of bX_{j-1} and pushes each boundary point $x \in bX_{j-1}$ a distance approximately $1/j$ in a direction perpendicular to the position vector of x in $\mathbb{C}^n$. In this way we increase the boundary distance from a fixed interior point of the curve an amount of approximately $1/j$, while the extrinsic diameter is increased, by Pythagoras' theorem, an amount of the order of $1/j^2$. Moreover, we ensure that the boundary of the new complex curve, X_j, lies inside the ball of radius

$$R_j = \sqrt{R_{j-1}^2 + 1/j^2} > R_{j-1}$$

but close to its boundary sphere. See Fig. 1. Since

$$\sum_{j\geq 1} 1/j = \infty \quad \text{and} \quad \sum_{j\geq 1} 1/j^2 < \infty,$$

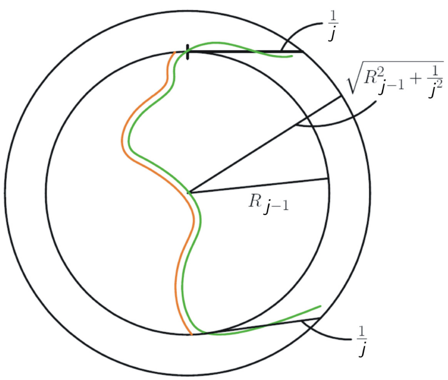

Fig. 1. Schematic representation of the geometry of the deformations used in the inductive construction in [15]. The key idea is to deform the submanifold near its boundary by pushing each boundary point in a direction orthogonal to its position vector.

if we arrange this process in the right way then we obtain in the limit a complete complex curve that is contained in the ball of radius $R = \lim_{j\to\infty} R_j < \infty$ and is proper in it. In order to prescribe the topology of the curve, we begin with a disc and at each step of the inductive construction apply a surgery which enables us to add either a handle or a boundary component to a given compact bordered complex curve in $\mathbb{C}^n$. Finally, if $n \geq 3$, then a general position argument allows to guarantee that all complex curves in the sequence are embedded, and hence the limit one can be granted to be embedded as well. The main tool in order to make the described deformations is the theory of uniform approximation for holomorphic functions on open Riemann surfaces; in particular, the Runge-Mergelyan theorem (see E. Bishop [21] or e.g. Theorem 5 in the survey on holomorphic approximation by J.E. Fornæss, F. Forstnerič, and E.F. Wold [33]).

A different construction of complete bounded complex curves in $\mathbb{C}^2$ can be found in [59], where F. Martín, M. Umehara, and K. Yamada gave examples with arbitrary finite genus and finitely many ends. Their technique relies on the existence of a simply connected complete bounded holomorphic null curve in $\mathbb{C}^3$ (such a curve was first constructed in [15], an alternative construction was given later by L. Ferrer, F. Martín, M. Umehara, and K. Yamada in [31]; see the mentioned sources or e.g. [12, §2.3] for definitions) and modifies a method developed by F.J. López in [56] for constructing complete minimal surfaces in $\mathbb{R}^3$ of hyperbolic conformal type.

