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## Hilbert-Samuel Formula and Equidistribution Theorem

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## Hilbert-Samuel Formula and Equidistribution Theorem

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Huayi Chen
Institute for Theoretical Sciences & School of Science
Westlake University
Hangzhou, Zhejiang, China

Atsushi Moriwaki
School of General Education
Chubu University
Kasugai, Aichi, Japan

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# Chapter 1
# Introduction

**Abstract** In this chapter, we describe the background of our work presented in the monograph and explain the main results obtained.

## 1.1 Backgrounds

The aim of this book is to explore positivity conditions and establish the Hilbert-Samuel formula and the equidistribution theorem in the context of adelic curves. This exploration traces its roots back to the foundational work of Dedekind and Weber [51], which revealed the striking resemblance between number fields and function fields, laying the groundwork for a prosperous research domain of arithmetic geometry. In particular, Arakelov's work [4, 5] established the arithmetic intersection theory on a relative curve over Spec $\mathbb{Z}$, which is analogous to the classic intersection theory of divisors on a projective surface. Arakelov's work has been generalized by Gillet and Soulé [60, 61] to a higher-dimensional case, leading to a rich theory of arithmetic schemes.

### *1.1.1 Positivity Conditions*

Positivity conditions are important for a multitude of reasons. They act as a conduit for deeper insights into the structure and properties of algebraic varieties, play a crucial role in many fundamental theorems and conjectures, and establish vital connections with other branches of mathematics.

In algebraic geometry, one of the main notions of positivity is ampleness, which is essential for the embedding of varieties into projective spaces. Thanks to the foundational paper of Serre [84], its importance has been widely recognized. Weaker positivity conditions, such as nefness, bigness and pseudo-effectivity, which have better behaviours by birational pullbacks, play a crucial role in birational algebraic geometry.



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In arithmetic geometry, the arithmetic ampleness has been studied by Zhang [102, 103]. In particular, a criterion of Nakai-Moishezon type has been proved for arithmetic ampleness and has been applied in the arithmetic approach to equidistribution problem and Bogomolov conjecture over number fields. The bigness in Arakelov geometry has been introduced in [79] and further studied in [80, 96]. An arithmetic analogue of Fujita's approximation theorem has then been proved [35, 97]. As for the arithmetic version of the pseudo-effectivity, a link with Dirichlet's unit theorem in algebraic number theory has been discovered in [81].

### 1.1.2 Hilbert Function

In algebraic geometry, Hilbert function measures the growth of graded linear series of a line bundle on a projective variety. Let $k$ be a field, $X$ be an integral projective scheme of dimension $d \in \mathbb{N}$ $(= \mathbb{Z}_{\geqslant 0})$ over $\operatorname{Spec} k$, and $L$ be an invertible $\mathcal{O}_X$-module. The Hilbert function of $L$ is defined as

$$\begin{aligned} H_L : \mathbb{N} &\longrightarrow \mathbb{N}, \\ n &\longmapsto \dim_k(H^0(X, L^{\otimes n})). \end{aligned}$$

If $L$ is ample, then the following asymptotic estimate holds:

$$H_L(n) = \frac{(L^d)}{d!} n^d + o(n^d). \tag{1.1.1}$$

This formula, which relates the asymptotic behaviour of the Hilbert function and the self-intersection number of $L$, is, for example, a consequence of Hirzebruch-Riemann-Roch theorem and Serre's vanishing theorem. It turns out that the construction and the asymptotic estimate of Hilbert function have analogue in various contexts, such as graded algebra, local multiplicity, relative volume of two metrics, etc.

### 1.1.3 Arithmetic Hilbert-Samuel Function

In Arakelov geometry, an arithmetic analogue of Hilbert function has been introduced by Gillet and Soulé [60], and an analogue of the asymptotic formula (1.1.1) has been deduced from their arithmetic Riemann-Roch theorem. This result is called an arithmetic Hilbert-Samuel theorem. Let $\mathscr{X}$ be a flat regular integral projective scheme of dimension $d+1$ over $\operatorname{Spec} \mathbb{Z}$, and $\overline{\mathscr{L}} = (\mathscr{L}, \varphi)$ be a Hermitian line bundle on $\mathscr{X}$, namely an invertible $\mathcal{O}_{\mathscr{X}}$-module $\mathscr{L}$ equipped with a smooth metric

$\varphi$ on $\mathscr{L}(\mathbb{C})$. For any integer $n \in \mathbb{N}$, let $\|\cdot\|_{n\varphi}$ be the norm on the real vector space $H^0(\mathscr{X}, \mathscr{L}^{\otimes n}) \otimes_{\mathbb{Z}} \mathbb{R}$ defined as

$$\forall\, s \in H^0(\mathscr{X}, \mathscr{L}^{\otimes n}) \otimes_{\mathbb{Z}} \mathbb{R} \subseteq H^0(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes n}), \quad \|s\|_{n\varphi} = \sup_{x \in \mathscr{X}(\mathbb{C})} |s|_{n\varphi}(x).$$

Then the couple $(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi})$ forms a lattice in a normed vector space. Recall that its arithmetic Euler-Poincaré characteristic is

$$\chi(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi}) = \ln \frac{\mathrm{vol}(\{s \in H^0(\mathscr{X}, \mathscr{L}^{\otimes n}) \otimes_{\mathbb{Z}} \mathbb{R} \mid \|s\|_{n\varphi} \leqslant 1\})}{\mathrm{covol}(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi})}$$

where $\mathrm{vol}(\cdot)$ denotes a Haar measure on the real vector space $H^0(\mathscr{X}, \mathscr{L}) \otimes_{\mathbb{Z}} \mathbb{R}$, and

$$\mathrm{covol}(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi})$$

is the covolume of the lattice $H^0(\mathscr{X}, \mathscr{L}^{\otimes n})$ with respect to the Haar measure $\mathrm{vol}(\cdot)$, namely the volume of any fundamental domain of this lattice. In this setting, the arithmetic Hilbert-Samuel theorem shows that if $\mathscr{L}$ is relatively ample and the metric $\varphi$ is positive, then the sequence

$$\frac{\chi(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi})}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N},\ n \geqslant 1$$

converges to the arithmetic intersection number $(\overline{\mathscr{L}}^{d+1})$. In the case where $\overline{\mathscr{L}}$ is ample, the arithmetic Hilbert-Samuel theorem also permits to relate the asymptotic behaviour (when $n \to +\infty$) of

$$\mathrm{card}(\{s \in H^0(\mathscr{X}, \mathscr{L}^{\otimes n}) \mid \|s\|_{n\varphi} \leqslant 1\})$$

to the arithmetic intersection number of $\overline{\mathscr{L}}$. These results have various applications in arithmetic geometry, such as Vojta's proof of Mordell conjecture [17, 88], equidistribution problem, and Bogomolov conjecture [86, 87, 101],. The arithmetic Hilbert-Samuel theorem has then been reproved in various settings and also been generalized in works such as [1, 48, 82].

