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Coordinated by
Dmitri Koroliouk, Sergiy Lyashko
and Nikolaos Limnios

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Preface

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Mathematical methods in engineering are characterized by a wide range of techniques and methods for approaching problems. Moreover, completely different analysis techniques can be applied to the same problem, which is justified by the difference in specific applications.

Therefore, the study of the analysis and solution of specific problems leads the researcher to generate their own techniques and approaches for analyzing similar problems that continuously arise in the process of technical development.

This book provides solutions to specific problems in current areas of computational engineering and cyberphysics, which are considered as examples of ideas and approaches to practical tasks.

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1

The Hydrodynamic-type Equations and the Solitary Solutions

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This chapter analyzes the problem of solitary waves for hydrodynamic-type equations. It mainly focuses on the Korteweg–de Vries (KdV) equation with a small singular perturbation. It should be noted that in the study of various problems associated with the singularly perturbed KdV and Kdv-like equations, linear and nonlinear Wentzel–Kramers–Brillouin (WKB) methods are used. A description of the main idea of this technique is given, and its effectiveness for constructing asymptotic soliton-like solutions for KdV and KdV-like equations with variable coefficients and a singular perturbation is provided.

1.1. Introduction

Nonlinear hydrodynamic-type equations are associated with important models of modern physics and applied mathematics (Whitham 1974; Ablowitz 2011). These

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equations arise in the study of various wave processes and phenomena in hydrodynamics and many other fields of natural science. They are quite difficult to analyze because of their nonlinear nature and the fundamental impossibility of their integration in a closed form. Therefore, it is natural to study either their particular solutions, for example, solutions in the form of a traveling wave, or such equations, using numerical analysis.

These equations include the Boussinesq equation, the Korteweg–de Vries equation (the KdV equation), the modified Korteweg–de Vries equations, the Burgers' equation and the Benjamin–Bona–Mahony equation (Bullough and Caudry 1980; Lamb 1980; Dodd et al. 1982; Newell 1985). The equations have the traveling wave solution $u = f(x - at)$.

The Boussinesq equation:

$$u_{tt} = c_s^2 \frac{\partial^2}{\partial x^2}\left(u + \frac{3}{2}\frac{u^2}{h} + \frac{h^2}{3}u_{xx} \right) \qquad [1.1]$$

and the Korteweg–de Vries equation:

$$u_{xxx} - 6uu_x + u_t = 0 \qquad [1.2]$$

appeared in the 19th century as a result of scientific discussions by many outstanding scientists about the propagation of long waves in rectangular channels and the search for mathematical models to describe the solitary waves that were observed by John Scott Russell in 1834 (Russell 1844).

Equations [1.1] and [1.2] have solutions that describe the waves with profile $A\cosh^{-2}(c(x - vt))$. A characteristic feature of waves is localization in space and time, as well as the dependence of the propagation velocity on their amplitude (Bullough and Caudry 1980).

The Burgers' equation (Burgers 1939)

$$u_t + uu_x = \nu u_{xx}, \quad \nu > 0, \quad x \in R, \ t \geq 0, \qquad [1.3]$$

is one of the simple models describing the evolution of shock waves. It is deduced from the well-known Navier–Stokes system in the one-dimensional case.

The Benjamin–Bona–Mahony equation or regularized long wave equation (Peregrin 1966)

$$u_t + u_x + uu_x - u_{xxt} = 0 \qquad [1.4]$$

was proposed in 1966 because of a search for an alternative to the KdV equation for describing long waves on the surface of a liquid. Although equation [1.4] has one-soliton solutions (Benjamin et al. 1972), it does not have m-soliton $(m \geq 2)$ solutions. Moreover, [1.4] is not an integrable system because it only has three conservation laws.

The integrability of the Korteweg–de Vries (KdV) equation and the KdV-like equations makes it possible to use powerful algebro-geometric approaches (Blackmore et al. 2011) and methods of the Hamiltonian analysis (Faddeev and Takhtajan 1987) to study the properties of the equations and construct a wide set of their exact solutions.

However, the consideration of wave processes in media with variable characteristics and, in particular, with a small dispersion, requires the study of nonlinear equations with variable coefficients and small perturbations. The presence of variable coefficients in nonlinear equations significantly complicates the researching of the corresponding systems. As a result, either the methods of asymptotic analysis are the only mathematical tool for studying such systems or such equations must be studied using numerical analysis (Lyashko 1991, 1995).

