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MATHEMATICS

Mathematics in Engineering

Computational Methods and Mathematical Modeling in Cyberphysics and Engineering Applications 1

**Coordinated by
Dmitri Koroliouk, Sergiy Lyashko
and Nikolaos Limnios**

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Contents

Preface	xi
Dmitri KOROLIUK	
Chapter 1. The Hydrodynamic-type Equations and the Solitary Solutions	1
Sergiy LYASHKO, Valerii SAMOILENKO, Yuliia SAMOILENKO and Ihor GAPYAK	
1.1. Introduction	1
1.2. The Korteweg–de Vries equation and the soliton solutions	3
1.3. The Korteweg–de Vries equation with a small perturbation	4
1.4. The linear WKB technique and its generalization	7
1.5. Acknowledgments	11
1.6. References	11
Chapter 2. The Nonlinear WKB Technique and Asymptotic Soliton-like Solutions to the Korteweg–de Vries Equation with Variable Coefficients and Singular Perturbation	15
Sergiy LYASHKO, Valerii SAMOILENKO, Yuliia SAMOILENKO and Evgen VAKAL	
2.1. Introduction	16
2.2. Main notations and definitions	18
2.3. The structure of the asymptotic one-phase soliton-like solution.	19
2.4. The KdV equation with quadratic singularity	20
2.5. Equations for the regular part of the asymptotics and their analysis	22
2.6. Equations for the singular part of the asymptotics and their analysis	24
2.6.1. The main term of the singular part	25
2.6.2. The higher terms of the singular part and the orthogonality condition	26
2.6.3. The orthogonality condition and the discontinuity curve	29
2.6.4. Prolongation of the singular terms from the discontinuity curve	34
2.7. Justification of the algorithm	38

2.8. Discussion and conclusion	44
2.9. Acknowledgments	45
2.10. References	45

Chapter 3. Asymptotic Analysis of the vcKdV Equation with Weak Singularity 49

Sergiy LYASHKO, Valerii SAMOILENKO, Yuliia SAMOILENKO and Nataliia LYASHKO

3.1. Introduction	50
3.2. The asymptotic soliton-like solutions	51
3.3. The examples of the asymptotic soliton-like solutions	56
3.3.1. The asymptotic step-wise solutions	57
3.3.2. The asymptotic solutions of soliton type	61
3.4. Discussion and conclusion	66
3.5. Acknowledgments	66
3.6. References	66

Chapter 4. Modeling of Heterogeneous Fluid Dynamics with Phase Transitions and Porous Media 69

Gennadiy V. SANDRAKOV

4.1. Introduction	69
4.2. The large particle method	72
4.3. The particle-in-cell method	79
4.4. Modeling of heterogeneous fluid dynamics	83
4.5. Modeling of heterogeneous fluid dynamics with phase transitions	88
4.6. Modeling of viscous fluid dynamics and porous media	94
4.7. References	98

Chapter 5. Mathematical Models and Control of Functionally Stable Technological Process 101

Volodymyr PICHKUR, Valentyn SOBCHUK and Dmytro CHERNIY

5.1. Introduction	101
5.2. Analysis of production process planning procedure	104
5.3. Mathematical model of the production process management system of an industrial enterprise.	108
5.4. Control design	111
5.5. Algorithm of control of production process	115
5.6. Conclusion	116
5.7. Acknowledgments	117
5.8. References	118

Chapter 6. Alternative Direction Multiblock Method with Nesterov Acceleration and Its Applications 121

Vladislav HRYHORENKO, Nataliia LYASHKO, Sergiy LYASHKO and Dmytro KLYUSHIN

6.1. Introduction	121
6.2. Proximal operators.	122
6.3. ADMM (alternating direction method of multipliers)	128
6.4. Bregman iteration	131
6.5. Forward-backward envelope (FBE)	132
6.6. Douglas–Rachford envelope (DRE)	133
6.7. Proximal algorithms for complex functions	134
6.8. Fast alternative directions methods.	137
6.9. Numerical experiments	142
6.9.1. Exchange problem	142
6.9.2. Basis pursuit problem.	143
6.9.3. Constrained LASSO problem	144
6.10. Conclusion	145
6.11. References	145

Chapter 7. Modified Extragradient Algorithms for Variational Inequalities 149

Vladimir V. SEMENOV and Sergey V. DENISOV

7.1. Introduction	149
7.2. Preliminaries	149
7.3. Overview of the main algorithms for solving variational inequalities and approximations of fixed points	156
7.4. Modified extragradient algorithm for variational inequalities	164
7.5. Modified extragradient algorithm for variational inequalities and operator equations with a priori information	173
7.6. Strongly convergent modified extragradient algorithm.	177
7.6.1. Algorithm variant for variational inequalities	178
7.6.2. Variant for problems with a priori information.	193
7.7. References	199