### *1.1.4 Adelic Curves*

Recently, a new framework of Arakelov geometry has been proposed in [43], allowing for the consideration of arithmetic geometry over any countable field. Let $K$ be a field. A structure of a proper adelic curve with the underlying field $K$ is given by a family of absolute values $(|\cdot|_\omega)_{\omega\in\Omega}$ of $K$ parametrized by a measure space $(\Omega, \mathcal{A}, \nu)$, which satisfies a product formula of the form

$$\forall\, a \in K^\times, \quad \int_\Omega \ln |a|_\omega \, \nu(\mathrm{d}\omega) = 0.$$

This notion is a very natural generalization of Weil's adelic approach of number theory [89] (see also the work of Chevalley [47] in the case of function fields). The foundation of height theory and Arakelov geometry for projective varieties over an adelic curve has been established in the works of Gubler [65] (in a slightly different setting of $M$-fields) and Chen-Moriwaki [43], respectively; see also the model theoretical approach of Ben Yaacov and Hrushovski [67]. More recently, the arithmetic intersection theory in the setting of adelic curves has been developed in [45].

In general, it is not possible to consider global integral models of an adelic curve. Several classic notions and constructions, such as integral lattice and its covolume, do not have adequate analogue over adelic curves. It turns out that a modified and generalized form of normed lattice—adelic vector bundle—has a natural avatar in this setting. An adelic vector bundle consists of a finite-dimensional vector space $V$ over $K$ equipped with a family of norms $(\|\cdot\|_\omega)_{\omega\in\Omega}$ on vector spaces $V_\omega = V \otimes_K K_\omega$ (where $K_\omega$ denotes the completion of $K$ with respect to the absolute value $|\cdot|_\omega$), which satisfy dominancy and measurability conditions [43, §§4.1.2-4.1.3]. In the framework of adelic curves, adelic vector bundles have been studied in [43, Chapter 4], generalizing previous works of Bost [21, Appendix A] and Gaudron [59] in the classic context of usual global fields.

## 1.2 Results

In this section, we present the main results of the book. We fix a proper adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), (|\cdot|_\omega)_{\omega\in\Omega})$, where $K$ is a field, $(\Omega, \mathcal{A}, \nu)$ is a measure space, and each $|\cdot|_\omega$ is an absolute value on $K$, such that, for any $a \in K^\times = K \setminus \{0\}$, the function $(\omega \in \Omega) \mapsto \log |a|_\omega$ is $\nu$-integrable and of integral 0 on $\Omega$.

### *1.2.1 Arithmetic χ-Volume*

Let $\pi : X \to \operatorname{Spec} K$ be a projective $K$-scheme. For any $\omega \in \Omega$, let $X_\omega = X \times_{\operatorname{Spec} K} \operatorname{Spec} K_\omega$ and let $X_\omega^{\mathrm{an}}$ be the analytic variety associated with $X_\omega$ (in the sense of Berkovich [13] if $|\cdot|_\omega$ is non-Archimedean). If $E$ is a vector bundle on $X$, namely a locally free $\mathcal{O}_X$-module of finite rank, we denote by $E_\omega$ the pull-back of $E$ on $X_\omega$. As an *adelic vector bundle* on $X$, we refer to the data $\overline{E} = (E, (\psi_\omega)_{\omega\in\Omega})$ consisting of a vector bundle $E$ on $X$ and a family $(\psi_\omega)_{\omega\in\Omega}$ of metrics on $E_\omega$ with

$\omega \in \Omega$, such that the tautological line bundle $\mathcal{O}_E(1)$ on $\mathbb{P}(E)$ equipped with the Fubini-Study metric family $\psi^{\mathrm{FS}}$ is composed of continuous metrics satisfying the dominancy and measurability conditions [43, §6.1.2 and §6.1.4]. It turns out that if $X$ is geometrically reduced, then the vector space of global sections $H^0(X, E)$ equipped with supremum norms $(\|\cdot\|_{\psi_\omega})_{\omega\in\Omega}$ forms an adelic vector bundle on the base adelic curve, denoted as $\pi_*(\overline{E})$.

Let $\pi : X \to \operatorname{Spec} K$ be an integral and geometrically reduced projective $K$-scheme of dimension $d$ and $\overline{L} = (L, \varphi)$ be an adelic line bundle on $X$, that is, an adelic vector bundle of rank 1 on $X$. Assume that the line bundle $L$ is ample. We introduce the notion of arithmetic $\chi$-volume as

$$\widehat{\mathrm{vol}}_\chi(\overline{L}) = \limsup_{n\to+\infty} \frac{\widehat{\deg}(\pi_*(\overline{L}^{\otimes n}))}{n^{d+1}/(d+1)!}.$$

In the above formula, the Arakelov degree of the adelic vector bundle

$$\overline{V} = (V, (\|\cdot\|_\omega)_{\omega\in\Omega})$$

is defined as

$$\widehat{\deg}(\overline{V}) := -\int_\Omega \ln \|s_1 \wedge \cdots \wedge s_r\|_{\omega,\det} \, \nu(\mathrm{d}\omega),$$

where $(s_i)_{i=1}^r$ is an arbitrary basis of $E$ over $K$. This notion is a good candidate to replace the Euler-Poincaré characteristic (which is not defined in the adelic curve setting).

In view of the similarity between Arakelov degree and Euler-Poincaré characteristic of Euclidean lattices, the notion of $\chi$-volume is analogous to that of sectional capacity introduced in [83] or to that of volume in [96]. Moreover, similarly to the number field case, we show in Theorem-Definition 4.2.1 that the above superior limit defining the $\chi$-volume is actually a limit. However, from the methodological point of view, we do not follow the classic approaches, which are difficultly implantable in the adelic curve setting. Our strategy consists in casting the Arakelov geometry over an adelic curve to that in the particular case where the adelic curve contains a single copy of the trivial absolute value on $K$, that is, the absolute value $|\cdot|_0$ such that $|a|_0 = 1$ for any $a \in K \setminus \{0\}$. More precisely, to each adelic vector bundle $\overline{V} = (V, (\|\cdot\|_\omega)_{\omega\in\Omega})$, we associate an ultrametric norm $\|\cdot\|_0$ on $V$ (where we consider the trivial absolute value $|\cdot|_0$) via Harder-Narasimhan theory in the form of $\mathbb{R}$-filtrations such that

$$\left|\widehat{\deg}(V, (\|\cdot\|_\omega)_{\omega\in\Omega}) - \widehat{\deg}(V, \|\cdot\|_0)\right| \leqslant \frac{1}{2}\nu(\Omega_\infty)\dim_K(V)\ln(\dim_K(V)),$$

where $\Omega_\infty$ denotes the set of $\omega \in \Omega$ such that $|\cdot|_\omega$ is Archimedean. Then the convergence of the sequence defining $\widehat{\mathrm{vol}}_\chi(\overline{L})$ follows from a limit theorem of normed graded linear series as follows (see Theorem 3.4.3 and Corollary 3.4.4 for this result in a more general form and for more details):