1.2. The Korteweg–de Vries equation and the soliton solutions

Most of the above-mentioned equations are models of fluid motion. The Korteweg–de Vries equation is used to mathematically describe the dynamics of solitary waves on a liquid surface. Later, in the middle of the 20th century, it was found that equation [1.2] and the KdV-like equations arise while studying many various phenomena and processes, in particular, in plasma, solid body theory, optic, biology, telecommunication systems, etc. Equation [1.2] is currently one of the fundamental equations of modern physics. If we consider this equation as an example, then we can find many different characteristic phenomena and properties that are inherent in hydrodynamic-type equations.

An in-depth study of the Korteweg–de Vries equation began in the late 1960s after the publication of the paper by Zabusky and Kruskal (1965). The authors pointed out the connection between the equation and the Fermi–Pasta–Ulam problem. Zabusky and Kruskal found that the Korteweg–de Vries equation possesses solutions that have the property of retaining their waveform after collision with waves of the same nature. Such solutions are called *soliton solutions* or *solitons* (Zabusky and Kruskal 1965).

The amazing discovery drew great attention from both mathematicians and physicists to the Korteweg–de Vries equation. This attention increased significantly after the creation of a new technique called the inverse scattering transform, which has been successfully applied to the study of many integrable systems (Gardner et al. 1967). Subsequently, the inverse scattering transform became a powerful tool for studying numerous nonlinear systems of modern theoretical and mathematical physics.

Many monographs are devoted to different mathematical aspects of the theory of nonlinear integrable systems, including the Korteweg–de Vries equation. In particular, one-, two-, multi-soliton solutions, finite-gap solutions, etc. have been constructed for the KdV equation. For this equation and its generalizations, problems on existence, uniqueness and other properties (smoothness, exponential decay, etc.) of its solutions, and solutions to the Cauchy problem for various classes of initial functions were also studied (Sjoberg 1970; Hruslov 1976; Kato 1979; Faminskii 1990).

It has been found that properties of the solutions to the KdV equation essentially depend on the initial conditions (Pokhozhayev 2010; Turbal et al. 2015). In addition to soliton, periodic, finite-gap and shock wave solutions, the KdV equation has solutions with some singularities on the curves (Arkadyev et al. 1984), as well as solutions that are destroyed roughly or destroyed according to the gradient catastrophe scenario (Pokhozhayev 2010).

1.3. The Korteweg–de Vries equation with a small perturbation

The discovery of new classes of solutions to the KdV equation had a great influence on the development of the perturbation theory of nonlinear systems. Such research began with the study of the perturbed Korteweg–de Vries equation by asymptotic methods. In the paper by Miura and Kruskal (1974), the authors generalized the linear Wentzel–Kramers–Brillouin (WKB) method for the nonlinear equations and constructed the leading term of the asymptotic series for the quasi-periodic solution to the KdV equation with a small dispersion:

$$u_t + uu_x + \delta^2 u_{xxx} = 0, \qquad [1.5]$$

where δ is a small parameter. Thus, the nonlinear WKB method was created. It has become a powerful mathematical tool for studying nonlinear systems of the hydrodynamic-type (Maslov 1982; Samoilenko and Samoilenko 2008, 2019; Lyashko et al. 2021) with a singular perturbation.

By means of the Bogoliubov M.M. averaging method (Bogoliubov and Mitropolsky 1961), Flaschka et al. (1980) studied the modulation of nonlinear waves, which are described by the Korteweg–de Vries equation. Although the equation under consideration does not contain a small parameter δ explicitly, the dependence of its finite-gap solutions on the small parameter manifests through a small deformation of the associated Riemann surfaces.

Furthermore, Lax and Levermore (1983a, 1983b, 1983c) considered the problem on the limit of the solution to the Cauchy problem for the singularly perturbed Korteweg–de Vries equation [1.5], as a small parameter tends to zero. While solving the problem they used the inverse scattering transform. In particular, the authors constructed eigenfunctions for the Lax operator with a small perturbation by the linear WKB method. Subsequently, this problem under other initial conditions was considered in many papers and studied mainly by numerical methods.

Asymptotic methods turned out to be an effective means of investigating other problems for the Korteweg–de Vries equation. So, for example, the equation with regular perturbation was considered by de Kerf (1988), who constructed asymptotic solutions to the Cauchy problem of the following form for the non-homogeneous Korteweg–de Vries equation

$$u_t - 6uu_x + u_{xxx} = \varepsilon f[u], \quad u(x,0) = u_0(x), \qquad [1.6]$$

where $f[u]$ is a functional and the initial function $u_0(x)$ is a soliton-like one.

