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Fractional-Order Sliding Mode Control: Methodologies and Applications

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Preface

Fractional-order calculus, which has a historical memory capacity can be viewed as the generalization of its traditional integer-order counterpart. Due to its special properties, considerable physical systems can be modeled by fractional-order calculus, such as viscoelastic systems, polymeric chemistry systems, biomedical systems, and electrode processes, and even the electromagnetic theory can be modeled by the fractional-order calculus. Various researchers have been dedicated to fractional-order control theory in recent years. Compared with the integer-order control, fractional-order control preserves several advantages. One is that it is more suitable for flexible structures, especially for structures with viscoelastic characteristics. Another lies in that the robustness of the system can be effectively improved. The last owes to the faster response to input and the smaller overshoots, simultaneously. On the other hand, sliding mode control, an effective nonlinear control method, has attracted much attention and is widely applied to practical applications. To make full use of the advantages of fractional-order control and sliding mode control, fractional-order sliding mode control has been proposed by introducing fractional-order operation in sliding mode control. Such a control scheme has been applied to real engineering systems described by Lagrangian mechanics, such as the linear motor system and space tethered system, just to name a few. However, there still exist some problems to be solved for this kind of system. For example, since existing continuous sliding mode control strategies are not suitable for linear motor equipped with discrete digital controller and communication network, the discrete counterpart needs to be designed. Fractional-order sliding mode control schemes for the deployment of space tethered systems are still seldomly reported. To solve such problems, this monograph attends to introduce the fractional-order control theory and discusses how to design fractional-order sliding mode control schemes for linear motors and space tethered systems, respectively.

The fractional-order control and fractional-order sliding mode control problems of the linear motor and the deployment of space tethered system are thoroughly investigated in this work. Several novel fractional-order control schemes and fractional-order sliding mode control strategies including adaptive fractional-order sliding mode control, fractional-order terminal sliding mode control, and fractional-order fuzzy sliding mode control are proposed to improve the performance of the system. This monograph consists of three parts. One focuses on the problems of fractional-order control for rigid-flexible coupling space structures and space tethered systems. Another is concerned with the problems of fractional-order sliding mode control for linear motor systems. The last studies the problems of fractional-order sliding mode control for the deployment of space tethered systems.

Specifically, the brief view of fractional-order control strength in modeling and control includes the following:

1. Fractional-order dynamics and control of rigid-flexible coupling space structures,
2. Fractional-order controller of space tethered system.

The main contents of fractional-order sliding mode control for linear motor systems are as follows:

1. Practical tracking control of linear motor via discrete-time fractional-order sliding mode control,
2. Practical tracking control of linear motor by adaptive fractional-order Terminal sliding mode control,
3. Discrete-time fractional-order terminal sliding mode control for the tracking control of linear motor,
4. Fractional-order sliding mode contouring error control for multidimensional systems.

The main contents of fractional-order sliding mode control for space tethered systems are as follows:

1. Fractional-order fuzzy sliding mode control for the deployment of tethered satellite system under input saturation,
2. Fractional-order terminal sliding mode control for the deployment of tethered satellite system,
3. Fractional-order sliding mode control for the deployment of tethered spacecraft systems,
4. Fractional-order adaptive sliding mode control for the deployment of space tethered systems with input limitation.

Among these topics, both simulations and experiments are conducted to validate the effectiveness and advantages of the proposed fractional-order control schemes and fractional-order sliding mode control strategies in this monograph.

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Part III Fractional-Order SMC for the Deployment of Space Tethered System

Notations and Acronyms

$\|\cdot\|$	Euclidean norm
$\|\cdot\|_1$	1-norm
$\|\cdot\|_2$	2-norm
$\|\cdot\|_\infty$	∞-norm
$\in$	Belongs to
$\mathcal{L}_1$	$\mathcal{L}_1 = \{f(t) : \|f(t)\|_1 < \infty\}$
$\mathcal{L}_2$	$\mathcal{L}_2 = \{f(t) : \|f(t)\|_2 < \infty\}$
$\mathcal{L}_\infty$	$\mathcal{L}_\infty = \{f(t) : \|f(t)\|_\infty < \infty\}$
N^+	Set of positive integers
sgn($\cdot$)	Sign function
$\mathbb{R}$	Field of real numbers
$\mathbb{R}^n$	Space of n-dimensional real vectors
$\mathbb{R}^{n\times m}$	space of $n \times m$ real matrices
CSMC	Continuous-time sliding mode control
DSMC	Discrete-time sliding mode control
FNTSM	Fast nonsingular terminal sliding mode
FO	Fractional order
LMI	Linear matrix inequality
SMC	Sliding mode control
SMDO	Sliding mode disturbance observer
STS	Space tethered systems
TSS	Tethered satellite system