Despite the flexibility of the methods in [15, 59], none of them allows to control the complex structure on the curve, except of course in the simply connected case: every bounded immersed complex curve in $\mathbb{C}^n$ must have hyperbolic conformal type, and hence if it is simply connected it is biholomorphic to the disc $\mathbb{D}$. Indeed, since the construction in [15] relies on Runge's theorem, at a certain stage one does not have enough information on the placement in $\mathbb{C}^n$ of some parts of the curve, and hence one is forced to cut away some small pieces of the curve to ensure its boundedness. This makes one to lose the control of the complex structure of the curve. This difficulty was overcome by A. Alarcón and F. Forstnerič in [5], also published in 2013, where two additional complex analytic tools were introduced into the game, namely, the method of F. Forstnerič and E.F. Wold [43] for exposing boundary points of a complex curve in $\mathbb{C}^n$ and the use of approximate solutions to Riemann-Hilbert boundary value problems. (The former is a modern technique that has led to important progress in the classical Forster-Bell-Narasimhan Conjecture asking whether every open Riemann surface admits a proper holomorphic embedding into $\mathbb{C}^2$ [20, 34]; we refer to F. Forstnerič [37, §9.10–9.11] for a survey on this long-standing open problem. On the other hand, the use of the Riemann-Hilbert problem for constructing proper holomorphic maps has a long history; we refer to F. Forstnerič and J. Globevnik [41], B. Drinovec Drnovšek and F. Forstnerič [28], and the references therein.) The implementation of these new tools enabled to substantially simplify the construction in [15] and, moreover, to control the complex structure on the curve. Recall that a bordered Riemann surface is an open connected Riemann surface M that is the interior, $M = \overline{M} \setminus b\overline{M}$, of a compact one dimensional complex manifold $\overline{M}$ with smooth boundary $b\overline{M}$ consisting of finitely many closed Jordan curves; such an $\overline{M}$ is called a compact bordered Riemann surface.

Theorem 3 (Alarcón-Forstnerič [5]**).** *Let $n \geq 2$ be an integer. Every bordered Riemann surface M admits a complete proper holomorphic immersion $M \to \mathbb{B}_n$ that is an embedding if $n \geq 3$. The same holds true if we replace the ball by any pseudoconvex domain in $\mathbb{C}^n$.*

More generally, it is shown in [5] that if X is a Stein manifold of dimension $n \geq 2$ endowed with a hermitian metric, then every bordered Riemann surface M admits a complete proper holomorphic immersion $M \to X$ that can be chosen an embedding if $n \geq 3$. Recall that a Stein manifold is the same thing as a closed complex submanifold of a complex Euclidean space; we refer to [37] for a comprehensive monograph on the theory of Stein manifolds. A domain D in $\mathbb{C}^n$

($n \geq 2$) is a Stein manifold if and only if it is pseudoconvex, meaning that there is a strictly plurisubharmonic exhaustion function $D \to \mathbb{R}$. This happens if and only if D is a domain of holomorphy and if and only if D is holomorphically convex. For instance, every convex domain in $\mathbb{C}^n$ is pseudoconvex. We refer to the monographs by R.M. Range [61] (see also the introductory note [62]) and L. Hörmander [49,50] for background on the subject.

More recently, the construction technique in [5] has been refined to produce complete bounded complex curves with control on the complex structure and with some further control on the asymptotic behavior. In particular, there are such curves bounded by Jordan curves. The following is a compilation of results by A. Alarcón, I. Castro Infantes, B. Drinovec Drnovšek, F. Forstnerič, and F.J. López [3,4,9,17,40].

Theorem 4. *Let R be a compact Riemann surface and assume that $M = R \setminus \bigcup_{i \in I} D_i$ is a domain in R whose complement is a finite or countable union of pairwise disjoint, smoothly bounded closed discs (i.e., diffeomorphic images of $\overline{\mathbb{D}}$). The following assertions hold true:*

