Zhendong Luo

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Preface

About the Finite Element and Mixed Finite Element Methods

Many people ask: what is ***finite element*** (FE) ***method***? What is it used for?

We can only give rough answer herein. If you want to know more details about the FE method, you have to read the book well.

As we know, a large number of natural laws and phenomena can be described by partial differential equations (PDEs). Unfortunately, when their calculation domains are irregular geometric shape, or their initial and boundary values, or source terms are complicated, we cannot find out their analytical solutions, so we can only find their numerical solutions, i.e., approximate solutions.

The simplest idea is to substitute a polynomial with known basis functions and unknown coefficients into PDE and to solve out the unknown coefficients, and then multiplied by the corresponding basis functions to obtain an approximate solution. This method is known as the ***Galerkin method***, which is to adopt the same basis functions on the whole computing domain. But in the textbook of numerical methods, there is an example where a bell-shaped function such as $y = \exp(-\boldsymbol{x}^2)$ cannot be approximated with very high-degree polynomial defined on the whole computational domain, whereas segmenting linear functions in the one-dimensional (1D) space, piecewise linear functions in the two-dimensional (2D) space, or chunking linear functions in the three-dimensional (3D) space can uniformly converge to the bell function. Therefore, the Galerkon method is not suitable for solving the radically varied PDEs. Especially, the Galerkin method can only be used to solve PDEs defined on rectangular computational domain since its basis functions are identical on the whole computational domain.

In order to ensure that the approximate numerical solutions converge to the genuine solution of PDEs, ones divide the computation domain into ***some sub-domains***, which are known as ***finite elements***, and adopt lower degree polynomials (usually the linear functions are enough) on each sub-domain to approximate the genuine solution of PDEs so that the obtained approximate numerical solutions can uniformly converge to the genuine solutions of PDEs. This is just the basic idea of the FE method, which is come down to two steps that the computation domain is divided into several sub-domains and the solutions of PDEs are replaced with the low-degree polynomials on each sub-domain.

However, the lower degree (such as linear or quadratic) polynomials on each sub-domain are usually only guaranteed to be continuous on the common interface of two sub-domains, so they are only whole continuous on the whole computing domain Ω. Therefore, they usually only belong to $H^1(\Omega)$, namely they have only first-order generalized derivatives (see Theorem 1.3.1) and cannot be directly input into PDEs. Thus, we first employ Green's formula to convert PDEs with $2k$th-order classical derivatives into the equivalent ***variational problem***s with kth-order generalized derivatives. We then employ the lower degree (such as linear or quadratic) polynomials on each element to construct the ***element stiffness matrix*** by means of the equivalent variational problems. Finally, by compositing all element stiffness matrices into the total stiffness matrix, we obtain the system of FE algebraic equations on the whole calculating domain. As long as the unknown coefficient vectors in the system of FE algebraic equations are solved out, the FE approximate solutions for PDEs can be obtained. Hence, to find the FE approximate solutions for PDEs usually includes the following five aspects.

(1) By using Green's formula, we convert PDEs with $2k$th-order classical derivatives into the equivalent variational problems with kth-order generalized derivatives and prove that the equivalent variational problems have a unique generalized solutions by means of the theories of functional analysis and Sobolev spaces. This is basic part which will be provided in this book.
(2) Since the system of FE algebraic equations consists of the lower degree (such as linear or quadratic) polynomials on each sub-domain (finite element), we need to study function interpolation polynomials in various cases, i.e., to learn the constructions for interpolation polynomials and the error estimates of function interpolation polynomials. The function interpolations and their error estimates on the 2D triangle element and the 3D tetrahedron element are classical and can be found in FE textbooks (see, e.g., [25]), but they are not practical. The author of this book proposed the function interpolations and error estimates on the 2D arbitrarily quadrilateral element and the 3D arbitrarily hexahedron element in his Master's Thesis in 1989 (see [76]) or his papers [78, 84] and books [81, 90, 96]; these works belonged to original at the time. They will be provided in this book.
(3) The existence and uniqueness of FE solution for the FE equation need to be proved theoretically, which are the main basic theory in this book.
(4) For complex systems of PDEs, such as Burgers equation, Stokes equation, Navier-Stokes equation, and Boussinesq equation, their systems of FE equations are composed of several sets of FE equations with mutual constraints. The FE method with restricted conditions is known as the ***mixed FE*** (***MFE***) ***method***. The MFE method plays an important role in reducing the derivative orders of PDEs, too. It can convert the higher order PDEs into the system of lower order PDEs, which can be solved with the lower degree polynomials.