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Chapter 1
Introduction

FO calculus has a long history which can be traced back to 300 years ago [77]. It can not only describe the objects better than integer-order calculus in natural environment [56], but also has an effect of memory which means that it contains both the present and the past information.

To quickly understand the relationship between FO calculus and classical calculus, Let us review the definition process of classical calculus. Considering a continuous function $y = f(x)$, the well-known calculus definition can be defined by

$$f'(t) = \frac{df}{dt} = \lim_{h \to 0} \frac{f(t) - f(t-h)}{h}. \tag{1.1}$$

Applying the rules twice gives the second-order derivative

$$\begin{aligned} f''(t) = \frac{d^2 f}{dt^2} &= \lim_{h \to 0} \frac{f'(t) - f'(t-h)}{h} \\ &= \lim_{h \to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}. \end{aligned}$$

Similarly, we can get

$$\begin{aligned} f'''(t) = \frac{d^3 f}{dt^3} &= \lim_{h \to 0} \frac{f''(t) - f''(t-h)}{h} \\ &= \lim_{h \to 0} \frac{f(t) - 3f(t-h) + 3f(t-2h) + f(t-3h)}{h^3}. \end{aligned}$$

Moreover, we can obtain

$$f^n(t) = \frac{d^n f}{dt^n} = \lim_{h \to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t - rh),$$

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where

$$\binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r!}.$$

In the last equation, the parameters n and r are integer number. Assuming that the step size is small and the order n is generalized to any rational fractional number. Then, a less rigorous definition for FO differentiation can be stated as follows.

$$f_h^n(t) = \frac{1}{h^p} \sum_{i=0}^{n} (-1)^i \binom{p}{i} f(t - ih).$$

Of course, the above process is unscientific and non-causality, but useful for us to understand the definition process. What's more, we can use it in engineering application because the discrete form is more suitable for modern digital controller. The equation is famously called Gr*ü*unwald-Letnikov FO definition (abbreviate as G-L definition). A more rigorous and mathematical process can be found in [77].

The above section is just a first view of FO calculus. In fact the concept of FO calculus has been widely implemented in controller design [58, 111], which attracts attention from researchers in academia and industry. In the past decades, researchers applied FO calculus to SMC schemes and conducted them in industrial fields [100], such as the hydraulic manipulators [91], permanent magnet synchronous motors [106] and so on, which have proved that these control schemes are more effective than their integer-order counterparts. In what follows, preliminaries of FO calculus are provided, and its applications in the linear motor and space tethered system are discussed.

1.1 The Definition of Fraction Order Calculus

The history of FO calculus, which is deemed as the generalization of traditional integer-order calculus, can go back to 1690s. Many great mathematician have contributed to the field, such as Euler, Lagrange, Riemann and so on. Liouville expanded the functions in series of exponentials and defined the n*th*-order derivative. Riemann proposed a different definition that involved a definite integral and was applicable to power series with non-integer exponents. Gr*ü*unwald definite generalized derivative, as stated above, from the starting point of the integer derivative. Several centuries have been past. FO calculus has been slowly developed in the field of mathematics society. The main reason for limiting in engineering applications is the lack of physical meaning. Until now, There are three kinds of common used definitions of FO derivative, which are Riemann-Liouville, Grünwald-Letnikor, and Caputo definitions. More details about the FO calculus can be found in [41, 71, 77]. Here, both the Riemann-Liouville definition and Caputo definition used in this monograph are given.