Ilyin and Kalyakin studied the problem [1.6] when the initial condition described the perturbation of multisoliton solutions. The authors constructed perturbed soliton solutions to the Korteweg–de Vries equation (Kalyakin 1992) using the Krylov–Bogoliubov–Mitropolsky method (Bogoliubov and Mitropolsky 1961).

Later Glebov et al. (2005) and Glebov and Kiselev (2005) considered the Korteweg–de Vries equation with forced perturbation of the following form:

$$\varepsilon^2 f(\varepsilon x)\cos\left(\frac{S\left(\varepsilon^2 x, \varepsilon^2 t\right)}{\varepsilon^2}\right).$$

They found asymptotic solutions using the nonlinear WKB method. This made it possible to study the phenomenon of passage through resonance.

In the papers mentioned above, the Korteweg–de Vries equation with a small parameter was considered. It is important that under a regular perturbation, the unperturbed equation is the classical Korteweg–de Vries equation, while under a singular perturbation it is an equation with constant coefficients.

The presence of singular perturbations in such problems requires the study of challenging mathematical problems. The first papers on singularly perturbed nonlinear partial differential equations with variable coefficients are due to Maslov and his students, who considered equations that are integrable in particular cases. We should also mention the paper by Maslov and Omel'yanov (1981), in which an asymptotic solution of a differential equation with variable coefficients was constructed. They considered an equation describing the propagation of waves in shallow water with variable depth. It was the KdV-like equation of the following form:

$$u_t + (\rho_1 + 3\rho_2 u)u_x + \varepsilon^2 \rho_3 u_{xxx} + \rho_4 u = 0, \qquad [1.7]$$

where the coefficients of the equation depend on the depth function $H(x) > 0$ of the non-perturbed liquid and have the form:

$$\rho_1 = \sqrt{gH(x)}, \quad \rho_2 = \sqrt{gH^{-1}(x)}\Big/2, \quad \rho_3 = \sqrt{gH^5(x)}\Big/6, \quad \rho_4 = \rho_{2x}/2,$$

where g is the acceleration of gravity, and ε is a small parameter that characterizes the value of the dispersion.

Using the nonlinear WKB method, the authors constructed asymptotic solutions to equation [1.7]. In structure, the solutions are close to solutions of the soliton type. These solutions are called *asymptotic soliton-like solutions* (Maslov and Omel'yanov 1981, 2001).

Furthermore, the problem of constructing asymptotic soliton-like solutions was considered in a more general statement when the coefficients of the KdV equation are variable and have a rather general form (Samoilenko and Samoilenko 2012a, 2012b).

Finally, we note that singularly perturbed partial differential equations with variable coefficients naturally arise in the study of mathematical models of wave processes and phenomena in media whose characteristics depend on time and spatial variables, and singular perturbation. These problems are generally non-integrable, although they have soliton solutions in the case of constant coefficients. Hence, the problem of constructing asymptotic soliton-like solutions for a singularly perturbed partial differential equation with variable coefficients is of great interest (Whitham 1974; Ablowitz 2011; Samoilenko and Samoilenko 2019).

1.4. The linear WKB technique and its generalization

To construct an asymptotic soliton-like solution to the Korteweg–de Vries equation with a singular perturbation, the nonlinear WKB method (Miura and Kruskal 1974) is used. This technique is generalization of the WKB method.

The WKB method is named after the German scientists Wentzel, Kramers and Brillouin. At the beginning of 1926, in fact, at the same time, they proposed this method (Brillouin 1926; Kramers 1926; Wentzel 1926) for the construction of approximate solutions to the one-dimensional stationary Schrödinger equation of quantum mechanics (Schrödinger 1926)

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{d\,x^2}+V(x)\Psi(x)=E\Psi(x)\,. \qquad [1.8]$$

This was an efficient approach to studying the equation, since [1.8] is a non-integrable differential equation in general.

Here, the Planck constant $\hbar$ is supposed to be a small parameter, and $\Psi = \Psi(x)$ is a wave (complex-valued) function which, according to its physical content, satisfies the conditions of regularity, i.e. the wave function is finite, single-valued and continuous with its first derivatives. These assumptions are satisfied, for example, under condition $E - V(x) > 0$.