The earliest MFE works are, respectively, the reduced derivative-order method for solving biharmonic equation (see [26, 59]), the Raviart and Thomas' elements for solving 2D second-order elliptic equation (see [167]), and the MFE methods for solving Stokes equation and Navier-Stokes equation (see [15, 46, 175, 196]), the MFE methods for solving plane elasticity equation (see [163]), and the more general and more adaptable MFE method (see [37]). The author of this book found that the above-mentioned MFE methods have too more degrees of freedom and the argument process is too complicated, so he proposed some MFE methods with the fewer degrees of freedom and very simple argument process for the second-order elliptic equation, plane elasticity equation, Stokes equation, Navier-Stokes equation, and Boussinesq equation (see [76, 78, 81, 84, 90, 96]). These methods will be partly provided in this book. Readers can learn about the MFE method and its application, and make use of the MFE method to settle more real-world problems.

(5) Most PDEs in practical engineering are time-dependent unsteady. To establish the FE or MFE equations for unsteady PDEs is also main task in this book, which is the important applications of FE and MFE methods. Readers may learn about the various techniques for solving the unsteady PDEs and make use of the techniques to solve more complex problems.

The above five aspects are just classical FE method, which has been widely used in scientific engineering computations since it was proposed by Turner et al. in 1956 in order to solve a complex structural problem (see [193]). It has emerged as a powerful approach for solving PDEs including elliptic type, parabolic type, hyperbolic type, and complex hydrodynamics equations. At present, the basic theory of FE method has been developed perfectly. However, when it is used to solve the real-world engineering problems, the FE equation usually includes hundreds of thousands or even tens of millions unknowns (degrees of freedom). Even if it is computed on some advanced computers, it takes days or even tens of days to obtain the numerical results. Owing to the FE method containing a lot of unknowns, the round-off errors in the calculation process are rapidly accumulated, resulting in that the obtained numerical solutions appear very large deviation, which is very difficult to obtain the desired numerical solutions. A key question for employing the FE method to solve the real-world engineering problems is to lessen the unknowns in the FE method so as to be able to retard the accumulation of round-off errors in the computation process, save CPU runtime, lighten the calculating load, and enhance the real-time calculating accuracy of numerical solutions. This is the ultimate goal of the book.

About the Reduced-Dimension of Finite Element and Mixed Finite Methods

The ultimate goal in this book is to reduce the dimension of FE and MFE methods for the unsteady PDEs from two aspects about the reduced-dimension for FE subspaces and for unknown FE solution coefficient vectors in FE equations, where mainly includes the author Z.D. Luo's study results on the FE reduced-dimension in 20 years from 2003 to now. The above five aspects are prepared for the reduced-dimension of FE equations of unsteady PDEs. Of course, they can also serve as the best reference materials for learning FE method since they are exactly main FE basic theory, which are basic part in this book.

The reduced-dimension for the FE subspaces was first initiated by the author of this book and his coworkers in 2007 (see [132]); for more details, see [107]. The author Z.D. Luo's most contribution is to link the classical FE method with the reduced-dimension FE method skillfully by the Hahn-Banach theorem (i.e., bounded linear functional continuation theorem in functional analysis) and to establish the theory of existence, stability, and convergence of reduced-dimension FE solutions, which should be original and is introduced in Chap. 4.

The reduced-dimension for the unknown solution coefficient vectors of FE equations was first initiated by the author of this book in 2020 (see [103, 109]). The biggest highlight of reduced-dimension for the unknown solution coefficient vectors is only to lower the dimension of unknown solution coefficient vectors, but to maintain the FE subspace unchanged in the reduced-dimension FE method, namely the reduced-dimension FE method has the same basis functions as the classical FE method so that it has the same accuracy as the classical FE method. In addition, the stability and convergence of the reduced-dimension solutions are discussed by the matrix analysis, which makes the theoretical analysis more convenient. The reduced-dimension method is introduced in Chap. 5.

About Proper Orthogonal Decomposition Method

The two reduced-dimension methods mentioned above are established by using, respectively, the continuous and discrete ***proper orthogonal decomposition*** (POD) methods to lower the dimension for the FE equations. The POD method is very old; its predecessor is principal vector analysis, now it is still used in data mining. The POD method essentially provides a set of orthogonal bases for representing a given set of data in a certain least squares optimal sense, i.e., it offers a way to find optimal lower dimensional approximations for the set of given data. It was an eigenvector analysis method, which was initially presented by Pearson in 1901 and was used to extract targeted main ingredients of huge amounts of data (see [162]). Pearson's data mining and sample analysis as well as data processing are still relevant even today. The fashionable name of such data is called "***Big Data***."