Chapter 8. On Multivariate Algorithms of Digital Signatures on Secure El Gamal-Type Mode 205

Vasyl USTIMENKO

8.1. On post-quantum, multivariate and non-commutative cryptography	206
8.2. On stable subgroups of formal Cremona group and privatization of multivariate public keys based on maps of bounded degree	208
8.3. Multivariate Tahoma protocol for stable Cremona generators and its usage for multivariate encryption algorithms.	211

8.4. On multivariate digital signature algorithms and their privatization scheme	214
8.5. Examples of stable cubical groups	216
8.5.1. Simplest graph-based example.	216
8.5.2. Other stable subgroups defined via linguistic graphs	219
8.5.3. Special homomorphisms of linguistic graphs and corresponding semigroups	222
8.5.4. Example of stable subsemigroups of arbitrary degree	223
8.6. Conclusion	225
8.7. References	227

Chapter 9. Metasurface Model of Geographic Baric Field Formation 231
Dmitri KOROLIUK, Maksym ZOZIUK, Pavlo KRYSENKO and Yuriy YAKYMENKO

9.1. Introduction	231
9.2. The parametric scalar field model principle	233
9.3. Local isobaric scalar field model.	234
9.4. Modeling Chladni figures based on the proposed model.	235
9.5. The frequency of forcing influences and the problem of its detection	237
9.6. Conclusion	239
9.7. References	241

Chapter 10. Simulation of the Electron–Hole Plasma State by Perturbation Theory Methods 245
Andrii BOMBA, Sergiy LYASHKO and Ihor MOROZ

10.1. Introduction. Nonlinear boundary value problems of the p–i–n diodes theory	245
10.2. Construction of an asymptotic solution of a boundary value problem for the system of the charge carrier current continuity equations and the Poisson equation	249
10.3. Simulation of the charge carriers’ stationary distribution in the electron–hole plasma of the p–i–n diode assembly elements	262
10.4. Modeling the charge carriers stationary distribution in the active region of the integrated surface-oriented p–i–n structures	264
10.5. Final considerations	270
10.6. References	271

Chapter 11. Diffusion Perturbations in Models of the Dynamics of Infectious Diseases Taking into Account the Concentrated Effects 273
Serhii BARANOVSKY, Andrii BOMBA, Sergiy LYASHKO and Oksana PRYSHCHEPA

11.1. Introduction	273
11.2. Model problem of infectious disease dynamics taking into account diffusion perturbation and asymptotics of the solution	277

11.3. Modeling of diffusion perturbations of infectious disease process taking into account the concentrated effects and immunotherapy	282
11.4. Modeling the influence of diffusion perturbations on development of infectious diseases under convection	288
11.5. Numerical experiment results	292
11.6. Conclusion	300
11.7. References	301
Chapter 12. Solitary Waves in the “Shallow Water” Environments	305
Yurii TURBAL, Mariana TURBAL and Andrii BOMBA	
12.1. Introduction	305
12.2. T-forms for the solitary wave approximation	307
12.3. Existence of the solution of the gas dynamics equations in the form of solitary waves	313
12.4. Analysis of the localized wave trajectories.	332
12.5. Numerical results.	338
12.6. Conclusion	341
12.7. References	342
Chapter 13. Instrument Element and Grid Middleware in Metrology Problems	345
Pavlo NEYEZHMAKOV, Stanislav ZUB, Sergiy LYASHKO, Irina YALOVEGA and Nataliia LYASHKO	
13.1. Introduction	345
13.2. Security in the grid	347
13.3. Grid element for measuring instruments	347
13.4. Grid and some problems of metrology	350
13.5. Discussion and conclusion.	352
13.6. References	352
Chapter 14. Differential Evolution for Best Uniform Spline Approximation	355
Larysa VAKAL and Evgen VAKAL	
14.1. Introduction	356
14.2. Problem statement	356
14.3. Review of methods for spline approximation	357
14.4. Algorithm	359
14.5. Experimental results and discussion	362
14.6. Conclusion	364
14.7. References	365

Chapter 15. Finding a Nearest Pair of Points Between Two Smooth Curves in Euclidean Space	367
Vladimir V. SEMENOV, Nataliia LYASHKO, Stanislav ZUB and Yevhen HAVRYLKO	
15.1. Introduction	367
15.2. Define the problem and notations.	368
15.3. Lagrange function with energy dissipation.	369
15.4. Lagrange equation	370
15.5. Hamiltonian equations	372
15.6. Numerical experiments	376
15.7. Concluding remarks	378
15.8. References	379
 Chapter 16. Constrained Markov Decision Process for the Industry	 381
Michel BOUSSEMARY and Nikolaos LIMNIOS	
16.1. Introduction	381
16.2. Introduction to constrained Markov decision processes	382
16.2.1. Introduction	382
16.2.2. Model.	383
16.2.3. Economic criteria	384
16.2.4. Infinite horizon expected discounted reward	386
16.2.5. Infinite horizon expected average reward	392
16.3. Markov decision process with a constraint on the asymptotic availability . .	396
16.3.1. Introduction	396
16.3.2. Model.	397
16.3.3. Algorithm	399
16.3.4. Application.	399
16.4. Markov decision process with a constraint on the asymptotic failure rate. . .	408
16.4.1. Introduction	408
16.4.2. Model.	409
16.4.3. Algorithm	413
16.4.4. Application.	413
16.5. Conclusion	418
16.6. References	419
 List of Authors	 423
 Index.	 427