**Theorem A** *Assume that the graded $K$-algebra $\bigoplus_{n\in\mathbb{N}} H^0(X, L^{\otimes n})$ is of finite type. For any integer $n \geqslant 1$, let $\|\cdot\|_n$ be a norm on $H^0(X, L^{\otimes n})$ ($K$ is equipped with the trivial absolute value). Assume that*

(a) $\inf\limits_{s\in V_n\setminus\{0\}} \ln\|s\|_n = O(n)$ *when* $n \to +\infty$,

(b) *for any* $(n, m) \in \mathbb{N}^2_{\geqslant 1}$ *and any* $(s_n, s_m) \in V_n \times V_m$, *one has*

$$\|s_n \cdot s_m\|_{n+m} \leqslant \|s_n\|_n \cdot \|s_m\|_m.$$

*Then the sequence*

$$\Big(\frac{\widehat{\deg}(V_n, \|\cdot\|_n)}{n^{d+1}/(d+1)!}\Big)_{n\in\mathbb{N}_{\geqslant 1}}$$

*converges in* $\mathbb{R}$.

## *1.2.2 Hilbert-Samuel Formulas*

In view of the classic Hilbert-Samuel theorems in algebraic geometry and in Arakelov geometry, it is natural to compare the $\chi$-volume to the arithmetic intersection number of adelic line bundles introduced in [45] (see also the work [65] on heights of varieties over $M$-fields under the assumption of integrability of local heights). Let $\pi : X \to \operatorname{Spec} K$ be a projective $K$-scheme of dimension $d \geqslant 0$ and $\overline{L} = (L, \varphi)$ be an adelic line bundle on $X$ such that $L$ is ample and the metrics in the family $\varphi$ are semi-positive. Then the arithmetic self-intersection number $(\overline{L}^{d+1})$ of $\overline{L}$ is written in a recursive way as

$$\frac{1}{N}\bigg[(\overline{L}|^{d}_{\mathrm{div}(s)}) - \int_{\Omega}\int_{X^{\mathrm{an}}_{\omega}} \ln|s|_{\varphi_\omega}(x)\, c_1(L_\omega, \varphi_\omega)^d(\mathrm{d}x)\,\nu(\mathrm{d}\omega)\bigg], \tag{1.2.1}$$

where $N$ is a positive integer and $s$ is a global section of $L^{\otimes N}$, which intersects properly with all irreducible components of the projective scheme $X$. One of the main results of the book is the following theorem (see Theorem 5.5.1).

**Theorem B** *Let $X$ be an integral projective $K$-scheme. Assume that either $X$ is geometrically integral, or the field $K$ is perfect. Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on $X$ such that $L$ is ample and that all metrics in the family $\varphi$ are semi-positive, then the following equality holds:*

$$\widehat{\mathrm{vol}}_\chi(\overline{L}) = (\overline{L}^{d+1}).$$

Note that in the literature, there exists a local version of the Hilbert-Samuel theorem which establishes an equality between the relative volume of two metrics and the relative Monge-Ampère energy between them. We refer the readers to [14] for the Archimedean case and to [23, 26] for the non-Archimedean case (see also [24]). These results show that, for a fixed ample line bundle $L$ on $X$, the difference between $\widehat{\mathrm{vol}}_\chi(\overline{L})$ and $(\overline{L}^{d+1})$ does not depend on the choice of the metric family on $L$ (see Proposition 5.1.4 and Remark 5.1.6). Moreover, by an argument of the projection to a projective space (on which the arithmetic Hilbert-Samuel theorem can be proved by explicit computation, see Proposition 5.2.5), one can show that the inequality $\widehat{\mathrm{vol}}_\chi(\overline{L}) \geqslant (\overline{L}^{d+1})$ holds (see Step 2 of the proof of Theorem 5.5.1).

In view of the recursive formula (1.2.1) defining the self-intersection number, a natural idea to prove Theorem B could be an argument of induction, following the approach of [1] by using an adaptation to non-Archimedean setting of some techniques of complex analytic geometry developed in [23, 55]. However, it seems that a refinement in the form of an asymptotic expansion of the function defining the local relative volume is necessary to realize this strategy. Unfortunately, such refinement is not yet available.

Our approach consists in casting the arithmetic data of $\overline{L}$ to a series of metrics over a trivially valued field. This could be considered as a higher-dimensional generalization of the approach of Harder-Narasimhan $\mathbb{R}$-filtration mentioned above. What is particular in the trivial valuation case is that the local geometry becomes automatically global, thanks to the trivial "product formula". In this case, the arithmetic Hilbert-Samuel theorem follows from the equality between the relative volume and the relative Monge-Ampère energy with respect to the trivial metric (see Theorem 5.3.2). Note that this result also shows that, in the case of a projective curve over a trivially valued field, the arithmetic intersection number defined in [45] coincides with that constructed in a combinatoric way in [44] (see Remark 5.3.3). The comparison of diverse invariants of $\overline{L}$ with respect to those of its casting to the trivial valuation case provides the opposite inequality $\widehat{\mathrm{vol}}_\chi(\overline{L}) \leqslant (\overline{L}^{d+1})$. As a sequel to the above arguments in terms of trivially valued fields, our way towards the arithmetic Hilbert-Samuel theorem over an adelic curve gives a new approach even for the classic case.

As an application, we prove the following higher-dimensional generalization of Hodge index theorem (see Corollaries 6.5.1 and 6.5.2).

**Theorem C** *Let $X$ be an integral projective $K$-scheme. Assume that either $X$ is geometrically integral, or the field $K$ is perfect. Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on $X$. Assume that $L$ is nef and all metrics in the family $\varphi$ are semi-positive, then the inequality $\widehat{\mathrm{vol}}(\overline{L}) \geqslant (\overline{L}^{d+1})$ holds. In particular, if $(\overline{L}^{d+1}) > 0$, then $L$ is big.*

Theorem B naturally leads to the following refinement of the arithmetic Hilbert-Samuel theorem, by introducing a tensor product by an adelic vector bundle on $X$ (see Corollary 5.5.2). Same as in Theorem B, we assume that either $X$ is geometrically integral, or the field $K$ is perfect.