The method of snapshots for POD was first proposed by Sirovich in 1987 (see [174]). The POD method has been widely and successfully applied in many fields, including signal analysis and pattern recognition (see [41]), statistics (see [61]), geophysical fluid dynamics or meteorology (see also [61] or [74]), and biomedical engineering (see [48]). For a long time since 1987, the POD method was mainly used to perform the principal component analysis in statistical computations and to search for certain main behavior of dynamic systems. It was not until 2001 that Kunisch-Volkweind used the POD method to deal with the reduced-order Galerkin methods for PDEs (see [62, 63]). From that moment forth, the model order reduction or reduced-basis of the numerical computational methods based on the POD technique for PDEs began rapidly to develop and had brought on high efficiency for finding numerical solutions to PDEs (see, e.g., [2, 9, 12, 16, 44, 48, 51, 54, 55, 164–166, 168, 194, 195, 199, 200, 208–210]).

Although Kunisch-Volkweind's POD reduced-order Galerkin methods in [62, 63] included the error estimates of reduced-order Galerkin numerical solutions, those error estimates consist of some matrix norms without concrete orders of convergence. Especially, they took *all* the numerical solutions of classical Galerkin method at all time nodes on the total time span $[0, t_e]$ to construct the POD basis functions and used them to establish the POD-based reduced-order models, and then recompute the numerical solutions at all same time nodes. This is completely repetitive computation but not much extra reward and gain. The author Luo begin to ponder how to improve on this and further extend the methodology initiated by Kunisch-Volkweind's work beyond the Galerkin method to other numerical methods, such as the FE method, MFE method, time-space FE (TSFE) method, finite difference (FD) scheme, collection spectral (CS) method, finite spectral element (FSE) method, natural boundary element (NBE) method, and finite volume element (FVE) method, and construct the reduced-dimension methods without recomputing.

The author of this book was attracted to study the reduced dimensionality of numerical methods based on the POD method for PDEs at the beginning of 2003. At that time, few or none comprehensive accounts existed and only fragmentary introductions about POD were available. He spent more than 3 years (2003∼2005) studying the underlying optimization methods, statistical principles, and numerical solutions for POD. Then in 2006, he and his coworkers published their first two papers based on the POD method (see [19, 20]). They dealt with oceanic models and data assimilation.

Afterwards, the author of this book and his coauthors have established some POD-based reduced-dimension FD schemes (see [4, 33, 36, 130, 131, 137, 140, 178]) and reduced-dimension FE methods (see [32, 35, 68, 132–136, 138, 139, 141, 144, 146, 147, 149, 150, 188]) successively. They deduced the error estimates for POD-based reduced-dimension solutions for PDEs of various types since 2007 in a series of papers. They also proposed some POD-based reduced-dimension FVE methods and relevant error estimates of reduced-dimension FVE solutions (see [70, 142, 145, 149, 151]) for PDEs in another series of papers beginning in 2011. These reduced-dimension methods based on POD technique were specific to the classical FD schemes, FE methods, and FVE methods for the construction of the reduced-dimension models, from where they extracted one from every ten classical numerical solutions as *snapshots*, which are significantly different from Kunisch-Volkweind's methods, which had extracted numerical solutions from the classical Galerkin method at all time nodes on the total time span $[0,\ t_e]$. Hence, these reduced-dimension methods constitute improvements, generalizations, and extensions for/from/on Kunisch-Volkweind's methods in [62, 63]. However, the reduced-dimension methods in the above cited work need yet to repeat *part* of the calculations on the same time span $[0, t_e]$.

Since 2012, the author of this book and his coworkers further improve the foregoing reduced-dimension methods based on POD and have established the following five main reduced-dimension ***extrapolating*** (***recursive***) ***methods***:

(1) The reduced-dimension extrapolating FD schemes (see [5, 6, 30, 97, 108, 117, 121, 148, 152–157, 159, 179, 190, 218])
(2) The reduced-dimension extrapolating FE (RDEFE) methods about the FE spaces (see [69, 72, 99, 100, 111, 118, 181, 182, 185, 189, 202])
(3) The reduced-dimension extrapolating FVE methods about the FE spaces (see [98, 101, 102, 119, 143, 158, 183, 184])
(4) The RDEFE methods about the unknown FE solution coefficient vectors (see [103, 104, 109, 110, 122, 160, 186, 207, 212])
(5) The reduced-dimension extrapolating NBE methods about the FE spaces (see [187, 191, 192])

In addition, the author of this book and his coworkers also established some reduced-dimension extrapolating models of TSFE, CS, and FSE methods about the FE subspaces and unknown FE solution coefficient vectors for the unsteady PDEs. These reduced-dimension extrapolating methods need only to employ the standard numerical solutions on some initial rather short time span $[0, t_0]$ ($t_0 \ll t_e$) of, respectively, the classical FD, FE, FVE, TSFE, CS, FSE, and NBE schemes as snapshots in order to construct the POD bases. Thereupon, they have significantly improved the previous and existing versions of the reduced-order methods. They do not have to repeat large-scale computations. The physical significance is that one can use the existing data to forecast the future evolution of nature. Moreover, the reduced-dimension extrapolating methods can be treated by in a similar way as the classical FD, FE, FVE, TSFE, CS, FSE, and NBE methods, resulting in the error estimates with concrete orders of convergence. These reduced-dimension extrapolating numerical methods are far superior to the POD reduced-dimension methods mentioned earlier including reduced-order Galerkin methods.

