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51

Pablo Amster
Pierluigi Benevieri
Editors

Topological Methods for Delay and Ordinary Differential Equations

With Applications to Continuum Mechanics

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Editors

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Preface

Continuum Mechanics is a set of space-time field theories, each a class of mathematical models that are representations of natural phenomena involving bodies extended in space and considered at a scale in which the quantum structure of the matter can be reasonably left momentarily apart. The theory of isothermal elasticity of simple bodies is a paradigmatic example; the adjective “simple” is used here in W. Noll’s sense, meaning that the energy depends only on the deformation gradient. The story does not end here, of course, because Continuum Mechanics has a very broad horizon that includes even the general relativity theory. With its polymorphic nature, Continuum Mechanics is in itself a playground for researchers in mathematical analysis, differential geometry, and numerical analysis, not only in terms of applications of abstract results but—and perhaps above all—for the challenging problems frequently offered by the setting. When we look at evolutionary processes, be them described by hyperbolic or parabolic partial differential equations, and think of them in terms of numerical methods, after discretization in space, we have in the hands a system of (typically nonlinear) ordinary differential equations, an analytical structure pertaining also to lattice-type systems. Such equations (typically nonlinear and at times integro-differential in time, namely when memory effects are involved) may be analyzed at least qualitatively per se, besides proceeding with a discretization in time and the pertinent integration algorithms. When we opt for this aspect, topological methods enter into play as essential tools.

Topology has been a key tool in the study of dynamical systems since its birth in the nineteenth century with the celebrated Henri Poincare’s works on celestial mechanics. Also, topological methods are essential in classifying and discussing defect structures at microscopic scales in solids or qualitatively describing peculiar features of flows pertaining to simple or complex fluids. Pertinent techniques include fixed point theory, the analysis of periodic orbits and/or chaotic attractors, and so on. Here, we focus attention on evolutionary phenomena and look at nonlinear ordinary and delay differential equations, with emphasis in the applications to mechanics and biological sciences. The book is designed for a wide audience, however interested in differential equations, functional analysis, topology and related topics, as well as in the applied sciences.

Specifically, Chap. 1 offers a clear example of how topological methods help in the analysis of ordinary differential equations. At first the focus is mainly on the history of ideas, with special attention to the celebrated Poincaré-Birkhoff theorem, from its precursors to recent developments. Then, the chapter deals with some results concerning the multiplicity of periodic solutions to a special class of Hamiltonian systems with symmetries.

Chapter 2 discusses Kepler's problem in the plane in two different relativistic versions. First, the relativistic differential operator is involved. Then, the same operator appears as a correction to the usual gravitational potential as suggested by Tullio Levi-Civita. Bifurcation results of periodic solutions for the two relativistic problems are then discussed. The approach rests on an abstract bifurcation theory developed by Alan Weinstein, applied in the case of nearly integrable Hamiltonian systems that satisfy the usual KAM isoenergetic non-degeneracy condition.

Chapter 3 collects recent results on the existence of T-periodic solutions for the relativistic pendulum for both conservative and damped circumstances. The non-variational structure of the second case implies in the application of topological degree methods a crucial difference between classical and relativistic frameworks.

The topic of T-periodic solutions reappears also in Chap. 4, dedicated to existence, multiplicity, and stability problems for a large class of micro-electro-mechanical parallel plate actuators, whose motion is governed by a Duffing equation. Mathematical tools include the method of upper and lower solutions, averaging theory and a method proposed by Rafael Ortega.

Chapter 5 deals with the analysis of chemostats linked in series. Assuming that the input of limiting nutrients is regulated by a T-periodic function, at least one T-periodic solution is obtained by means of the Poincaré translation operator. In particular, it is proven that the chain of chemostats guarantees the periodical coexistence of the microbial species.

Chapter 6 discusses a parameter-dependent thermostat problem subjected to deviated arguments. Temperature steady-states in a heated bar of finite length satisfy a second-order nonlinear differential equation. Solvability of the system rests on a recent Birkhoff-Kellogg type theorem in affine cones.