The main idea of the WKB method is that a solution of equation [1.8] is represented in the special form $\Psi(x) = e^{\Phi(x)}$, where the complex-valued function $\Phi(x)$ according to equation [1.8] satisfies the second-order nonlinear differential equation

$$\Phi''(x) + \Phi'^2(x) = \frac{2m}{\hbar^2}\left(V(x) - E\right). \qquad [1.9]$$

It is natural to consider the real-valued functions $A(x) = \operatorname{Re}\Phi'(x)$, $B(x) = \operatorname{Im}\Phi'(x)$, characterized amplitude and phase of the oscillating solutions. From relation [1.8], it follows that these functions satisfy the first-order nonlinear differential equation

$$A' + A^2 - B^2 = \frac{2m}{\hbar^2}\left(V(x) - E\right), \quad B' + 2AB = 0, \qquad [1.10]$$

the solutions of which can be constructed as asymptotic expansions in the value $\hbar$:

$$A(x) = \frac{1}{\hbar}\sum_{k=0}^{\infty} \hbar^k A_k(x), \quad B(x) = \frac{1}{\hbar}\sum_{k=0}^{\infty} \hbar^k B_k(x). \qquad [1.11]$$

The coefficients of asymptotic series [1.11] are determined recursively from the equations found in the standard way from equation [1.10]. In particular, the main terms $A_0(x)$ and $B_0(x)$ in equation [1.11] have to satisfy the differential equations

$$A_0^2(x) - B_0^2(x) = 2m\left(V(x) - E\right), \quad A_0(x)B_0(x) = 0, \qquad [1.12]$$

and the terms $A_1(x)$ and $B_1(x)$ satisfy the following differential equations:

$$A_0'(x) + 2A_0A_1 - 2B_0B_1 = 0, \qquad [1.13]$$

$$B_0'(x) + 2A_0B_1 + 2A_1B_0 = 0. \qquad [1.14]$$

Obviously, to find the coefficients $A_0(x)$, $B_0(x)$, $A_1(x)$ and $B_1(x)$ from relations [1.12]–[1.14], both cases $A_0(x) = 0$ and $B_0(x) = 0$ must be considered separately.

Taking into account the condition of regularization, we first consider the case $A_0(x) = 0$. This equality means that the amplitude of the oscillation changes more slowly than its phase. Therefore, from equations [1.12] to [1.14], we have

$$B_0(x) = \pm\sqrt{2m(E - V(x))},$$

$$A_1(x) = -\frac{1}{4}\frac{d}{dx}\ln\left(2m(E - V(x))\right), \quad B_1(x) = 0.$$

As a result, taking into account the first two terms of the asymptotic for the functions $A(x)$, $B(x)$, the asymptotic approximation for the solution to equation [1.8] is given in the form

$$\Psi(x) = \frac{C_+}{\sqrt[4]{2m(E - V(x))}}\exp\left(\frac{i}{\hbar}\int\sqrt{2m(E - V(x))}\,dx\right) +$$

$$+\frac{C_-}{\sqrt[4]{2m(E - V(x))}}\exp\left(-\frac{i}{\hbar}\int\sqrt{2m(E - V(x))}\,dx\right), \qquad [1.15]$$

where C_+, C_- are arbitrary constants.

If $B_0(x) = 0$, i.e. the phase changes more slowly than the amplitude, then from equations [1.12]–[1.14], we have

$$A_0(x) = \pm\sqrt{2m(E - V(x))}, \quad B_0(x) = B_1(x) = 0,$$

and the asymptotic approximation for the solution to equation [1.8] is written as

$$\Psi_1(x) = \frac{C_+}{\sqrt[4]{2m(E - V(x))}}\exp\left(\frac{1}{\hbar}\int\sqrt{2m(E - V(x))}\,dx\right) +$$

$$+\frac{C_-}{\sqrt[4]{2m(E - V(x))}}\exp\left(-\frac{1}{\hbar}\int\sqrt{2m(E - V(x))}\,dx\right), \qquad [1.16]$$

where C_+, C_- are arbitrary complex constants.

Note that while applying the WKB method, it is enough to construct only the first two terms of the asymptotics.

Formulas [1.15] and [1.16] asymptotically approximate the solutions to equation [1.8] for all real arguments x except for the neighborhoods of the turning points, where the equality $E - V(x) = 0$ holds.

In the neighborhoods of the turning points, the asymptotic solution to equation [1.8] is searched for in the other way, using the expansion of its right-hand side in a Taylor series in the neighborhood of the corresponding point. The main term of the asymptotic solution to equation [1.8] is defined from the Bessel equation and written as a linear combination of the Bessel functions.