The Main Content and Arrangement of the Book

The purpose of this book is to provide with some approaches for lessening the unknowns of the FE and MFE methods of the unsteady PDEs. It attempts to provide a detailed self-contained presentation for readers as follows.

- A detailed presentation of the basic theory of FE method
- A detailed presentation of the basic theory of MFE method
- A detailed presentation of the MFE methods for the unsteady PDEs
- A detailed presentation of reduced-dimension methods of FE subspaces for the unsteady PDEs
- A detailed presentation of reduced-dimension methods of unknown FE or MFE solution coefficient vectors for the unsteady PDEs

This book can also be used as both the introduction of FE method and the gateway to the FE frontier. The reason is that Chaps. 1 and 2 provide a very detailed theoretical foundation of FE and MFE methods and Chap. 3 provides the MFE methods for solving the unsteady PDEs. The reader can learn the FE and MFE methods for solving various steady and unsteady PDEs in Chaps. 1, 2, and 3. In Chaps. 4 and 5, the principle and applications of POD-based reduced-dimension for the FE subspaces and unknown FE or MFE solution coefficient vectors for the unsteady PDEs are introduced in detail, respectively. The readers who only care about engineering applications need only to learn the construction of reduced-dimension models, and then to apply in practical engineering calculations. This will greatly improve the calculation efficiency and save CPU runtime so as to do wonders for your engineering calculations.

In a word, this book can guide you from the entry (entrance) of FE method, and gradually leads you to the forefront of development for the reduced-dimension FE methods. Especially, the reduced-dimension numerical methods based on the POD method are just beginning to develop, just like children and teenagers, and still have a lot of room to grow. The author of this book sincerely hopes that this book can help you achieve something unexpected.

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Changsha, China Zhendong Luo
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Contents

List of Abbreviations

1D	One-dimensional
2D	Two-dimensional
3D	Three-dimensional
B-B	Babuška-Brezzi
CN	Crank–Nicolson
CNFE	Crank–Nicolson finite element
CNMFE	Crank–Nicolson mixed finite element
CS	Collection spectral
FE	Finite element
FD	Finite difference
FDCNMFE	Fully discretized Crank–Nicolson mixed finite element
FDMFE	Fully discretized mixed finite element
FSE	Finite spectral element
FVE	Finite volume element
MAE	Mean absolute errors
MFE	Mixed finite element
NBE	Natural boundary element
PDE	Partial differential equation
POD	Proper orthogonal decomposition
RDCNFE	Reduced-dimension Crank–Nicolson finite element
RDCNMFE	Reduced-dimension Crank–Nicolson mixed finite element
RDEFE	Reduced-dimension extrapolating finite element
RDMFE	Reduced-dimension mixed finite element
RDRCNF	Reduced-dimension recursive Crank–Nicolson finite element
RDRCNMFE	Reduced-dimension recursive Crank–Nicolson mixed finite element
RLW	Regularized long wave
RT	Raviart and Thomas
SDCN	Semi-discretized Crank–Nicolson
SDMFE	Semi-discretized mixed finite element
TSFE	Time-space finite element

Chapter 1
Basic Theory of Standard Finite Element Method

To discuss the existence of generalized solutions and finite element (FE) and mixed FE (MFE) solutions for partial differential equations (PDEs), we need to use the basic principles of functional analysis including generalized derivatives and Sobolev spaces, which are bases for the numerical analysis of PDEs. Therefore, the relevant theory and principles for the functional analysis are first provided in the following section, the more details see [1, 25, 67, 96, 216].

1.1 The Basic Principles of Functional Analysis

1.1.1 Linear Operator and Linear Functional

Definition 1.1.1 Let $\mathbb{S}$ be a subspace in normed or Hilbert space $\mathbb{H}$ and $\mathbb{S}'$ be made up of limit points of $\mathbb{S}$. The union set of $\mathbb{S}$ and $\mathbb{S}'$ is known as the ***closure*** of the subspace $\mathbb{S}$ and is denoted by $\bar{\mathbb{S}}$, in other words, $\bar{\mathbb{S}} = \mathbb{S} \cup \mathbb{S}'$. If $\bar{\mathbb{S}} = \mathbb{H}$, then $\mathbb{S}$ is known as ***dense*** in $\mathbb{H}$. If $\mathbb{S}$ is dense in $\mathbb{H}$ and $\mathbb{S}$ is denumerable, then $\mathbb{H}$ is known as ***separable***.