**Theorem D** *Let* $\overline{L} = (L, \varphi)$ *be an adelic line bundle on* $X$ *and* $\overline{E} = (E, \psi)$ *be an adelic vector bundle on* $X$*. Assume that* $L$ *is ample and the metrics in* $\varphi$ *are semi-positive. Moreover, we suppose that either* $\mathrm{rk}(E) = 1$ *or* $X$ *is normal. Then one has*

$$\lim_{n\to+\infty} \frac{\widehat{\deg}\left(H^0(X, L^{\otimes n} \otimes E), (\|\cdot\|_{n\varphi_\omega+\psi_\omega})_{\omega\in\Omega}\right)}{n^{d+1}/(d+1)!} = \mathrm{rk}(E)(\overline{L}^{d+1}).$$

## *1.2.3 Asymptotic Minimal Slope and Arithmetic Ampleness*

The second part of the book is devoted to the study of positivity conditions of adelic line bundles. Positivity of line bundles is one of the most fundamental and important notions in algebraic geometry. In Arakelov geometry, the analogue of ampleness and Nakai-Moishezon criterion have been studied by Zhang [102, 103]. The arithmetic bigness has been introduced in the works of Moriwaki and Yuan [79, 80, 96]. These positivity conditions and their properties have various applications in Diophantine geometry.

We assume that the underlying field $K$ of the adelic curve $S$ is perfect. Let $X$ be a projective scheme over $\mathrm{Spec}\,K$. Given an adelic line bundle $\overline{L}$ on $X$, we are interested in various positivity conditions of the adelic line bundle $\overline{L}$. We say that the adelic line bundle $\overline{L}$ is *relatively ample* if the invertible $\mathcal{O}_X$-module $L$ is ample and if the metrics of $\overline{L}$ are all semi-positive. The relative nefness can then be defined in a limit form of relative ampleness, similarly to the classic case in algebraic geometry. Recall that the global intersection number of relatively ample adelic line bundles (or more generally, integrable adelic line bundles) can be defined as the integral of local heights along the measure space in the adelic structure (cf. [65], [45, §4.4]). This construction is fundamental in the Arakelov height theory of projective varieties.

We first introduce a numerical invariant—asymptotic minimal slope—to describe the global positivity of an adelic line bundle $\overline{L}$ such that $L$ is ample. This invariant, which is denoted by $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L})$, describes the asymptotic behaviour (when $n \to +\infty$) of the minimal slopes of the sectional spaces $H^0(X, L^{\otimes n})$ equipped with sup norms, which are adelic vector bundles on $S$. It turns out that this invariant is super-additive with respect to $\overline{L}$. This convexity property allows us to extend the construction of the asymptotic minimal slope to the cone of adelic line bundles with nef underlying invertible $\mathcal{O}_X$-module (see Sect. 6.2 for the construction of the asymptotic minimal slope and its properties). The importance of this invariant can be shown by the following height estimate (see Theorem 6.3.2 for its proof and Proposition 6.4.8 for its generalization to the relatively nef case).

**Theorem E** *Assume that the field* $K$ *is perfect. Let* $X$ *be a reduced projective scheme of dimension* $d \geqslant 0$ *over* $\mathrm{Spec}\,K$ *and* $\overline{L}_0, \ldots, \overline{L}_d$ *be a family of relatively* *ample adelic line bundles on X. For any $i \in \{0, \ldots, d\}$, let*

$$\delta_i = (L_0 \cdots L_{i-1} L_{i+1} \cdots L_d).$$

*Then the following inequality holds:*

$$(\overline{L}_0 \cdots \overline{L}_d) \geqslant \sum_{i=0}^{d} \delta_i \, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i),$$

*where $(\overline{L}_0 \cdots \overline{L}_d)$ denotes the arithmetic intersection number of $\overline{L}_0, \ldots, \overline{L}_d$.*

The asymptotic minimal slope always increases if one replaces the adelic line bundle by its pullback by a projective morphism (see Theorem 6.6.6): if $g : X \to P$ is a projective morphism of reduced $K$-schemes of dimension $\geqslant 0$, then for any adelic line bundle $\overline{M}$ on $P$ such that $M$ is nef, one has $\widehat{\mu}_{\min}^{\mathrm{asy}}(g^*(\overline{M})) \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{M})$. Typical situations include a closed embedding of $X$ into a projective space, or a finite covering over a projective space, which allow to obtain lower bounds of $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i)$ in the applications of the above theorem. Note that the particular case where $\overline{L}_0, \ldots, \overline{L}_d$ are all equal to the same adelic line bundle $\overline{L}$ gives the following inequality

$$\frac{(\overline{L}^{d+1})}{(d+1)(L^d)} \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}), \tag{1.2.2}$$

which relates the normalized height of $X$ with respect to $\overline{L}$ and the asymptotic minimal slope of the latter.

This inequality is similar to the first part of [103, Theorem 5.2]. However, the imitation of the devissage argument using the intersection of hypersurfaces defined by small sections would not work in the setting of adelic curves. This is mainly due to the fact that the analogue of Minkowski's first theorem fails for adelic vector bundles on a general adelic curve. Although (in the case where $X$ is an integral scheme) the inequality (1.2.2) could be obtained in an alternative way by the arithmetic Hilbert-Samuel formula of $\overline{L}$ together with the fact that the minimal slope of an adelic vector bundle on $S$ is always bounded from above by its slope (see Proposition 6.7.1), the proof of Theorem E needs a new idea. Our approach consists in combining an analogue of the slope theory of Bost [20, 21] with the height of multi-resultant.

### *1.2.4 Applications in Equidistribution Problem*

By virtue of the relative positivity and the Hilbert-Samuel formula, one has natural applications in equidistribution problems. In Arakelov geometry, equidistribution of algebraic points of small height in an arithmetic projective variety has firstly been studied in the work of Szipro et al. [86] (see also the Bourbaki's seminar review [2] of Abbes), which has a fundamental importance in the resolution of Bogomolov conjecture [87, 101] by Arakelov geometry method (see [49] for another approach to the conjecture by Diophantine geometry).

Let us remind the statement of the arithmetic equidistribution theorem in its classic form. Let $A$ be an abelian variety over a number field and $\overline{L}$ be an adelic line bundle on $A$, which is composed of a symmetric ample line bundle $L$ and a positive adelic metric $\varphi$, such that the Arakelov height function with respect to $\overline{L}$ coincides with the Néron-Tate height. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of algebraic points of $A$ such that the Néron-Tate height of $x_n$ converges to 0 when $n \to +\infty$ (we say that such a sequence is *small*). Then the Zariski closure $X$ of $(x_n)_{n\in\mathbb{N}}$ is the translation of an abelian subvariety of $A$ by a torsion point. Moreover, if in addition any subsequence of $(x_n)_{n\in\mathbb{N}}$ is Zariski closed in $X$, then, for any Archimedean place $\sigma$ of the number field, the Borel measure $\delta_{x_n,\sigma}$ on $X_\sigma(\mathbb{C})$ of taking the average on the Galois orbit of $x_n$ converges weakly to the Monge-Ampère measure $c_1(L_\sigma, \varphi_\sigma)^{\dim(X)}$ on $X_\sigma(\mathbb{C})$.