- The Riemann-Liouville (R-L) definition is addressed as

$$ {}_{t_0}\mathrm{D}_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} \frac{f(\varepsilon)}{(t-\varepsilon)^{\beta-n+1}} \mathrm{d}\varepsilon, \tag{1.2} $$

where $n-1 < \beta < n$, n is an integer, $f(t)$ is a continuous function with respect to t, and $\Gamma(\cdot)$ is a Gamma function.

The Laplace transform of the FO derivative based on Riemann-Liouville definition is expressed as

$$ \mathscr{L}\left\{ {}_{t_0}\mathrm{D}_t^{\beta} f(t) \right\} = s^{\beta} F(s) - \sum_{k=0}^{n-1} s^k \, {}_{t_0}\mathrm{D}_t^{\beta-k-1} f(t)|_{t=t_0}, $$

where $n-1 < \beta < n$ and s is the Laplace operator. The Oustaloup's discretization algorithm shown in [75] is used to approximate s^{β} in numerical simulations. For notational simplicity, in what follows, ${}_0\mathrm{D}_t^{\beta}$ is simplified as D^{β}.

- The Caputo definition of the FO derivative is as follows.

$$ D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, n-1 < \alpha < n, \tag{1.3} $$

where a is a real number, n is a given integer number and $\Gamma(\cdot)$ is the Gamma function defined as $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$.

It is noted that the FO derivative of a function is related to not only the current state $f(t_n)$ but also the previous states $f(t_i)$, $i < n$. It indicates the FO derivative possesses histroical memory capacity. The stability of FO calculus is very important to its application in dynamics and control. Matignon pioneered in this area [67] by checking the condition of FO system stability by the poles of FO system in the complex plane of its pseudo-state space. Subsequently, many stability conditions have been derived [19, 73], for instance, the LMI condition for linear time-delay systems [19, 73], Mittag-Leffler stability for nonlinear FO systems [50] and so on.

1.2 Realization of Fractional-Order Calculus in Engineering

In order to obtain a FO calculus in engineering application, there are two kinds of methods which are called the discrete numerical method and approximation method in Laplace domain. A natural approach for computing can be got by using the G-L definition which has the discrete form. Using the commonly used integer-order transfer function to approximate the fraction order operators in a given frequency range is the main feature for the approximation method in Laplace domain. Some results have been summarized in Vinagre's work [89].

In this book, considering the discussion background mainly focusing on the control science and engineering, the FO differential equation can be written as

$$D_t^{\alpha} y(t) = f(t, y(t)). \tag{1.4}$$

For the common sense linear control system, the above equation can be also rewritten as

$$\begin{aligned} a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \cdots + a_0 D^{\alpha_0} y(t) = \\ b_m D^{\beta_m} f(t) + b_{m-1} D^{\beta_{m-1}} f(t) + \cdots + b_0 D^{\beta_0} f(t), \end{aligned} \tag{1.5}$$

or by a continuous transfer function of the form

$$G(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}}. \tag{1.6}$$

It's clear that for the above equation, the first and most important thing is how to solve the equation. Therefore, in this section, we will introduce two kinds of realization examples to compute the FO calculus in our applications.

1.2.1 Discrete Numerical Methods

- **G-L definition based discrete method**

Based on the Grünwald-Letnikov FO definition, for a given FO operator $D^{\alpha} f(t)$, we can use the following approximation

$$D^{\alpha} f(t) \approx \Delta_h^{\alpha} f(t), \tag{1.7}$$

$$\Delta_h^{\alpha} f(t)|_{t=kh} = h^{-\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f(kh - jh). \tag{1.8}$$

As we can see in the former equation, as the step $t = kh \to \infty$, more and more items will be computed to solve the equation. In other words, as the time/step grows, more and more memories are needed to store every step value (from $t = 0 \to kh$) to get current solution. It is unpractical to require infinite memories in our application no matter engineering or numerical simulation. To solve this problem, the short memory principle or finite term truncation method is the natural way for computing. Before introducing the principle, define the coefficient parameter as follows

$$w_j^\alpha = (-1)^j \binom{\alpha}{j}. \tag{1.9}$$

It's clearly that the difference equation (1.8) can be viewed as a weighted sum of the values at all time $f(t - jh)$, $j = 0, 1, 2, ...$ with the coefficient equation (1.9). We can observe that the values of the w_j^α near $t = 0$ (or any other time step $t = t_0$ as initial step) have little influence for the current $t = kh \to \infty$. What's more, the initial value will influence the current value more and more small as the current time $t \to \infty$. Thus, The short memory principle or finite term truncation method allows us to approximate the numerical solution by using the information of the "recent past". By using the time interval $[t - L, t]$ instead of the long history $[0, t]$ to get the given length of memory, a window function liked method is easily proposed to compute the FO derivatives. Some results for the error of the approximation can be found in Monje's work [71].