1. *M is the complex structure of a complete bounded complex curve in $\mathbb{C}^2$. In fact, for any $n \geq 2$ there is a continuous map $\varphi \colon \overline{M} \to \mathbb{C}^n$ such that the restricted map $\varphi|_M \colon M \to \mathbb{C}^n$ is a complete holomorphic immersion (embedding if $n \geq 3$) and $\varphi|_{bM} \colon bM = \bigcup_{i \in I} bD_i \to \mathbb{C}^n$ is injective.* **(Alarcón-Drinovec Drnovšek-Forstnerič-López** [4], **Alarcón-Forstnerič** [9]**.)**
2. *If I is finite, then there is a continuous map $\varphi \colon \overline{M} \to \overline{\mathbb{B}}_n$ ($n \geq 3$) such that $\varphi(bM) \subset b\mathbb{B}_n$ and $\varphi|_M \colon M \to \mathbb{B}_n$ is a complete proper holomorphic immersion (embedding if $n \geq 3$). The same holds true with $\mathbb{B}_n$ replaced by any convex domain in $\mathbb{C}^n$.* **(Alarcón-Drinovec Drnovšek-Forstnerič-López** [4]**.)**
3. *If I is finite, then for any domain D in $\mathbb{C}^n$ there is a complete holomorphic immersion $\varphi \colon M \to D$ whose image is a dense subset of D. If $n \geq 3$ then φ can be chosen injective.* **(Alarcón-Castro Infantes** [3]**.)**
4. *There is a Cantor set C in R whose complement admits a complete holomorphic immersion $R \setminus C \to \mathbb{C}^2$ with bounded image. There also exist a Cantor set C in R and a complete holomorphic embedding $R \setminus C \hookrightarrow \mathbb{C}^3$ with bounded image.* **(Forstnerič** [40]**.)**
5. *There is a Cantor set C in M and a continuous map $\varphi \colon \overline{M} \setminus C \to \mathbb{C}^n$ ($n \geq 2$) such that $\varphi|_{M \setminus C} \colon M \setminus C \to \mathbb{C}^n$ is a complete holomorphic immersion and $\varphi|_{bM} \colon bM = \bigcup_{i \in I} bD_i \to \mathbb{C}^n$ is injective. If $n \geq 3$ then C can be chosen so that $\varphi \colon \overline{M} \setminus C \to \mathbb{C}^n$ is an injective map.* **(Forstnerič** [40]**.)**

Summarizing, by the year 2015 there were available in the literature several constructions of complete bounded complex curves immersed in $\mathbb{C}^2$ and embedded in $\mathbb{C}^3$, allowing a high control on the asymptotic behavior (proper in the ball or in a given pseudoconvex domain, bounded by Jordan curves, etc.), on the topology, and on the complex structure of the examples. However, the construction of complete bounded embedded complex curves in $\mathbb{C}^2$ turns out to be a much more challenging undertaking, and the question whether such curves exist remained open (see [5, Question 1]). Recall that complex curves are generically embedded in $\mathbb{C}^n$ for $n \geq 3$, meaning that self-intersections can be removed by applying small deformations, while self-intersections of complex curves in $\mathbb{C}^2$, which are generically double points, are stable under such deformations. That is the main reason why the task is a more difficult one.

The Yang problem for embedded complex curves in the affine plane $\mathbb{C}^2$ was finally settled by A. Alarcón and F.J. López in a paper published in 2016.

Theorem 5 (Alarcón-López [18]). *Every convex domain in $\mathbb{C}^2$ admits complete properly embedded complex curves.*

The proof goes by induction and involves an approximation process by embedded complex curves in $\mathbb{C}^2$. The main step in the construction is to prove that every compact embedded complex curve X in $\mathbb{C}^2$ with the boundary bX lying in the boundary bD of a regular strictly convex domain $D \subset \mathbb{C}^2$ may be approximated by another compact embedded complex curve $\tilde{X}$ with $b\tilde{X} \subset b\tilde{D}$, for any given convex domain $\tilde{D} \subset \mathbb{C}^2$ with $\overline{D} \subset \tilde{D}$. The new curve $\tilde{X}$ is ensured to contain a biholomorphic copy of X, which we denote by X as well, and the main point is to guarantee that $\tilde{X} \setminus X \subset \tilde{D} \setminus D$ and the intrinsic Euclidean distance in $\tilde{X}$ from X to $b\tilde{X}$ is suitably larger (in a Pythagorical way similar to that explained in Fig. 1) than the distance from D to $b\tilde{D}$. These conditions are the key for obtaining embeddedness, completeness, and properness of the limit complex curve in the limit convex domain. In order to guarantee the embeddedness of $\tilde{X}$ a standard self-intersection removal method consisting of replacing every normal crossing in an immersed complex curve in $\mathbb{C}^2$ by an embedded annulus is applied. This surgery may generate shortcuts in the arising desingularized curve $\tilde{X}$, thereby giving rise to divergent paths of shorter length. This is an important difficulty for ensuring completeness; for instance, if one applies this surgery at each step in the inductive construction in the proof of Theorem 2 or Theorem 3, then one still obtains in the limit a properly embedded complex curve in $\mathbb{B}_2$ (or, more generally, in any given pseudoconvex domain of $\mathbb{C}^2$), but it need not be complete. A main novelty in the construction in [18] is provided a good enough (say, in a Pythagorical sense) estimate of the growth of the intrinsic Euclidean diameter of the desingularized, embedded complex curve $\tilde{X}$; this is achieved by keeping a stronger control on the placement of $\tilde{X} \setminus X$ in $\tilde{D} \setminus D$. The main tool in this construction continues to be the classical Runge-Mergelyan approximation theorem for holomorphic functions on open Riemann surfaces.