This equidistribution theorem has then been generalized in various contexts. We refer the readers to [79] for the case where the base field is a finitely generated extension of $\mathbb{Q}$, to [32, 73] for the case of a semi-abelian variety, to [6, 8] for the equidistribution of a small sequence of sub-varieties, to [9, 10, 56] for the case of a dynamical system on a projective line, and to [31] for an equidistribution theorem of a small sequence of algebraic points in the analytic variety over a non-Archimedean place. We also refer to [54, 64] for similar results over function fields. In [96], an arithmetic analogue of Siu's inequality has been proved, which leads to an equidistribution theorem with a weaker condition on the metrics of the adelic line bundle.

We revisit the equidistribution of a small sequence of subvarieties in the setting of Arakelov geometry over an adelic curve. Assume that the underlying field $K$ is countable and perfect. Let $X$ be an integral projective scheme over $\operatorname{Spec} K$ and $d$ be $\dim(X)$. Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on $X$, namely an invertible $\mathcal{O}_X$-module $L$ together with a family $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ of metrics on $L_\omega$ satisfying dominancy and measurability conditions. We assume in addition that $L$ is semiample (namely a tensor power of $L$ is generated by global sections), $\deg_L(X) = (L^d) > 0$, and $\varphi$ is semi-positive. The data $\overline{L}$ permit to construct an arithmetic intersection number $\big(\overline{L}|_Y^{\dim(Y)+1}\big)$ for any integral closed subscheme $Y$ of $X$, which can be written as an integral over $\Omega$ of local intersection numbers. In the case where $\deg_L(Y) = \big(L|_Y^{\dim(Y)}\big) > 0$, the *normalized height* of $Y$ with respect to $\overline{L}$ is defined as

$$h_{\overline{L}}(Y) = \frac{\big(\overline{L}|_Y^{\dim(Y)+1}\big)}{(\dim(Y)+1)\deg_L(Y)}.$$

Let $Y$ be an integral closed subscheme of $X$ such that $\deg_L(Y) > 0$. For any $\omega \in \Omega$, we denote by $\delta_{\overline{L},Y,\omega}$ the Radon measure on $X_\omega^{\mathrm{an}}$ such that for any continuous function $f$ on the analytic space $X_\omega^{\mathrm{an}}$,

$$\int_{X_\omega^{\mathrm{an}}} f(x)\,\delta_{\overline{L},Y,\omega}(\mathrm{d}x) = \frac{1}{\deg_L(Y)} \int_{Y_\omega^{\mathrm{an}}} f(y)\,c_1(L_\omega|_{Y_\omega}, \varphi_\omega|_{Y_\omega})^{\dim(Y)}(\mathrm{d}y).$$

In the case where $|\cdot|_\omega$ is non-Archimedean, the Monge-Ampère measure

$$c_1(L_\omega|_{Y_\omega}, \varphi_\omega|_{Y_\omega})^{\dim(Y)}(\mathrm{d}y)$$

has been constructed in [31, Definition 2.4].

Note that if one modifies the metrics $\varphi_\omega$ for $\omega$ belonging to a set of measure 0, the height of subvarieties of $X$ does not change. However, the local Monge-Ampère measure can be modified by this procedure. Hence, it is not adequate to consider a local equidistribution problem with respect to a single place $\omega$ unless the set $\{\omega\}$ belongs to $\mathcal{A}$ and has a positive measure with respect to $\nu$. We therefore introduce the following global version of Monge-Ampère measure. Let $\Omega'$ be an element of $\mathcal{A}$ such that $\nu(\Omega') > 0$. We denote by $X_{\Omega'}^{\mathrm{an}}$ the disjoint union $\coprod_{\omega\in\Omega'} X_\omega^{\mathrm{an}}$ of local analytifications indexed by $\Omega'$. We equipped this set with a suitable $\sigma$-algebra $\mathcal{B}_{X,\Omega'}$ so that the canonical projection map $X_{\Omega'}^{\mathrm{an}} \to \Omega'$ sending the elements of $X_\omega^{\mathrm{an}}$ to $\omega$ gives a fibration of measurable spaces. It turns out that local Monge-Ampère measures mentioned above form a disintegration of a measure on $(X_{\Omega'}^{\mathrm{an}}, \mathcal{B}_{X,\Omega'})$ over $\nu|_{\Omega'}$: for any integral closed subscheme $Y$ of $X$ such that $\deg_L(Y) > 0$, we denote by $\delta_{\overline{L},Y,\Omega'}$ the measure on $(X_{\Omega'}^{\mathrm{an}}, \mathcal{B}_{X,\Omega'})$ which is defined as

$$\int_{X_{\Omega'}^{\mathrm{an}}} f(x)\,\delta_{\overline{L},Y,\Omega'}(\mathrm{d}x) := \int_{\Omega'} \left( \int_{X_\omega^{\mathrm{an}}} f(x)\,\delta_{\overline{L},Y,\omega}(\mathrm{d}x) \right) \nu(\mathrm{d}\omega).$$

It is worthwhile to say that the global adelic measure determines the local measures almost everywhere, that is, if the global measure $\delta_{\overline{L},Y,\Omega'}$ coincides with another global measure $\delta_{\overline{L}',Y,\Omega'}$, then $\delta_{\overline{L},Y,\omega} = \delta_{\overline{L}',Y,\omega}$ is verified almost everywhere on $\Omega'$ (cf. Proposition 7.7.1). From a functional point of view, one can consider $\delta_{\overline{L},Y,\Omega'}$ as a linear form on the vector space of adelic families of continuous functions on $X$. Denote by $\mathscr{C}_\mathrm{a}^0(X)$ the set of families $f = (f_\omega)_{\omega\in\Omega}$ of continuous functions on $X$ such that $(\mathcal{O}_X, (\mathrm{e}^{-f_\omega}|\cdot|_\omega)_{\omega\in\Omega})$ forms an adelic line bundle on $X$. Note that $f$ yields a measurable function $f_\Omega$ on $X_\Omega^{\mathrm{an}}$ given by $f_\Omega(x) = f_\omega(x)$ for $x \in X_\omega^{\mathrm{an}}$. We denote by $\mathscr{C}_\mathrm{a}^0(X;\Omega')$ the vector subspace of $\mathscr{C}_\mathrm{a}^0(X)$ consisting of $f \in \mathscr{C}_\mathrm{a}^0(X)$ such that $f_\omega = 0$ for any $\omega \in \Omega \setminus \Omega'$. Then

$$(f \in \mathscr{C}_\mathrm{a}^0(X;\Omega')) \longmapsto \int_{X_{\Omega'}^{\mathrm{an}}} f(x)\,\delta_{\overline{L},Y,\Omega'}(\mathrm{d}x)$$

defines a linear functional on $\mathscr{C}_\mathrm{a}^0(X;\Omega')$. One of the main results of the book is the following (see Theorem 8.11.2 and Corollary 8.11.4).