Chapter 7 deals with the analysis of forced oscillations described by non-linear implicit ordinary differential equations involving a generalized ϕ-Laplacian type differential operator. These analytical structures appear in non-Newtonian fluid-dynamics, nonlinear elasticity, diffusion of flows in porous media, theory of capillarity, and glaciology. The chapter illustrates bifurcation properties of periodic solutions, following a topological approach based on Brouwer's degree.

Chapter 8 deals also with bifurcation, now concerning a class of neutral delay differential equations, depending on a real parameter. The approach is based on a concept of oriented degree for noncompact perturbations of Fredholm maps with index zero between Banach spaces. The general result is applied to analyze Mackey–Glass-type and Duffing-type equations.

Finally, Chap. 9 shows how to combine topological analysis in phase space with dynamical systems theory. For deterministic systems, the topological structure of a flow is an invariant that provides information on the mechanisms acting in phase space to shape a flow. Appropriate topological analysis involves finding a topological representation of the underlying structure and obtaining an algebraic description, which may allow the computation of topological invariants. Developing methods to accomplish these two steps has involved several false starts as well as partially successful roads. Eventually, the attempts lead to what it is nowadays called a *templex*. The chapter introduces readers to these concepts, with emphasis on flows in phase space, the different manners of describing their topological structure, and the recently developed tools. Examples using ordinary and delay differential equations are presented.

We gratefully acknowledge Professor Paolo Maria Mariano of the University of Florence, Italy, for the invitation to edit this volume and the support he gave us throughout our work. We also thank the staff of Springer-Birkauser for the help we received and the solution to all the technical problems we encountered. In particular, we would like to mention Saveetha Balasundaram, Jeffrey Taub, Britta Rao, and Chris Eder.

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Chapter 1
Periodic Solutions of Hamiltonian Systems with Symmetries

Alessandro Fonda

Mathematics Subject Classification 34C25

1.1 A Historical Introduction

In this section we will provide a brief historical account on the development of the theory of periodic solutions of Hamiltonian systems that originated from the celebrated Poincaré–Birkhoff Theorem. We will start from the original conjecture by Poincaré, concerning fixed points of a planar area-preserving homeomorphism, and follow the efforts done to prove it and generalize it over more than one hundred years so to obtain a powerful tool to be used in the applications. We will try to explain the relation with the Lusternik–Schnirelmann theory and its generalizations, which have been used to extend the theorem to higher dimensions, so to provide multiple periodic solutions for Hamiltonian systems. Finally, we will emphasize a recent result of the author with R. Ortega on a two-point boundary value problem, which however is strongly related with the Poincaré–Birkhoff Theorem, and which we will then use to prove the multiplicity of periodic solutions to some Hamiltonian systems with symmetries.

1.1.1 The Poincaré–Birkhoff Theorem

Among the most influential mathematicians in history, we acknowledge Jules Henri Poincaré, who passed away 112 years ago, on July 17, 1912. Just three months before his death, in the May 1912 issue of the *Rendiconti del Circolo Matematico di Palermo*, he published his paper "Sur un théorème de géométrie," which asserts

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the existence of at least two fixed points for an area-preserving homeomorphism of a planar circular annulus onto itself, such that the points of the inner circle Γ_1 are moved along Γ_1 in the clockwise sense and the points of the outer circle Γ_2 are moved along Γ_2 in the counter-clockwise sense [103].

However, Poincaré was not able to prove this result, and he tried to justify himself with these words (in our translation):

> *I have never presented such an incomplete work to the public; therefore, I think it necessary to briefly explain the reasons which convinced me to publish it, and, above all, those which drove me to start it. I have already proved in the past the existence of periodic solutions for the three body problem; however, the result was still unsatisfactory [...] While thinking at this problem, I convinced myself that the answer should depend on the truth or falseness of a geometric theorem [...] So, I was led to research the veracity of this theorem, but I met some unexpected difficulties [...] It seems that, in such a situation, I should refrain from any publication until I have solved the problem; but, after all the pointless efforts made over many months, I thought that the wiser choice was to leave the problem to mature, while resting for some years; this would have been fine if I had been sure to be able to take it up again one day; but at my age I cannot be so sure. On the other hand, the importance of the subject is too great and the quantity of results so far obtained too considerable, to resign myself to let them definitively unfruitful [...] I think that these considerations are sufficient to justify me.*

The existence of one fixed point was proved by George David Birkhoff [10] the year later, in 1913, while the proof of the existence of a second fixed point was provided by Birkhoff himself [12] only in 1925 (see also [14, 26] for a modern exposition). Since then, the "théorème de géométrie" is known as the Poincaré–Birkhoff Theorem.