- **Generalized Adams-Bashforth-Moulton method**

Another effective discretization method can be viewed as generalized method from the classical integer-order Adams-Bashforth-Moulton method [21]. Considering the FO differential (1.4) in the time interval $0 \leqslant t \leqslant T$ with initial value $y^{(k)}(0) = y_0^{(k)}$ [20, 22, 77], the equivalent Volterra integral is:

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \tag{1.10}$$

Define $h = T/N$, $t_n = nh$, $n = 0, 1, \cdots N \in Z^+$, the following Adams-Bashforth predictive formula is firstly used.

$$y_h^p(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j)), \tag{1.11}$$

where the parameter $b_{j,n+1} = h^\alpha((n + 1 - j)^\alpha - (n - j)^\alpha)/\alpha$, and then the Adams-Moulton correction formula is introduced,

$$\begin{aligned} y_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha + 2)} f(t_{n+1}, y_h^p(t_{n+1})) \\ + \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j)), \end{aligned} \tag{1.12}$$

where the parameter $a_{j,n+1}$ can be stated as

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha} & , j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1} & , 1 \leqslant j \leqslant n \\ 1 & , j = n+1 \end{cases} \tag{1.13}$$

And the error is

$$\max_{j=0,1,\ldots,N} \left| y(t_j) - y_h(t_j) \right| = O(h^p), \tag{1.14}$$

where $p = \min(2, 1+\alpha)$ [21]. We can find that the generalized Adams-Bashforth-Moulton method has the same disadvantage in computing, which means that infinite memories are needed as the time/step grow. Thus, the same finite term truncation method to get "recent past" effect is also important.

1.2.2 Approximations Method in Laplace Domain

The above discrete methods are very useful in numerical computing. However, many of the engineering problems can be indeed well expressed by transfer functions. Many engineers in the control field are familiar with the transfer function. Considering the FO transfer function (1.6), many researchers have proposed approximation methods in Laplace domain based on the classical transfer function. The main feature is using the classical integer-order transfer function to approximate FO operator s^α for a given frequency range. This kind of approximation method can be roughly divided into two categories. One is the approximation method based on continued fraction expansions (CFE) [16] and interpolation techniques, such as Carlson method [9] and Matsuda's algorithm [68]. The other is the approximate method based on Curve fitting or Identification techniques, such as Oustaloup's and Chareff's algorithms [11, 75].

- **Continued fraction expansions and interpolation method**

The CFE is a very famous method for evaluation of functions, which frequently converges much more rapidly than power series expansions, and converges in a much larger domain in the complex plane. The result of such approximation for an irrational function $G(s)$, can be expressed in the form,

$$\begin{aligned} G(s) &\approx a_0(s) + \cfrac{b_1(s)}{a_1(s) + \cfrac{b_2(s)}{a_2(s) + \cfrac{b_3(s)}{a_3(s)+\cdots}}} \\ &= a_0(s) + \frac{b_1(s)}{a_1(s)+} \frac{b_2(s)}{a_2(s)+} \frac{b_3(s)}{a_3(s)+} \cdots , \end{aligned} \tag{1.15}$$

where $a_i(s)$ and $b_i(s)$ are rational functions of the variable s, or are constant. The application of the method yields a rational function $\hat{G}(s)$, which is an approximation of the irrational function $G(s)$. Based on this principal, three brief methods are presented

(1) **Piecewise linear approximation**: In Laplace domain, FO operator $s^{-\alpha}, 0 < \alpha < 1$ can be approximated by following rational function

$$G_h(s) = \frac{1}{(1+sT)^{\alpha}}, \tag{1.16}$$

$$G_l(s) = \left(1 + \frac{1}{s}\right)^{\alpha}, \tag{1.17}$$

where $G_h(s)$ is the approximation for high frequencies ($\omega \gg 1$), and $G_l(s)$ is the approximation for low frequencies ($\omega \ll 1$).