**Theorem F** *Let $X$ be an integral projective scheme of dimension $d$ over* Spec $K$ *and $\overline{L} = (L, \varphi)$ be an adelic line bundle on $X$ such that $L$ is semiample, $(L^d) > 0$, and $\varphi$ is semi-positive. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of closed points of $X$ such that each of its subsequences is Zariski dense in $X$ and that $h_{\overline{L}}(Y_n)$ is well defined and converges to $h_{\overline{L}}(X)$ when $n \to +\infty$. Then, for any $\Omega' \in \mathcal{A}$ such that $\nu(\Omega') > 0$, the sequence of measures $(\delta_{\overline{L},Y_n,\Omega'})_{n\in\mathbb{N}}$, viewed as a sequence of linear functionals on $\mathscr{C}^0_{\mathrm{a}}(X;\Omega')$, converges pointwise to $\delta_{\overline{L},X,\Omega'}$.*

The proof of the theorem is inspired by the original work of Szipro, Ullmo, and Zhang and the subvariety version of Autissier, together with the differentiability interpretation introduced in [37]. The idea relies on the following simple observation. Let $V$ be a real vector space, $x_0$ be an element of $V$, and $f$ and $g$ be two real-valued functions on $V$ such that $f(x) \geqslant g(x)$ for any $x \in V$. Assume $f$ is concave on $V$, $g$ is Gâteaux differentiable at $x_0$, and $f(x_0) = g(x_0)$. Then the function $f$ is also Gâteaux differentiable at $x_0$ and its differential identifies with that of $g$. Concretely in the case of the equidistribution problem, we consider, for any integral closed subscheme $Y$ of $X$ such that $\deg_L(Y) > 0$, the functional $\Phi_Y : \mathscr{C}^0_{\mathrm{a}}(X;\Omega') \to \mathbb{R}$ which sends $f \in \mathscr{C}^0_{\mathrm{a}}(X;\Omega')$ to

$$\widehat{\mu}^{\mathrm{asy}}_{\max}((L, \varphi + f)|_Y) := \lim_{n\to+\infty} \frac{\widehat{\mu}_{\max}\big(H^0(Y, L|_Y^{\otimes n}), (\|\cdot\|_{(n\varphi_\omega+nf_\omega)|_{Y_\omega}})_{\omega\in\Omega}\big)}{n},$$

This functional is concave. Consider now a generic sequence $(Y_n)_{n\in\mathbb{N}}$ of integral closed subschemes of $X$. For any $f \in \mathscr{C}^0_{\mathrm{a}}(X;\Omega')$, let

$$\Phi_{Y_\bullet}(f) := \liminf_{n\to+\infty} \Phi_{Y_n}(f).$$

Since the functionals $\Phi_{Y_n}$ are concave, so is $\Phi_{Y_\bullet}$. The sequence $(Y_n)_{n\in\mathbb{N}}$ being generic, the functional $\Phi_{Y_\bullet}$ is bounded from below by $\Phi_X$ (see Remark 8.10.2). Therefore, if $\Phi_X$ is Gâteaux differentiable at 0 and if $\Phi_{Y_\bullet}(0)$ coincides with $\Phi_X(0)$, then the functional $\Phi_{Y_\bullet}$ is also Gâteaux differentiable at 0. Note that $\Phi_X$ is differentiable at 0 notably when $\Phi_X(0) = h_{\overline{L}}(0)$. Therefore, the particular case where $Y_n$ are closed points leads to Theorem F.

Moreover, the lower bound $\widehat{\mu}^{\mathrm{asy}}_{\max}(L,\varphi)$ of $\Phi_{Y_\bullet}(0)$ is attained by a certain generic sequence $(Y_n)_{n\in\mathbb{N}}$ (see Sect. 8.10). In particular, if the function

$$(f \in \mathscr{C}^0_{\mathrm{a}}(X;\Omega')) \longmapsto \widehat{\mu}^{\mathrm{asy}}_{\max}(L, \varphi + f)$$

is Gâteaux differentiable at 0, then the relation

$$\lim_{n\to+\infty}\lim_{t\to 0+} \frac{\widehat{\mu}^{\mathrm{asy}}_{\max}\big((L,\varphi+tf)|_{Y_n}\big) - \widehat{\mu}^{\mathrm{asy}}_{\max}\big((L,\varphi)|_{Y_n}\big)}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \widehat{\mu}^{\mathrm{asy}}_{\max}(L, \varphi + tf)$$

holds. Note that Theorem F gives a partial answer of [98, Conjecture 5.4.1] by Yuan-Zhang.

### 1.2.5 Bigness and Relative Fujita Approximation

The global adelic space that we use to study the equidistribution problem permits to extend the construction of arithmetic intersection product in allowing one of the adelic line bundles to be possibly not integrable. This construction has applications in the study of weak relative positivity conditions. Bigness is another type of positivity condition which describes the growth of the total graded linear series of a line bundle. In Arakelov geometry of number fields, the arithmetic bigness describes the asymptotic behaviour of the number of small sections in the graded sectional algebra of adelic vector bundles. This notion can be generalized to the setting of Arakelov geometry over adelic curves in replacing the logarithm of the number of small sections by the positive degree of an adelic vector bundle (namely the supremum of the Arakelov degrees of adelic vector subbundles).

In [43, Proposition 6.4.18], the arithmetic bigness is related to an arithmetic sectional invariant—asymptotic maximal slope, which is quite similar to asymptotic minimal slope: for any integral projective $K$-scheme and any adelic line bundle $\overline{L}$ on $X$ such that $L$ is big, we introduce a numerical invariant $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ which describes the asymptotic behaviour (when $n \to +\infty$) of the maximal slopes of $H^0(X, L^{\otimes n})$ equipped with sup norms (see Sect. 8.3 for its construction and properties). It turns out that this invariant is also super-additive with respect to $\overline{L}$, which allows to extend the function $\widehat{\mu}^{\mathrm{asy}}_{\max}(\cdot)$ to the cone of adelic line bundles $\overline{L}$ such that $L$ is pseudo-effective. Moreover, in the case where $L$ is nef, the inequality $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \leqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ holds.