Applications of the Poincaré–Birkhoff Theorem to dynamical systems coming from nonlinear mechanics and geometry were already suggested by Poincaré in [103] and studied by Birkhoff in [11, 13]. As mentioned by Zehnder in [127], Arnold considered this theorem as "the seed of symplectic topology."

1.1.2 First Extensions in the Planar Case

In the case of planar Hamiltonian systems one often looks for the existence of periodic solutions as fixed points of the Poincaré map. However, a major difficulty in the application of the Poincaré–Birkhoff Theorem is the construction of invariant annular regions. Hence, a modification of the theorem not assuming the invariance conditions for the annulus and its inner and outer boundaries became necessary for the applications.

In 1976, Jacobowitz [82, 83], following a suggestion of Moser [99], proposed a modified version of the Poincaré–Birkhoff Theorem for a topological pointed disc, showing how to apply it to the search of periodic solutions to some superlinear second-order differential equations. Applications in this direction were also given by Hartman [79] and Butler [27]. Based on Jacobowitz's result, W.-Y. Ding [41, 42] obtained a new version of the Poincaré–Birkhoff Theorem where the boundary invariance assumption was removed. Moreover, in [42] the circular annulus was replaced by a topological annulus whose inner and outer boundaries Γ_1 and Γ_2 are Jordan curves, assuming only Γ_1 to be star-shaped. It has been shown by Martins and Ureña in [94] that this star-shapedness assumption is not eliminable. Later on, Le Calvez and Wang in [88] provided an example showing that a star-shapedness assumption on the outer boundary is also needed, thus proving that Ding's theorem, hence most probably also Jacobowitz's, was not correct.

Meanwhile, Ding's theorem had been used by many authors to prove existence and multiplicity of periodic solutions of some nonautonomous Hamiltonian systems. See, e.g., [15, 16, 20, 22, 27, 28, 36–40, 46, 48, 51, 52, 56, 62, 74, 80, 82, 93, 104, 105, 109–112, 124–126]. Fortunately, all these papers deal with annuli for which both boundaries are star-shaped, and Rebelo [109] was able to save the situation proving a safer version of the Poincaré–Birkhoff Theorem where such an assumption was made. We refer to [61, 87] for further references and a complete historical account until the year 2012, the centennial of Poincaré's original paper.

Let us now state a modern version of the Poincaré–Birkhoff Theorem providing the existence and multiplicity of periodic solutions to a planar Hamiltonian system

$$x' = \partial_y H(t, x, y)\,, \qquad y' = -\partial_x H(t, x, y)\,, \tag{1.1}$$

where $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, with continuous partial derivatives $\partial_x H(t, x, y)$ and $\partial_y H(t, x, y)$. The following theorem is contained in [72].

Theorem 1.1 *Assume $H(t, x, y)$ to be T-periodic in t for some $T > 0$, and τ-periodic with respect to x, for some $\tau > 0$. Let $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}$ be two continuous and T-periodic functions, with $\gamma_1(x) < \gamma_2(x)$ for every $x \in \mathbb{R}$, satisfying the following property: All solutions (x, y) of (1.1) starting with $y(0) \in [\gamma_1(x(0)), \gamma_2(x(0))]$ are defined on $[0, T]$ and are such that*

$$\begin{cases} y(0) = \gamma_1(x(0)) & \Rightarrow \quad x(T) - x(0) < 0\,, \\ y(0) = \gamma_2(x(0)) & \Rightarrow \quad x(T) - x(0) > 0\,. \end{cases} \tag{1.2}$$

Then, system (1.1) has at least two geometrically distinct T-periodic solutions (x, y), with $y(0) \in]\gamma_1(x(0)), \gamma_2(x(0))[$.