Let $\mathbb{R}^n$ $(n = 1, 2, 3)$ be the n-dimensional Euclidean space and Ω be an open domain in $\mathbb{R}^n$. The boundary of Ω is denoted by $\partial\Omega$. The space formed by all functions defined on Ω, whose absolute value pth powers are Lebesgue integrable, is denoted by $L^p(\Omega)$ $(1 \leqslant p < \infty)$, and the space consisting of all essentially bounded (i.e., bounded except on a zero measure set) measurable functions defined

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on Ω is denoted by $L^\infty(\Omega)$. $L^p(\Omega)$ becomes a Banach space equipped with the norm

$$\|u\|_{0,p,\Omega} = \begin{cases} \left(\int_\Omega |u(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x}\right)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{\boldsymbol{x}\in\Omega} |u(\boldsymbol{x})|, & p = \infty, \end{cases}$$

where $\operatorname*{ess\,sup}_{\boldsymbol{x}\in\Omega} |u(\boldsymbol{x})| \equiv \inf_{\mathrm{mes}(\mathbb{K})=0} \sup_{\boldsymbol{x}\in\Omega\setminus\mathbb{K}} |u(\boldsymbol{x})|$ and $\mathrm{mes}(\mathbb{K})$ represents the Lebesgue measure of set $\mathbb{K}$. When $p = 2$, $L^2(\Omega)$ is a Hilbert space.

The space formed by functions with continuous mth-order derivatives defined on Ω is denoted by $C^m(\Omega)$ and the space formed by functions with continuous any order derivatives defined on Ω is denoted by $C^\infty(\Omega)$. The space $C^0(\Omega)$ is simply denoted as $C(\Omega)$. It is obvious that $C^\infty(\Omega)$ is dense in $L^2(\Omega)$ and $C^\infty(\Omega) \subset C^m(\Omega)$ $(m = 0, 1, 2, \cdots)$. While the space consisting of polynomials with rational coefficients is dense in $C^\infty(\Omega)$ under the norm of $L^2(\Omega)$ and is denumerable. Therefore, $L^2(\Omega)$ is a separable Hilbert space and its subspace is also separable Hilbert space. Thus, in order to study the properties of some Hilbert spaces, we need only to discuss the properties of its dense subspace such as the space consisting of polynomials, and then, by taking the limit we can deduce that the properties hold in total Hilbert space.

Definition 1.1.2 Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in linear normed space $\mathbb{H}$ are said to be ***equivalent***, if there exist two positive constants α and β such that

$$\alpha\|u\|_1 \leqslant \|u\|_2 \leqslant \beta\|u\|_1, \quad \forall u \in \mathbb{H}.$$

Remark 1.1.1 There may be two different norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in the same normed or Hilbert space $\mathbb{H}$, but $(\mathbb{H}, \|\cdot\|_1)$ and $(\mathbb{H}, \|\cdot\|_2)$ is considered as to be two different spaces. If the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, the convergence of the sequence $\{u_n\}_{n=1}^\infty$ in $\mathbb{H}$ is equivalent, in other words, if the sequence $\{u_n\}_{n=1}^\infty$ is convergent under the norm $\|\cdot\|_1$, then it is also convergent under the norm $\|\cdot\|_2$, and the reverse is also true.

The following conclusion has been proven in [216, Theorem 1.4.18].

Theorem 1.1.1 *Any two norms in finite-dimensional linear normed space are equivalent.*

In order to study the solvability for PDEs, it is necessary to introduce the linear operator, which is generalized for traditional function.

Definition 1.1.3 Let $\mathbb{X}$ and $\mathbb{Y}$ be two linear spaces defined in the real number field $\mathbb{R}$ and $\mathbb{D} \subset \mathbb{X}$ be a linear subspace of $\mathbb{X}$. The map $T : \mathbb{D} \to \mathbb{Y}$ is said to be the ***linear operator***, if there holds the following equality

$$T(\alpha u + \beta v) = \alpha T u + \beta T v, \quad \forall \alpha, \beta \in \mathbb{R}; \quad \forall u, v \in \mathbb{D},$$

where $\mathbb{D}$ is known as the ***definition domain*** of T and is written as $\mathbb{D} = \mathbb{D}(T)$, the set $\mathbb{R}(T) := \{y : \ y = Tv, \ v \in \mathbb{D}(T)\}$ is known as the ***value domain*** of T. If the linear space $\mathbb{Y} \subset \mathbb{R}$, the linear operator T is known as the ***linear functional***.

Definition 1.1.4 Let $\mathbb{X}$ and $\mathbb{Y}$ be two linear normed spaces equipped with the norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$, respectively. The linear operator $T : \mathbb{X} \to \mathbb{Y}$ is known as ***continuous***, if for any $v_0 \in \mathbb{X}$ and any $\{v_m\}_{m=1}^{\infty} \subset \mathbb{X}$, when $\|v_m - v_0\|_{\mathbb{X}} \to 0$ $(m \to \infty)$, there holds $\|Tv_m - Tv_0\|_{\mathbb{Y}} \to 0$ $(m \to \infty)$.