Recall that Fujita's approximation theorem asserts that a big line bundle can be decomposed on a birational modification into the tensor product of two $\mathbb{Q}$-line bundles which are, respectively, ample and effective, with a good approximation of the volume function. In this book, we establish the following relative version of Fujita's approximation theorem for the asymptotic maximal slope (see Theorem 8.5.6 and Corollary 8.5.7).

**Theorem G** *Assume that the field $K$ is perfect and the scheme $X$ is integral. Let $\overline{L}$ be an adelic line bundle on $X$ such that $L$ is big. For any real number $t < \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$, there exist a positive integer $p$, a birational projective $K$-morphism $g : X' \to X$, a relatively ample adelic line bundle $\overline{A}$, and an effective adelic line bundle $\overline{M}$ on $X'$ such that $g^*(\overline{L}^{\otimes p})$ is isomorphic to $\overline{A} \otimes \overline{M}$ and $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) \geqslant pt$.*

As an application, in the case where $X$ is an integral scheme, we can improve the height inequality in Theorem E in relaxing the positivity condition of one of the adelic line bundles and in replacing the asymptotic minimal slope of this adelic line bundle by the asymptotic maximal slope (see Theorem 8.6.1).

**Theorem H** *Assume that the field $K$ is perfect. Let $X$ be an integral projective scheme of dimension $d$ over* Spec $K$ *and $\overline{L}_0, \ldots, \overline{L}_d$ be adelic line bundles on $X$ such that $\overline{L}_1, \ldots, \overline{L}_d$ are relatively ample and $L_0$ is big. For any $i \in \{0, \ldots, d\}$, let*

$\delta_i = (L_0 \cdots L_{i-1} L_{i+1} \cdots L_d)$. *Then the following inequality holds:*

$$(\overline{L}_0 \cdots \overline{L}_d) \geqslant \delta_0\, \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}_0) + \sum_{i=1}^{d} \delta_i\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i).$$

In the case where $\overline{L}_0, \ldots, \overline{L}_d$ are all equal to the same adelic line bundle $\overline{L}$, the above inequality leads to

$$\frac{(\overline{L}^{d+1})}{(L^d)} \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) + d\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}).$$

### *1.2.6 Successive Minima*

In the case where the adelic curve $S$ comes from the canonical adelic structure of a number field, if $\overline{L}$ is a relatively ample adelic line bundle, then $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})$ is equal to the absolute minimum of the Arakelov (absolute) height function $h_{\overline{L}}(\cdot)$ on the set of closed points of $X$. This is essentially a consequence of [103, Corollary 5.7]. Similarly, the asymptotic maximal slope $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ is equal to the essential minimum of the height function $h_{\overline{L}}$. This is a result of Ballaÿ [11, Theorem 1.1]. In this book, we show that these results can be extended to the case of general adelic curves if we consider the heights of all integral closed subschemes of $X$. More precisely, we obtain the following result (see Theorem 8.8.3 and Proposition 8.10.1).

**Theorem I** *Assume that the field $K$ is perfect. Let $X$ be a non-empty reduced projective scheme over* Spec $K$ *and $\Theta_X$ be the set of integral closed subschemes of $X$. For any relatively ample adelic line bundle $\overline{L}$ on $X$, the following equalities hold:*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) = \inf_{Y \in \Theta_X} \frac{(\overline{L}|_Y^{\dim(Y)+1})}{(\dim(Y)+1)(L|_Y^{\dim(Y)})} = \inf_{Y \in \Theta_X} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Y).$$

*Moreover, if $X$ is an integral scheme, the following equality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) = \sup_{\substack{Y \in \Theta_X \\ Y \neq X}} \inf_{\substack{Z \in \Theta_X \\ Z \not\subseteq Y}} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Z).$$

We also show that a property similar to Minkowski's first theorem permits to recover the link between the asymptotic maximal and minimal slopes, and the Arakelov height of closed points in the number field case. More precisely, we say that a relatively ample adelic line bundle $\overline{L}$ is *strongly Minkowskian* if for any $Y \in \Theta_X$ one has

$$\lim_{n\to+\infty}\frac{1}{n}\sup_{\substack{s\in H^0(Y,L|_Y^{\otimes n})\\ s\neq 0}}\widehat{\deg}(s)\geqslant\frac{(\overline{L}|_Y^{\dim(Y)+1})}{(\dim(Y)+1)(L|_Y^{\dim(Y)})}.$$

This condition is always satisfied notably when the adelic curve $S$ comes from a number field (consequence of Minkowski's first theorem) or the function field of a projective curve (consequence of Riemann-Roch theorem). We then establish the following result (see Corollary 8.9.3).

**Theorem J** *Assume that the field $K$ is perfect. Let $X$ be an integral projective scheme over* Spec $K$ *and $\overline{L}$ be a relatively ample adelic line bundle on $X$ which is strongly Minkowskian. Denote by $X^{(0)}$ the set of closed points of $X$. Then the equality $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L})=\inf_{x\in X^{(0)}}h_{\overline{L}}(x)$ holds.*

Motivated by Theorem I, we propose the following analogue of successive minima for relatively ample adelic line bundles. Let $f: X\to \operatorname{Spec} K$ be an integral projective $K$-scheme of dimension $d$ and $\overline{L}$ be a relatively ample adelic line bundle on $X$. For $i\in\{1,\ldots,d+1\}$, let

$$e_i(\overline{L})=\sup_{\substack{Y\subseteq X\text{ closed}\\ \operatorname{codim}(Y)\geqslant i}}\ \inf_{\substack{Z\in\Theta_X\\ Z\not\subseteq Y}}\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Z).$$

With this notation, one can rewrite the assertion of Theorem I as

$$e_1(\overline{L})=\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}),\quad e_{d+1}(\overline{L})=\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}).$$

We show in Remark 8.10.3 that, in the number field case, one has

$$\forall\, i\in\{1,\ldots,d+1\},\quad e_i(\overline{L})=\sup_{\substack{Y\subseteq X\text{ closed}\\ \operatorname{codim}(Y)\geqslant i}}\ \inf_{x\in(X\setminus Y)^{(0)}}h_{\overline{L}}(x).\tag{1.2.3}$$

Thus, we recover the definition of successive minima in the sense of [102, §5]. We propose several fundamental questions about these invariants:

(1) Do the equalities (1.2.3) hold in the case of a general adelic curve, under the assumption that $\overline{L}$ is strongly Minkowskian?
(2) What is the relation between the invariants $e_2(\overline{L}),\ldots,e_d(\overline{L})$ and the sectional algebra $\bigoplus_{n\in\mathbb{N}}f_*(\overline{L}^{\otimes n})$?
(3) Does the analogue of some classic results in Diophantine geometry concerning the successive minima, such as the inequality

$$\frac{(\overline{L}^{d+1})}{(L^d)}\geqslant\sum_{i=1}^{d+1}e_i(\overline{L}),$$

still hold for general adelic curve?