Let us specify what we mean by *geometrically distinct* solutions. By the periodicity assumption, once a solution $z(t) = (x(t), y(t))$ of (1.1) has been found, infinitely many others appear by just adding an integer multiple of τ to $x(t)$. We will call *geometrically distinct* two solutions that cannot be obtained from each other in this way.

Assumption (1.2) is usually called *twist condition*, and the same conclusion also holds if the inequalities in (1.2) are reversed. The original Poincaré's setting on an annulus can be recovered by a change of variables, choosing some suitable polar coordinates. Notice that in Theorem 1.1 no uniqueness assumption is made for the solutions of initial value problems (hence the Poincaré map could be multivalued).

As an illustrative example, consider the planar system

$$x' = \psi(y + E(t)), \qquad y' = g(x), \tag{1.3}$$

where all functions involved are continuous on the whole real line $\mathbb{R}$.

Corollary 1.1 *Let the following sign assumption hold.*

$$\exists d > 0 : \quad |\sigma| \geq d \quad \Rightarrow \quad \sigma\psi(\sigma) > 0.$$

Moreover, let the function $E(t)$ be T-periodic, and the function $g(x)$ be 2π-periodic, with $\int_0^{2\pi} g(s)\, ds = 0$. Then, system (1.3) has at least two geometrically distinct T-periodic solutions.

Proof Since $g(x)$ is bounded, Theorem 1.1 directly applies, taking as constant the functions $\gamma_1(x) = -R$ and $\gamma_2(x) = R$, with $R > 0$ large enough. □

Corollary 1.2 *Let $\phi : I \to \mathbb{R}$ be an increasing homeomorphism, with I an open interval containing 0, and $\phi(0) = 0$. Moreover, let the function $g(x)$ be continuous and 2π-periodic, with $\int_0^{2\pi} g(s)\, ds = 0$, and the function $e(t)$ be T-periodic, with $\int_0^T e(t)\, dt = 0$. Then, the equation*

$$(\phi(x'))' = g(x) + e(t) \tag{1.4}$$

has at least two geometrically distinct T-periodic solutions.

Proof Setting $\psi = \phi^{-1}$ and $E(t) = \int_0^t e(\tau)\, d\tau$, equation (1.4) can be written in the form of system (1.3), and Corollary 1.1 directly applies. □

The case $\phi(y) = y$ was first proved in [97] by a variational method, taking as a model the forced pendulum equation. For the relativistic pendulum, when $\phi(y) = y/\sqrt{1 - y^2}$, the result has been proved in [7, 25]. See also [65].

1.1.3 The Relation with Lusternik–Schnirelmann Theory

The theory of Lusternik and Schnirelmann was first published in Russian in 1930. The French translation [92] appeared in the "Exposés sur l'analyse mathématique et ses applications," published under the direction of J. Hadamard, who introduces it with the following words (our translation):

> [...] *we will admire the novelty and breadth of views, the power and fecundity of ideas expressed. We considered it appropriate not to allow the reader ignore a work of this value.*

As the authors say in their introduction, they were motivated by some problems raised by Poincaré in a field connecting Analysis and Topology, a domain where the most important advances at that time had been achieved by Birkhoff.

In the following years, the ideas of Lusternik and Schnirelmann were extended and generalized in several directions. In 1964, Schwartz [114] provided a first infinite-dimensional version of the theory, thus starting the development of variational methods, in view of the applications to different boundary value problems. Indeed, both ODEs and PDEs could be handled using those methods, providing several multiplicity results. In 1978, Rabinowitz [107] showed us how the periodic problem for a Hamiltonian system could be treated using a variational method. In this case, a major difficulty lies in the fact that the associated functional is strongly indefinite.

On the other hand, there have been many attempts to generalize the Poincaré–Birkhoff Theorem to higher dimensions starting with Birkhoff himself, who considered this an *outstanding question* [12, 13]. In the sixties Arnold proposed some famous conjectures, some of which are still open problems. Since then, various higher-dimensional versions of the Poincaré–Birkhoff Theorem were claimed, for maps that are close to the identity and also for monotone twist maps [2, 3, 24, 100–102, 123].