The aforementioned surgery may increase the topological genus of the curve, and so there is no control on the topology of the examples in Theorem 5. They actually seem to have infinite genus, and can be ensured to have infinite topology. It therefore remained an open question whether there are complete bounded embedded complex curves in $\mathbb{C}^2$ of finite topology (see [18, Question 1.5]).

3 Submanifolds of Arbitrary Dimension

All examples of complete bounded complex submanifolds we have discussed so far are of complex dimension one; i.e., complex curves in $\mathbb{C}^n$. The first known such submanifolds of higher dimension were also given in [5].

Corollary 1 (Alarcón-Forstnerič [5]**).** *If D is a relatively compact, strongly pseudoconvex domain in a Stein manifold of dimension $k \geq 1$, then there are a complete proper holomorphic immersion $D \to (\mathbb{B}_2)^{2k} \subset \mathbb{C}^{4k}$ and a complete proper holomorphic embedding $D \hookrightarrow (\mathbb{B}_3)^{2k+1} \subset \mathbb{C}^{6k+3}$.*

The corollary is obtained by the following simple trick which was pointed out by J.E. Fornæss. In [29], B. Drinovec Drnovšek and F. Forstnerič proved that every domain D as in the statement admits a proper holomorphic immersion $g\colon D \to \mathbb{D}^{2k}$ into the polydisc $\mathbb{D}^{2k} = \mathbb{D}\times \overset{2k}{\cdots} \times\mathbb{D} \subset \mathbb{C}^{2k}$. Choose a complete proper holomorphic immersion $\varphi\colon \mathbb{D} \to \mathbb{B}_2$ provided by Theorem 3, and consider the proper holomorphic map $\varphi^{2k}\colon \mathbb{D}^{2k} \to (\mathbb{B}_2)^{2k}$ given by

$$\varphi^{2k}(\zeta_1, \ldots, \zeta_{2k}) = (\varphi(\zeta_1), \ldots, \varphi(\zeta_{2k})), \quad (\zeta_1, \ldots, \zeta_{2k}) \in \mathbb{D}^{2k}.$$

It is then easily checked that $\varphi^{2k} \circ g\colon D \to (\mathbb{B}_2)^{2k}$ is a complete proper holomorphic immersion. A slight modification of this argument using a proper holomorphic embedding $D \hookrightarrow \mathbb{D}^{2k+1}$ (existence of such is also proved in [29]) provides a proper holomorphic embedding $D \hookrightarrow (\mathbb{B}_3)^{2k+1}$. This same trick together with Theorem 5 allows to prove the following.

Corollary 2 (Alarcón-López [18]**).** *For any $k \in \mathbb{N}$ there is a complete bounded embedded k-dimensional complex submanifold $M^k \hookrightarrow \mathbb{C}^{2k}$.*

The ad hoc construction of the high dimensional examples in Corollaries 1 and 2 seemed to give very particular solutions to the Yang problem for complex submanifolds of dimension ≥ 2. This led to some new questions, as whether the dimension $2k$ in Corollary 2 is optimal, and exposed the need of looking for new construction methods other than those based on the existence of complete bounded complex curves. For instance, the natural question whether the ball $\mathbb{B}_k$ $(k \geq 2)$ admits a complete proper holomorphic embedding into the ball $\mathbb{B}_n$ for some $n > k$, and, in particular, for $n = k + 1$ appeared; see [5, Question 3] and [44, Question 13.2]. An affirmative answer to this question in the case of sufficiently high codimension was given in 2015 by B. Drinovec Drnovšek.