Definition 1.1.5 Let $\mathbb{X}$ and $\mathbb{Y}$ be two linear normed spaces equipped with the norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$, respectively. The linear operator $T : \mathbb{X} \to \mathbb{Y}$ is said to be ***bounded***, if there is a positive constant $M > 0$ such that

$$\|Tv\|_{\mathbb{Y}} \leqslant M\|v\|_{\mathbb{X}}, \quad \forall v \in \mathbb{X}.$$

For the continuity and boundedness for the linear operator, we have the following important property (see [216, Proposition 2.1.11]).

Proposition 1.1.2 *Let $\mathbb{X}$ and $\mathbb{Y}$ be two linear normed spaces equipped with the norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$, respectively. The linear operator $T : \mathbb{X} \to \mathbb{Y}$ is continuous, if and only if $T : \mathbb{X} \to \mathbb{Y}$ is bounded.*

Definition 1.1.6 The linear space consisting of all bounded linear functionals defined in the linear normed space $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ is denoted by $\mathbb{H}'$ equipped with the following norm

$$\|f\|_{\mathbb{H}'} = \sup_{\theta \neq v \in H} \frac{|f(v)|}{\|v\|_{\mathbb{H}}} = \sup_{\|v\|_{\mathbb{H}}=1} |f(v)|, \quad \forall f \in \mathbb{H}'.$$

The linear space consisting of all bounded linear operators from $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ to $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ is denoted by $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ equipped with the following norm

$$\|T\| = \sup_{\theta \neq v \in \mathbb{X}} \frac{\|Tv\|_{\mathbb{Y}}}{\|v\|_{\mathbb{X}}} = \sup_{\|v\|_{\mathbb{X}}=1} \|Tv\|_{\mathbb{Y}}, \quad \forall T \in \mathbb{L}(\mathbb{X}, \mathbb{Y}).$$

The bounded linear operator has the following result (see [216, Section 2.2]).

Theorem 1.1.3 (The Inverse Operator Existence Theorem) *The bounded linear operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ has a unique inverse operator T^{-1} if and only if T is lower bounded, namely there is a positive constant $\beta > 0$ such that*

$$\beta\|v\|_{\mathbb{X}} \leqslant \|Tv\|_{\mathbb{Y}}, \quad \forall v \in \mathbb{X}.$$

Further, there holds the following inequality

$$\|T^{-1}\| \leqslant \frac{1}{\beta}.$$

1.1.2 Orthogonal Projection and Riesz Representation Theorem

The optimal approximation for the analytical solutions of PDEs is one of methods for finding approximated solution, which is defined as follows.

Definition 1.1.7 (The Optimal Approximation) Supposed that $\mathbb{X}$ is a linear normed space, equipped with $\|\cdot\|$, and $\mathbb{M} \subset \mathbb{X}$ is a subset in $\mathbb{X}$. If for a given element $u \in \mathbb{X}$, there exists an element $u_0 \in \mathbb{M}$ such that

$$\|u - u_0\| \leqslant \|u - v\|, \quad \forall v \in \mathbb{M},$$

then u_0 is said to be an ***optimal approximation*** of u in $\mathbb{M}$.

Definition 1.1.8 Two elements u and v in the inner product space $\mathbb{H}$ equipped with inner product $(\cdot,\cdot)$ is said to be ***orthogonal*** (***vertical***), if $(u, v) = 0$. If $\mathbb{M}$ is a subset in the inner product space $\mathbb{H}$ and the set $\mathbb{M}^{\perp}$ is made up of elements that are orthogonal to every member of $\mathbb{M}$, in other word, $\mathbb{M}^{\perp} = \{u \in H : (u, v) = 0, \forall v \in \mathbb{M}\}$, then $\mathbb{M}^{\perp}$ is said to be ***orthocomplement*** of the subset $\mathbb{M}$ in $\mathbb{X}$.

Remark 1.1.2 It is easily proven (see, e.g., [216, Proposition 1.6.18]) that the orthocomplement $\mathbb{M}^{\perp}$ of the set $\mathbb{M}$ is a closed subspace in the inner product space $\mathbb{H}$.

The orthocomplement $\mathbb{M}^{\perp}$ of the subset $\mathbb{M}$ in the inner product $\mathbb{H}$ has the following useful conclusion (see [216, Corollary 1.6.35]).