Using a different approach, Conley and Zehnder [34, Theorem 3] proved another possible version of the Poincaré–Birkhoff Theorem in higher dimensions. They obtained the multiplicity of periodic solutions for a Hamiltonian system assuming that the C^2-smooth Hamiltonian function $H = H(t, x, y)$ is periodic in t and in the space variables x_k, and quadratic in y on a neighborhood of infinity. Precisely, they assumed

$$|y| \geq R \quad \Rightarrow \quad H(t, x, y) = \frac{1}{2}\langle \mathbb{B}y, y\rangle + \langle a, y\rangle \,, \tag{1.5}$$

for some $R > 0$, some vector $a \in \mathbb{R}^N$, and some regular symmetric matrix $\mathbb{B}$. Remarkably, their result does not need the Poincaré time map to be close to the identity, nor to have a monotone twist.

The development of infinite-dimensional Lusternik–Schnirelmann methods would allow Szulkin [115, Theorem 4.2] to generalize the Conley and Zehnder theorem by replacing the term $\langle a, y\rangle$ by nonlinearities $G(t, x, y)$ with bounded first-order derivatives. Further results along these lines can be found in [8, 9, 31, 47, 57, 58, 75–77, 81, 84, 85, 89, 95, 96, 98, 108, 116].

1.1.4 A Higher-Dimensional Poincaré–Birkhoff Theorem for Hamiltonian Systems

Despite all this ample literature, it seems that, for the time being, there is still no genuine generalization of the Poincaré–Birkhoff theorem to higher dimensions. However, in 2017, the author together with A.J. Ureña proved a higher-dimensional version of the Poincaré–Birkhoff Theorem while considering the Hamiltonian system

$$x' = \nabla_y H(t, x, y), \qquad y' = -\nabla_x H(t, x, y), \tag{1.6}$$

where $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is a continuous function, with continuous partial gradients $\nabla_x H(t, x, y)$ and $\nabla_y H(t, x, y)$. Here,

$$x = (x_1, \dots, x_N), \qquad y = (y_1, \dots, y_N).$$

Let us see how Theorem 1.1 can be generalized in the simpler case when $\gamma_1(x)$ and $\gamma_2(x)$ are constant functions.

Theorem 1.2 *Let $H(t, x, y)$ be T-periodic in t for some $T > 0$, and that, for every $k = 1, \dots, N$, it is τ_k-periodic with respect to the variable x_k, for some $\tau_k > 0$. Let $\mathscr{D}$ be a convex body in $\mathbb{R}^N$ with a smooth boundary, and assume that all the solutions (x, y) of (1.6) starting with $y(0) \in \mathscr{D}$ are defined on $[0, T]$ and are such that*

$$y(0) \in \partial\mathscr{D} \quad \Rightarrow \quad \langle x(T) - x(0), \nu_{\mathscr{D}}(y(0))\rangle > 0. \tag{1.7}$$

Then, system (1.6) has at least $N + 1$ geometrically distinct T-periodic solutions (x, y), with $y(0) \in \mathrm{int}(\mathscr{D})$.

In the above statement, $\nu_{\mathscr{D}}(y)$ denotes the outward unit normal vector to $\mathscr{D}$, for any $y \in \partial\mathscr{D}$. Theorem 1.2 extends Theorem 1.1, taking $\mathscr{D} = [\gamma_1, \gamma_2]$. The twist condition (1.7) can be further generalized in several different directions (see, e.g., [33, 53, 54, 72, 73] for the details).

This result has already found several applications (see [17, 19, 23, 29, 32, 35, 44, 45, 49, 50, 63, 64, 66–71, 78, 90, 91, 106, 118–122]). It was applied to vortex dynamics [5] and to Keplerian dynamics [18]. Finally, it has also been extended to infinite-dimensional Hamiltonian systems [21, 55].

1.1.5 A New Functional Setting

A crucial step in the application of variational methods is the choice of the space where to define the functional to be studied. For the T-periodic problem associated with the Hamiltonian system (1.6), the natural space seems to be the one made of those functions $z = (x, y)$ with both x and y belonging to $H_T^{1/2}$.