Theorem 6 (Drinovec Drnovšek [27]**).** *Every bounded strictly pseudoconvex domain $D \subset \mathbb{C}^k$ $(k \in \mathbb{N})$ with $\mathcal{C}^2$ boundary admits a complete proper holomorphic embedding $D \hookrightarrow \mathbb{B}_n$ for any large enough $n \in \mathbb{N}$.*

The proof in [27] continues to use the geometric idea of deforming a compact submanifold near the boundary in orthogonal directions to the position vector (see Fig. 1), but it exploits different tools. The main new ingredients are holomorphic peak functions, going back to ideas of M. Hakim and N. Sibony [48] and E. Løw [57], and the construction of inner functions on the ball, as well as J.E.

Fornæss embedding theorem [32]. The construction relies on suitably modifying earlier methods by F. Forstnerič [35] and E. Løw [58] for constructing proper holomorphic maps from a strictly convex domain with $\mathcal{C}^2$ boundary in $\mathbb{C}^k$ into a unit ball of some Euclidean space of higher dimension, in order to make them complete. The construction method in [27] requires a sufficiently high codimension, and hence the following question remains open; see [5, Question 3] and [44, Question 13.2].

Problem 2. Does there exist a complete proper holomorphic embedding $\mathbb{B}_k \hookrightarrow \mathbb{B}_{k+1}$ for $k \geq 2$?

4 Hypersurfaces

Recall that complex submanifolds of dimension $k \in \mathbb{N}$ in $\mathbb{C}^n$ are generically embedded for $n \geq 2k+1$ (meaning that one can get rid of their self-intersections by applying small deformations on compact pieces), while for $n \leq 2k$ self-intersections of k-dimensional complex submanifolds in $\mathbb{C}^n$ are stable under small deformations. This is the main reason why the Yang problem for embedded submanifolds is much more difficult in low codimension; in particular, for hypersurfaces. In the lowest dimensional case of $n = 2$ (i.e., for complex curves in $\mathbb{C}^2$), this question was first solved by A. Alarcón and F.J. López in [18] (see Theorem 5 above), but the question whether there are complete bounded embedded complex hypersurfaces in $\mathbb{C}^n$, or even whether there are such complex submanifolds of dimension k with $2k \geq n$, remained open for every $n \geq 3$. Indeed, note that the codimension in all examples of complete bounded embedded complex submanifold of dimension ≥ 2 which have been mentioned so far is high.

It was J. Globevnik who, in a pair of landmark papers in 2015–2016, positively settled Yang's question in Problem 1 for embeddings in arbitrary dimension and codimension; in particular, for hypersurfaces.

Theorem 7 (Globevnik [44,46]). *For any pair of integers $1 \leq k < n$ there is a complete closed embedded k-dimensional complex submanifold of $\mathbb{B}_n$. In particular, $\mathbb{B}_n$ admits a complete properly embedded complex hypersurface.*

The same holds true if we replace the ball by any pseudoconvex domain in $\mathbb{C}^n$.

Globevnik's approach is completely different from any previous method used in the study of the Yang problem, and it was a major breakthrough in this topic. In particular, his construction of a complete closed complex hypersurface in a given pseudoconvex domain $D \subset \mathbb{C}^n$ is implicit: the examples are obtained as level sets of highly oscillating holomorphic functions on D. (The existence of complete closed complex submanifolds of any higher codimension is then an obvious consequence.) To be more precise, Globevnik proved the following.