(4) In the case where $(X, L)$ is a polarized toric variety and the metrics in $\varphi$ are toric metrics (namely metrics that are stable by the action of the torus), is it possible to describe in a combinatoric way the positivity conditions of $\overline{L}$ and express the invariants $e_i(\overline{L})$ in terms of the combinatoric data of $(X, \overline{L})$, generalizing some results of [27, 28], for example?

### *1.2.7 Global Positivity and Applications*

The last chapter of the book is devoted to the study of global positivity of adelic line bundles. Motivated by Nakai-Moishezon criterion of ampleness, we say that an adelic line bundle $\overline{L}$ on $X$ is *ample* if it is relatively ample and if the normalized height with respect to $\overline{L}$ of integral closed subschemes of $X$ has a positive lower bound. We show that this condition is equivalent to the relative ampleness together with the positivity of the invariant $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L})$. Therefore, we deduced from Theorem E that if $\overline{L}_0, \ldots, \overline{L}_d$ are ample adelic line bundles on $X$, where $d$ is the dimension of $X$, then one has (see Proposition 9.1.3)

$$(\overline{L}_0 \cdots \overline{L}_d) > 0.$$

In the case where $\overline{L}$ is strongly Minkowskian, $\overline{L}$ is ample if and only if it is relatively ample and the height function $h_{\overline{L}}$ on the set of closed points of $X$ has a positive lower bound (see Proposition 9.1.4). Once the ample cone is specified, one can naturally define the nef cone as its closure. It turns out that the nefness can also be described in a numerical way: an adelic line bundle $\overline{L}$ is nef if and only if it is relatively nef and $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) \geqslant 0$ (see Proposition 9.1.7).

Bigness and pseudo-effectivity are also described in a numerical way by the invariant $\widehat{\mu}_{\max}^{\mathrm{asy}}(\cdot)$: an adelic line bundle $\overline{L}$ is big if and only if $L$ is big and $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) > 0$ (which coincides with the bigness in [43]); it is pseudo-effective if and only if $L$ is pseudo-effective and $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) \geqslant 0$ (see [43, Proposition 6.4.18] and Proposition 9.2.5). We deduce from Theorem H that if $\overline{L}_0, \ldots, \overline{L}_d$ are adelic line bundles on $X$ such that $\overline{L}_0$ is pseudo-effective and that $\overline{L}_1, \ldots, \overline{L}_d$ are nef, then the inequality $(\overline{L}_0, \ldots, \overline{L}_d) \geqslant 0$ holds (see Proposition 9.2.6).

As an application of the equidistribution theorem together with the global positivity properties of adelic line bundles, we consider Bogomolov conjecture over a countable field of characteristic zero (see Theorem 9.4.1). We assume that $K$ is an algebraically closed field of characteristic zero, $\nu(\Omega_\infty) > 0$, and $\nu(\mathcal{A}) \not\subseteq \{0, +\infty\}$. The following theorem is a generalization of [79, Theorem 8.1].

**Theorem K** *Let $A$ be an abelian variety over $K$, $L$ be an ample and symmetric line bundle on $A$, and $\varphi$ be a family of semi-positive metrics of $A$ such that $(A, \varphi)$ is nef and $\varphi_\omega$ is the canonical metric of $L_\omega$ for each $\omega \in \Omega$. If the essential minimum of $(L, \varphi)|_X$ is zero, then $X$ is a translation of an abelian subvariety of $A$ by a closed* *point of Néron-Tate height* 0*, which is a torsion point provided that any finitely generated subfield of* $K$ *has Northcott's property (cf. [45, Theorem 2.7.18]).*

We also discuss arithmetic dynamical systems in the adelic curve setting. We assume that $K$ is algebraically closed. Let $X$ be a projective integral scheme over $\operatorname{Spec} K$ and $L$ be an ample line bundle on $X$. We denote by $\operatorname{End}(X; L)$ the set of all endomorphisms $f : X \to X$ such that $f^*(L)$ is isomorphic to a tensor power of $L$ with exponent $> 1$. Note that $f$ forms a polarized dynamical system with respect to $L$. For any $f \in \operatorname{End}(X; L)$ with $f^*(L) \cong L^{\otimes d}$ for some $d > 1$, there exists a unique metric family $\varphi_f$ such that $(L, \varphi_f)$ forms an adelic line bundle and $f^*(\overline{L})$ is isometric to $\overline{L}^{\otimes d}$. We call it the *global canonical compactification* of $L$. It is easy to see that any $f$-preperiodic rational point of $X$ is of height 0. The converse is also true if the adelic curve $S$ has Northcott property. We establish the following result (see Theorem 9.5.1).

**Theorem L** *Let* $L$ *be an ample line bundle on* $X$ *and* $f$ *and* $g$ *be two elements of* $\operatorname{End}(X; L)$*. Then the following statements are equivalent:*

(1) *The adelic line bundles* $(L, \varphi_f)$ *and* $(L, \varphi_g)$ *define the same height function on the set of rational points of* $X$*.*
(2) $\{x \in X(K) \mid h_{(L,\varphi_f)}(x) = 0\} = \{x \in X(K) \mid h_{(L,\varphi_g)}(x) = 0\}$.
(3) $\{x \in X(K) \mid h_{(L,\varphi_f)}(x) = h_{(L,\varphi_g)}(x) = 0\}$ *is Zariski dense in* $X(K)$*.*

*Moreover, when these conditions are satisfied, there exist an integrable function* $\ell$ *on* $\Omega$ *and* $\Omega' \in \mathcal{A}$ *such that* $\nu(\Omega \setminus \Omega') = 0$ *and that*

$$\forall\, \omega \in \Omega', \quad \varphi_{g,\omega} = \mathrm{e}^{\ell(\omega)} \varphi_{f,\omega}.$$

### *1.2.8 Organization of the Book*

The rest of the book is organized as follows:

In Chap. 2, we consider metric families on vector bundles and discuss their dominancy and measurability. We also remind the notation that we use all through the book.

In Chap. 3, we study normed graded linear series over a trivially valued field and prove the limit theorem of their volumes. Then in Chap. 4, we deduce the limit theorem for graded algebra of adelic vector bundles over a general adelic curve, which proves in particular that the sequence defining the arithmetic volume function actually converges. We also show that the arithmetic Hilbert-Samuel theorem in the original form implies the generalized form with tensor product by an adelic vector bundle.

In Chap. 5, we prove the arithmetic Hilbert-Samuel theorem. We first prove that the difference of the arithmetic $\chi$-volume and the arithmetic intersection product does not depend on the choice of the metric family. Then we prove the arithmetic