(2) **Carlson's method** [9]: This method is an iterative approximation algorithm. Derive regular Newton iterative process to approximate the α-th root. The starting point of the method is the statement of the following relationships:

$$(H(s))^{1/\alpha} - G(s) = 0;\ H(s) = G(s)^{\alpha}. \tag{1.18}$$

Defining $\alpha = 1/q, m = q/2$, in each iteration, starting from the initial value $H_0(s) = 1$, an approximated rational function is obtained in the form

$$H_i(s) = H_{i-1}(s)\frac{(q-m)(H_{i-1}(s))^2 + (q+m)G(s)}{(q+m)(H_{i-1}(s))^2 + (q-m)G(s)}. \tag{1.19}$$

(3) **Matsuda's method** [68]: This method is based on the approximation of an irrational function by a rational one, obtained by CFE and fitting the original function in a set of logarithmically spaced points. Assuming that the selected points are $s_k, k = 0, 1, 2$, the approximation takes on the form:

$$H(s) = a_0 + \frac{s-s_0}{a_1+}\frac{s-s_1}{a_2+}\frac{s-s_2}{a_3+}\cdots, \tag{1.20}$$

where $a_i = \upsilon_i(s_i), \upsilon_0(s) = H(s), \upsilon_{i+1}(s) = \frac{s-s_i}{\upsilon_i(s)-a_i}$.

- **Curve fitting based method**

The curve fitting and identification method is based on the response of fractional integral operator in the frequency domain, and the curve fitting method is adopted to ensure that the following cost function is minimum in the sense of least square criterion:

$$J = \int W(\omega)\left|G(\omega) - \hat{G}(\omega)\right|^2 d\omega, \tag{1.21}$$

where $W(\omega)$ is a weighting function, $G(\omega)$ is the original frequency response, and $\hat{G}(\omega)$ is the frequency response of the approximated rational function. There are two important methods in applications.

(1) **Oustaloup's method** [75]: This method is a proposed by CRONE research group in France. Its highlight is to approximate fractional integral operators in the frequency band of interest through the following approximation formula $H(s) = s^{\mu}$.

$$H(s) \approx C \prod_{k=-N}^{N} \frac{1 + s/\omega_k}{1 + s/\omega'_k}, \tag{1.22}$$

where

$$\begin{aligned} &\omega'_0 = \alpha^{-0.5}\omega_u, \omega_0 = \alpha^{0.5}\omega_u, \omega_u = \sqrt{\omega_h \omega_l}, \\ &\frac{\omega'_{k+1}}{\omega'_k} = \frac{\omega_{k+1}}{\omega_k} = \alpha\eta > 1, \frac{\omega'_{k+1}}{\omega_k} = \eta > 0, \frac{\omega_k}{\omega'_k} = \alpha > 0, \\ &N = \frac{\log(\omega_N/\omega_0)}{\log(\alpha\eta)}, \mu = \frac{\log\alpha}{\log(\alpha\eta)}, \end{aligned} \tag{1.23}$$

where ω_u is the unit gain frequency and the central frequency of a band of frequencies geometrically distributed around it. ω_b, ω_h are the high and low transitional frequencies, respectively.

(2) **Chareff's method** [11]: This method is very similar with Oustaloup's method. The difference is using the following approximation function instead of fractional operator s^{α}.

$$H(s) = \frac{1}{(1 + s/P_T)^{\alpha}}. \tag{1.24}$$

In the frequency domain, above $H(s)$ can be instead by a quotient of polynomials in s in a factorized form:

$$\hat{H}(s) = \frac{\prod_{i=0}^{n-1} (1 + \frac{s}{z_i})}{\prod_{i=0}^{n} (1 + \frac{s}{p_i})}, \tag{1.25}$$

where the coefficients are computed for obtaining a maximum deviation from the original magnitude response in the frequency domain of y dB. Given the break frequency P_T and frequency range$[\omega_{min}, \omega_{max}]$, define

$$a = 10^{y/10(1-\alpha)}, b = 10^{y/10\alpha}. \tag{1.26}$$