Recently the author jointly with R. Ortega [60] has obtained the following multiplicity result for a two-point boundary value problem associated with system (1.6), precisely

$$y(a) = 0 = y(b). \tag{1.8}$$

Theorem 1.3 *Let $H : [a, b] \times \mathbb{R}^{2N} \to \mathbb{R}$ be a continuous function, with continuous partial gradients $\nabla_x H(t, x, y)$ and $\nabla_y H(t, x, y)$, and assume that, for every $k =$*

$1, \dots, N$*, it is* τ_k*-periodic with respect to the variable* x_k*, for some* $\tau_k > 0$*. Assume moreover that all solutions of* (1.6) *starting with* $y(a) = 0$ *are defined on* $[a, b]$*. Then, problem* (1.6)–(1.8) *has at least* $N + 1$ *geometrically distinct solutions.*

Surprisingly enough, *no twist condition is assumed* in the above statement. The main novelty in the proof of Theorem 1.3 lies in the fact that while for the periodic problem x and y are usually both taken in the same space $H_T^{1/2}$, in [60] we have assumed x and y to belong to some complementary spaces, which are closely related to fractional Sobolev spaces.

When the Hamiltonian function has the special form $H(t, x, y) = \frac{1}{2}|y|^2 + G(t, x)$, problem (1.6)–(1.8) becomes a Neumann boundary value problem for a second-order differential equation. We can find a multiplicity result for the Neumann problem by Castro [30] in 1980 and a similar one by Rabinowitz [108] in 1988. Both papers use variational methods.

The aim of this chapter is to provide the existence and multiplicity of T-periodic solutions for a Hamiltonian system of the type (1.6) whose Hamiltonian function is T-periodic in t and presents some symmetries. More precisely, we will assume $H(t, x, y)$ to be even, both in t and in y. In this setting, no twist condition will be needed.

We will then extend our result to infinite-dimensional systems, passing through a finite-dimensional approximation. However, the passage to the limit in the dimension will eventually only guarantee the existence of one periodic solution. We refer to [4, 21, 43, 55, 59, 86, 113] for related results in infinite dimension.

1.2 Finite-Dimensional Systems

1.2.1 The Main Result

We consider the Hamiltonian system (1.6), where $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is a continuous function, with continuous partial gradients $\nabla_x H(t, x, y)$ and $\nabla_y H(t, x, y)$. As above, we use the notation $z = (x, y)$, with

$$x = (x_1, \dots, x_N), \qquad y = (y_1, \dots, y_N).$$

Here are our assumptions:

(A_1) *The function* $H(t, x, y)$ *is* T*-periodic in the variable* t*, for some* $T > 0$.

(A_2) *For every* $k \in \{1, \dots, N\}$*, there is a* $\tau_k > 0$ *such that the function* $H(t, x, y)$ *is* τ_k*-periodic in the variable* x_k .

Let us now introduce the notation

$$\mathbb{T}^N = \prod_{k=1}^{N} [0, \tau_k].$$

In view of (A_2), the following assumptions will be made only for those x belonging to $\mathbb{T}^N$.

(A_3) *One has*

$$\nabla_y H\big(\tfrac{T}{2}, x, 0\big) = 0\,, \qquad \text{for every } x \in \mathbb{T}^N.$$

(A_4) *The function $H(t,x,y)$ is even in (t,y)*, i.e.,

$$H(-t, x, -y) = H(t, x, y)\,, \qquad \text{for every } (t, x, y) \in \mathbb{R} \times \mathbb{T}^N \times \mathbb{R}^N.$$

(A_5) *The solutions $z = (x, y)$ of* (1.6) *such that $z(0) \in \mathbb{T}^N \times \{0\}$ are defined on the whole time interval $\big[0, \frac{T}{2}\big]$.*

Here is our main result.