Theorem 1.1.4 (The Orthogonal Decomposition Theorem) *If $\mathbb{M}$ is a closed subspace in the Hilbert space $\mathbb{H}$, then for any $u \in \mathbb{H}$, there exists a unique $v \in \mathbb{M}$ and a unique $w \in \mathbb{M}^{\perp}$ such that*

$$u = v + w.$$

With the orthogonal decomposition theorem, namely Theorem 1.1.4, we can easily deduce the following Riesz representation theorem (see [216, Theorem 2.2.1])

Theorem 1.1.5 (The Riesz Representation Theorem) *For any bounded (i.e., continuous) linear functional f in the Hilbert space $\mathbb{H}$, namely for any $f \in \mathbb{H}'$, there exists a unique $u_f \in \mathbb{H}$ such that*

$$f(v) = (u_f, v), \quad \forall v \in \mathbb{H} \quad \text{and} \quad \|f\|_{H'} = \|u_f\|_{\mathbb{H}}.$$

The following Hahn-Banach theorem (see [216, Theorem 2.4.4]) is often used in the subsequent discussions.

Theorem 1.1.6 (The Hahn-Banach Theorem) *If $\mathbb{X}$ is the normed space and $\mathbb{X}_0$ is a linear subspace in $\mathbb{X}$, then for any bounded linear functional f_0 defined on $\mathbb{X}_0$,* *namely for any* $f_0 \in \mathbb{X}_0'$, *there exists a bounded linear functional* f *defined on* $\mathbb{X}$, *namely there exists a functional* $f \in \mathbb{X}'$ *such that*

$$f(v) = f_0(v), \quad \forall v \in \mathbb{X}_0 \quad \textit{and} \quad \|f\|_{\mathbb{X}'} = \|f_0\|_{\mathbb{X}_0'},$$

where f *is known as the extension on* $\mathbb{X}$ *of* f_0.

1.1.3 Smooth Approximation and Fundamental Lemma of Calculus of Variation

The closure of set $\{\boldsymbol{x} \in \Omega : u(\boldsymbol{x}) \neq 0\}$ is known as the ***support*** of the function u and denoted by $supp(u)$. The spaces $C_0^m(\Omega)$ and $C_0^\infty(\Omega)$ are, respectively, subsets of $C^m(\Omega)$ and $C^\infty(\Omega)$, consisting of functions with compact support in Ω.

There exists a function $j(\boldsymbol{x})$ satisfying the following conditions:

(i) $j(\boldsymbol{x}) \in C_0^\infty(\Omega)$;
(ii) $j(\boldsymbol{x}) \geqslant 0$ $(\forall \boldsymbol{x} \in \mathbb{R}^n)$ and $j(\boldsymbol{x}) = 0$ (if $|\boldsymbol{x}| > 1$);
(iii) $\displaystyle\int_{\mathbb{R}^n} j(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 1$.

For example, define

$$j(\boldsymbol{x}) = \begin{cases} \dfrac{1}{\gamma}\exp\left(-\dfrac{1}{1-|\boldsymbol{x}|^2}\right), & |\boldsymbol{x}| < 1; \\ 0, & |\boldsymbol{x}| \geqslant 1, \end{cases}$$

where $\gamma = \displaystyle\int_{|\boldsymbol{x}|<1} \exp\left(-\frac{1}{1-|\boldsymbol{x}|^2}\right)\mathrm{d}x$, then $j(\boldsymbol{x}) \in C_0^\infty(\mathbb{R}^n)$ and $\displaystyle\int_{\mathbb{R}^n} j(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 1$.

Definition 1.1.9 An integral operator J_ε:

$$J_\varepsilon u(\boldsymbol{x}) = \int_{\mathbb{R}^n} j_\varepsilon(\boldsymbol{x}, \boldsymbol{y})u(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

with a kernel

$$j_\varepsilon(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\varepsilon^n} j\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right), \quad \varepsilon > 0$$

is known as a ***smoothing operator*** and $J_\varepsilon u$ is known as an ***averaging function*** of u.

The averaging function has the following useful properties (see [67, 216]).

Theorem 1.1.7 (The Average Approximation Theorem) *For any function* $u \in L^p(\Omega)$ $(1 \leqslant p < \infty)$, *there hold the following results*

(i) $J_\varepsilon u \in C^\infty(\mathbb{R}^n) \cap L^p(\Omega)$, *and* $J_\varepsilon u \in C_0^\infty(\mathbb{R}^n) \cap L^p(\Omega)$ *if* $supp(u)$ *is bounded;*
(ii) $\|J_\varepsilon u\|_{0,p,\Omega} \leqslant \|u\|_{0,p,\Omega}$;
(iii) $\lim\limits_{\varepsilon \to 0} \|u - J_\varepsilon u\|_{0,p,\Omega} = 0$.

Remark 1.1.3 Theorem 1.1.7 shows that any function in $L^p(\Omega)$ $(1 \leqslant p < \infty)$ can be approximated by a smooth function. In other words, the space $C^\infty(\Omega)$ is dense in $L^p(\Omega)$ $(1 \leqslant p < \infty)$.