While using the linear WKB method, the solution of equation [1.8] is sought as an exponential function in some asymptotic expansion. Miura and Kruskal (1974) proposed to construct a solution to the nonlinear differential equation with a singular perturbation as an asymptotic series in a small parameter with quickly oscillating coefficients. They called this technique the nonlinear WKB method. This approach has proven to be quite effective (Samoilenko and Samoilenko 2005, 2008, 2019, 2012a, 2012b; Lyashko et al. 2021) for constructing the asymptotic soliton-like solutions to the following equation:

$$\varepsilon^n u_{xxx} = a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)uu_x, \quad n \in N, \qquad [1.17]$$

where $\varepsilon > 0$ is a small parameter, the coefficients $a(x,t,\varepsilon)$, $b(x,t,\varepsilon)$, $(x,t) \in R \times [0;T]$ are infinitely differentiable and are given with asymptotic (according to Poincaré) series

$$a(x,t,\varepsilon) = \sum_{k=0}^{N} \varepsilon^k a_k(x,t) + O\left(\varepsilon^{N+1}\right),$$

$$b(x,t,\varepsilon) = \sum_{k=0}^{N} \varepsilon^k b_k(x,t) + O\left(\varepsilon^{N+1}\right),$$

and the condition $a_0(x,t)b_0(x,t) \neq 0$, $(x,t) \in R \times [0;T]$, takes place.

Note that due to the presence of the variable coefficients in equation [1.17], the problem of finding its asymptotic solutions becomes much more complicated, and at the same time, a number of new interesting mathematical problems appear. In addition, the degree of the singularity in the equation [1.17] also has a significant impact on the structure of the constructed asymptotic soliton-like solution, the algorithm for finding it and the properties of the solutions.

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2

The Nonlinear WKB Technique and Asymptotic Soliton-like Solutions to the Korteweg–de Vries Equation with Variable Coefficients and Singular Perturbation

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Here we consider the problem of the mathematical description of soliton-like solutions of hydrodynamic models, such as the Korteweg–de Vries (KdV) equation and its generalizations, describing wave processes in inhomogeneous media with variable characteristics and a small dispersion. Mainly, attention is paid to the development of an algorithm for constructing approximate (asymptotic) solutions of soliton type. These solutions contain a regular part, which is a background function, and a singular part, which reflects the soliton properties of the solutions. They are

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constructed using the nonlinear WKB method. The construction of the regular and singular parts of the searched asymptotic solution is described in detail.

Problems that arise while using the algorithm as well as while its justification are discussed. Statements about the accuracy with which the constructed asymptotic solution satisfies the original equation are proved.

This approach provides us with effective tools for studying the influence of model parameters on the properties of the dynamical system under consideration. In particular, the obtained results can be used for the subsequent analysis of these solutions in different ways, for example, by using computer simulation.

2.1. Introduction

In the papers by Dobrokhotov and Maslov (1981), and Maslov and Omel'yanov (1981), the notion of an asymptotic soliton-like solution to equations of integrable type with a small singular perturbation is proposed. In accordance with the kind of small parameter, the nonlinear WKB method (Miura and Kruskal 1974) was used. Later, these results were detailed in the monograph (Maslov and Omel'yanov 2001). Much attention is also paid to other issues related to the properties of solutions to equations with a singular perturbation. In particular, the problem regarding discontinuous solutions to the unperturbed problems is considered. These solutions can be obtained from the formulas for the exact or asymptotic solutions in the limiting case as a small parameter tends to zero (Lax and Levermore 1983a, 1983b, 1983c; Samoilenko and Samoilenko 2010).

In the work on this subject (Maslov and Omel'yanov 1981), the equation of hydrodynamic KdV-like equation

$$u_t + (\rho_1 + 3\rho_2 u)u_x + \varepsilon^2 \rho_3 u_{xxx} + \rho_4 u = 0, \qquad [2.1]$$

was considered. Here, the coefficients of the equation depend on the depth function $H(x) > 0$ of the non-perturbed liquid and have the form

$$\rho_1 = \sqrt{gH(x)}, \quad \rho_2 = \sqrt{gH^{-1}(x)}\Big/2, \ \rho_3 = \sqrt{gH^5(x)}\Big/6, \quad \rho_4 = \rho_{2x}/2,$$

where g is the acceleration of gravity, and ε is a small parameter characterized value of dispersion.