Theorem 1.4 *Let the assumptions (A_1)-(A_5) hold true. Then, system* (1.6) *has at least $N+1$ geometrically distinct T-periodic solutions (x, y). These solutions satisfy*

$$(x(-t), y(-t)) = (x(t), -y(t))\,, \qquad \textit{for every } t \in \mathbb{R}\,,$$

and

$$y\big(n\tfrac{T}{2}\big) = 0\,, \qquad \textit{for every } n \in \mathbb{Z}\,.$$

Proof By Theorem 1.3, assumptions (A_2) and (A_5) guarantee the existence of $N+1$ geometrically distinct solutions (x, y) of (1.6) satisfying the two-point boundary condition

$$y(0) = 0 = y\big(\tfrac{T}{2}\big)\,. \tag{1.9}$$

Let $z = (x, y)$ be one of such solutions, defined on the interval $\big[0, \frac{T}{2}\big]$. We will extend it to the whole line $\mathbb{R}$ so to obtain the T-periodic solution we are looking for.

Before doing this, notice that

$$x'(0) = 0 = x'\big(\tfrac{T}{2}\big)\,.$$

Indeed, by (A_4),

$$\nabla_x H(-t, x, -y) = \nabla_x H(t, x, y)\,, \qquad \nabla_y H(-t, x, -y) = -\nabla_y H(t, x, y)\,; \tag{1.10}$$

hence

$$x'(0) = \nabla_y H(0, x(0), y(0)) = \nabla_y H(0, x(0), 0) = 0\,;$$

moreover, by (A_3),

$$x'\big(\tfrac{T}{2}\big) = \nabla_y H\big(\tfrac{T}{2}, x\big(\tfrac{T}{2}\big), y\big(\tfrac{T}{2}\big)\big) = \nabla_y H\big(\tfrac{T}{2}, x\big(\tfrac{T}{2}\big), 0\big) = 0\,.$$

First of all, we extend $z(t) = (x(t), y(t))$ to the interval $\big[-\frac{T}{2}, \frac{T}{2}\big]$ by setting

$$(x(-t), y(-t)) = (x(t), -y(t))\,.$$

It is easy to see that this function is continuously differentiable on the whole interval $\big[-\frac{T}{2}, \frac{T}{2}\big]$.

Now, since $x(t)$ is even on $\big[-\frac{T}{2}, \frac{T}{2}\big]$ and $y\big(-\frac{T}{2}\big) = y\big(\frac{T}{2}\big) = 0$, we have

$$z\big(-\tfrac{T}{2}\big) = z\big(\tfrac{T}{2}\big)\,.$$

We can then extend $z(t)$ to $\mathbb{R}$ by T-periodicity, thus obtaining a continuous function. Let us prove that it is differentiable. We just need to show that it is such at $t = \frac{T}{2}$. Indeed,

$$\begin{aligned}
\lim_{t\to\frac{T}{2}^+}(x'(t), y'(t)) &= \lim_{t\to-\frac{T}{2}^+}(x'(t+T), y'(t+T)) \\
&= \lim_{t\to-\frac{T}{2}^+}(x'(t), y'(t)) \\
&= \lim_{t\to-\frac{T}{2}^+}(-x'(-t), y'(-t)) \\
&= \lim_{t\to\frac{T}{2}^-}(-x'(t), y'(t)) = \big(0, y'\big(\tfrac{T}{2}\big)\big)\,;
\end{aligned}$$

hence

$$\lim_{t\to\frac{T}{2}^+}(x'(t), y'(t)) = \lim_{t\to\frac{T}{2}^-}(x'(t), y'(t)) = \big(0, y'\big(\tfrac{T}{2}\big)\big)\,.$$

Finally, let us prove that $z(t)$ is a solution of (1.6) on the whole line $\mathbb{R}$. We know that it is a solution on $\big[0, \frac{T}{2}\big]$ and that it is differentiable on $\mathbb{R}$. Then, on $\big[-\frac{T}{2}, 0\big]$, by (1.10) we have

$$\begin{aligned}
x'(t) &= -x'(-t) = -\nabla_y H(-t, x(-t), y(-t)) \\
&= -\nabla_y H(-t, x(t), -y(t)) = \nabla_y H(t, x(t), y(t))
\end{aligned}$$

and

$$\begin{aligned}
y'(t) &= y'(-t) = -\nabla_x H(-t, x(-t), y(-t)) \\
&= -\nabla_x H(-t, x(t), -y(t)) = -\nabla_x H(t, x(t), y(t))\,.
\end{aligned}$$