Moreover, there holds the following conclusion (see also [67, 216]).

Theorem 1.1.8 *If* $1 \leqslant p < \infty$, *then* $C_0^\infty(\Omega)$ *is dense in* $L^p(\Omega)$.

The following theorem can be easily proved by the average approximation theorem (see [67, 96, 216]).

Theorem 1.1.9 (The Fundamental Lemma of Calculus of Variations) *If* $u \in L^p(\Omega)$ $(1 \leqslant p < \infty)$ *satisfies*

$$\int_\Omega u(\boldsymbol{x}) \cdot \varphi(\boldsymbol{x}) d\boldsymbol{x} = 0, \quad \forall \varphi \in C_0^\infty(\Omega),$$

then $u(\boldsymbol{x}) = 0$ *holds almost everywhere on* Ω.

1.1.4 Generalized Derivatives and Sobolev Spaces

For $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$, the partial derivative of function $u(\boldsymbol{x})$ is denoted by

$$D^\alpha u = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} u = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}},$$

where α_i $(1 \leqslant i \leqslant n)$ are non-negative integers, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is known as an n-index, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and $\boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \cdots \cdot x_n^{\alpha_n}$. The integration of function $u(\boldsymbol{x})$ on an n-dimensional set Ω is denoted by

$$\int_\Omega u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \equiv \int \cdots \int_\Omega u(x_1, x_2, \cdots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

Definition 1.1.10 A function is known as ***locally integrable*** in Ω, if it is Lebesgue integrable on any bounded close subset in Ω. The set consisting of all locally integrable functions defined in Ω is denoted by $L^1_{\mathrm{loc}}(\Omega)$.

Definition 1.1.11 Let $L^1_{\text{loc}}(\Omega)$ be the space consisting of the locally integrable functions in Ω and $u \in L^1_{\text{loc}}(\Omega)$. If there exists a function $v \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega v(\boldsymbol{x})\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = (-1)^{|\alpha|}\int_\Omega u(\boldsymbol{x})D^\alpha\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \quad \forall\varphi \in C_0^\infty(\Omega),$$

then the function v is said to be the $|\alpha|$th-order ***generalized derivative*** of u and is denoted by $v = D^\alpha u$.

By the fundamental lemma of calculus of variations (Theorem 1.1.9) we can prove that a generalized derivative must be unique so long as it exists. Moreover, we can easily prove that if the traditional derivative of u exists and belongs to $L^2(\Omega)$, then its generalized derivative also exists and both are same. But a generalized derivative may not be a traditional derivative, see the relevant examples in [216, Formulas 3.3.4 and 3.3.5]. Therefore, the generalized derivative is indeed a generalization of concept for the traditional derivative.

The generalized derivatives have the same operational properties as those of the traditional derivatives as follows:

(i) For any constants a and b, $D^\alpha(au + bv) = aD^\alpha u + bD^\alpha v$;
(ii) $D^{\alpha+\beta}u = D^\alpha(D^\beta u)$;
(iii) $D(uv) = vDu + uDv \quad (D = \partial/\partial x_k, k = 1, 2, \cdots, n)$;
(iv) $D^\alpha u = 0$ for all α with $|\alpha| = m$, if only if u is an $(m-1)$th-degree polynomial almost everywhere.

Definition 1.1.12 Let m be a non-negative integer and $1 \leqslant p \leqslant \infty$. Set

$$W^{m,p}(\Omega) \equiv \left\{u \in L^p(\Omega) : \ D^\alpha u \in L^p(\Omega), \forall\alpha, 0 \leqslant |\alpha| \leqslant m\right\},$$

endowed with the following norm:

$$\|u\|_{m,p,\Omega} = \begin{cases} \left(\sum\limits_{0\leqslant|\alpha|\leqslant m} \|D^\alpha u\|^p_{0,p,\Omega}\right)^{1/p}, & \text{for } 1 \leqslant p < \infty, \\ \max\limits_{0\leqslant|\alpha|\leqslant m} \|D^\alpha u\|_{0,\infty,\Omega}, & \text{for } p = \infty. \end{cases}$$

Define $W_0^{m,p}(\Omega)$ as the closure of $C_0^m(\Omega)$ with respect to the norm $\|\cdot\|_{m,p,\Omega}$. Both the normed linear spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are known as ***Sobolev spaces*** on Ω.

Remark 1.1.4 If $p = 2$, $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are denoted as $H^m(\Omega)$ and $H_0^m(\Omega)$, respectively. Thus, $W^{0,p}(\Omega) = L^p(\Omega)$ and $H^0(\Omega) = L^2(\Omega)$. Obviously, $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are Banach spaces. But both $H^m(\Omega)$ and $H_0^m(\Omega)$ are