Theorem 8 (Globevnik [44,46]). *For any pseudoconvex domain $D \subset \mathbb{C}^n$ ($n \geq 2$) there is a holomorphic function on D whose real part is unbounded above on every divergent path $\gamma\colon [0,1) \to D$ of finite length.*

If a function $f\colon D \to \mathbb{C}$ on a domain $D \subset \mathbb{C}^n$ is holomorphic and nonconstant, then all its nonempty level sets $f^{-1}(c) = \{z \in D\colon f(z) = c\}$ ($c \in \mathbb{C}$) are closed complex hypersurfaces of D, possibly with singularities. Nevertheless, by Sard's theorem most of them are smooth, and hence properly embedded. So, if $c \in \mathbb{C}$ is such that $f^{-1}(c) \neq \varnothing$, then every divergent path $\gamma\colon [0,1) \to f^{-1}(c)$ diverges on D as well. If D is pseudoconvex and the function f is as in Theorem 8 then, since $\Re(f)$ is constant (and hence bounded) on γ, we have that γ has infinite length, and $f^{-1}(c)$ is thus complete. Therefore, the level sets $f^{-1}(c)$ ($c \in \mathbb{C}$) of f form a (possibly singular) holomorphic foliation of D by complete closed complex hypersurfaces. Theorem 8 thus implies the following corollary which, in turn, implies Theorem 7.

Corollary 3 (Globevnik [44,46]). *Every pseudoconvex domain $D \subset \mathbb{C}^n$ ($n \geq 2$) admits a (possibly singular) holomorphic foliation by complete closed complex hypersurfaces (most of which are smooth).*

Let us outline the proof of Theorem 8 in the case of $D = \mathbb{B}_n$ given in [44]. Recall that a convex polytope P in $\mathbb{R}^d$, $d \geq 2$, is a compact convex set which is the intersection of finitely many closed half-spaces. A face of P is a closed convex subset $F \subset P$ such that every closed segment in P whose relative interior intersects F is contained in F. The boundary bP of P is the union of its faces of dimension $d-1$, while the skeleton $\mathrm{skel}(P)$ of P is the union of all $(d-2)$-dimensional faces of P. Most of the work in the proof consists of constructing a sequence of convex polytopes P_j in $\mathbb{C}^n = \mathbb{R}^{2n}$ and positive numbers θ_j ($j \in \mathbb{N}$) satisfying the following conditions:

(i) $P_1 \subset \mathrm{Int}(P_2) \subset P_2 \subset \mathrm{Int}(P_3) \subset \cdots \subset \bigcup_{j\in\mathbb{N}} P_j = \mathbb{B}_n$.
(ii) Denote by U_j the θ_j-neighborhood of $\mathrm{skel}(P_j)$ in bP_j, and set $V_j = (bP_j)\setminus U_j$, $j \in \mathbb{N}$. If $\gamma\colon [0,1) \to \mathbb{B}_n$ is a divergent path such that $\gamma([0,1)) \cap V_j = \varnothing$ for all $j \geq j_0$ for some $j_0 \in \mathbb{N}$, then γ has infinite length.

Each set V_j ($j \in \mathbb{N}$) is compact and its connected components are closed convex sets in hyperplanes of $\mathbb{R}^{2n}$. The union

$$L = \bigcup_{j\in\mathbb{N}} V_j \tag{1}$$

of all of them is a sort of labyrinth of compact connected $(2n-1)$-dimensional convex sets in $\mathbb{B}_n \subset \mathbb{R}^{2n}$ with the property that every divergent path $\gamma\colon [0,1) \to \mathbb{B}_n$ meeting at most finitely many components of L has infinite length. See Fig. 2.

The construction of the labyrinth in [44] is very involved and belongs to convex geometry. With the labyrinth in hand, to complete the proof of Theorem 8 an idea of J. Globevnik and E.L. Stout from [47] is used in order to construct, via Runge's theorem, a sequence of holomorphic polynomials $f_j\colon \mathbb{C}^n \to \mathbb{C}$, $j \in \mathbb{N}$, such that the following conditions hold for each $j \in \mathbb{N}$:

(iii) $\Re(f_j(z)) \geq j+1$ for all $z \in V_j$.
(iv) $|f_{j+1}(z) - f_j(z)| \leq 1/2^{j+1}$ for all $z \in P_j$.