Preface

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Mathematical methods in engineering are characterized by a wide range of techniques and methods for approaching problems. Moreover, completely different analysis techniques can be applied to the same problem, which is justified by the difference in specific applications.

Therefore, the study of the analysis and solution of specific problems leads the researcher to generate their own techniques and approaches for analyzing similar problems that continuously arise in the process of technical development.

This book provides solutions to specific problems in current areas of computational engineering and cyberphysics, which are considered as examples of ideas and approaches to practical tasks.

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1

The Hydrodynamic-type Equations and the Solitary Solutions

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This chapter analyzes the problem of solitary waves for hydrodynamic-type equations. It mainly focuses on the Korteweg–de Vries (KdV) equation with a small singular perturbation. It should be noted that in the study of various problems associated with the singularly perturbed KdV and KdV-like equations, linear and nonlinear Wentzel–Kramers–Brillouin (WKB) methods are used. A description of the main idea of this technique is given, and its effectiveness for constructing asymptotic soliton-like solutions for KdV and KdV-like equations with variable coefficients and a singular perturbation is provided.

1.1. Introduction

Nonlinear hydrodynamic-type equations are associated with important models of modern physics and applied mathematics (Whitham 1974; Ablowitz 2011). These

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equations arise in the study of various wave processes and phenomena in hydrodynamics and many other fields of natural science. They are quite difficult to analyze because of their nonlinear nature and the fundamental impossibility of their integration in a closed form. Therefore, it is natural to study either their particular solutions, for example, solutions in the form of a traveling wave, or such equations, using numerical analysis.

These equations include the Boussinesq equation, the Korteweg–de Vries equation (the KdV equation), the modified Korteweg–de Vries equations, the Burgers' equation and the Benjamin–Bona–Mahony equation (Bullough and Caudry 1980; Lamb 1980; Dodd et al. 1982; Newell 1985). The equations have the traveling wave solution $u = f(x - at)$.

The Boussinesq equation:

$$u_{tt} = c_s^2 \frac{\partial^2}{\partial x^2} \left(u + \frac{3}{2} \frac{u^2}{h} + \frac{h^2}{3} u_{xx} \right) \quad [1.1]$$

and the Korteweg–de Vries equation:

$$u_{xxx} - 6uu_x + u_t = 0 \quad [1.2]$$

appeared in the 19th century as a result of scientific discussions by many outstanding scientists about the propagation of long waves in rectangular channels and the search for mathematical models to describe the solitary waves that were observed by John Scott Russell in 1834 (Russell 1844).

Equations [1.1] and [1.2] have solutions that describe the waves with profile $A \cosh^{-2}(c(x - vt))$. A characteristic feature of waves is localization in space and time, as well as the dependence of the propagation velocity on their amplitude (Bullough and Caudry 1980).

The Burgers' equation (Burgers 1939)

$$u_t + uu_x = \nu u_{xx}, \quad \nu > 0, \quad x \in R, \quad t \geq 0, \quad [1.3]$$

is one of the simple models describing the evolution of shock waves. It is deduced from the well-known Navier–Stokes system in the one-dimensional case.

The Benjamin–Bona–Mahony equation or regularized long wave equation (Peregrin 1966)

$$u_t + u_x + uu_x - u_{xx} = 0 \quad [1.4]$$

was proposed in 1966 because of a search for an alternative to the KdV equation for describing long waves on the surface of a liquid. Although equation [1.4] has one-soliton solutions (Benjamin et al. 1972), it does not have m -soliton ($m \geq 2$) solutions. Moreover, [1.4] is not an integrable system because it only has three conservation laws.

The integrability of the Korteweg–de Vries (KdV) equation and the KdV-like equations makes it possible to use powerful algebro-geometric approaches (Blackmore et al. 2011) and methods of the Hamiltonian analysis (Faddeev and Takhtajan 1987) to study the properties of the equations and construct a wide set of their exact solutions.

However, the consideration of wave processes in media with variable characteristics and, in particular, with a small dispersion, requires the study of nonlinear equations with variable coefficients and small perturbations. The presence of variable coefficients in nonlinear equations significantly complicates the researching of the corresponding systems. As a result, either the methods of asymptotic analysis are the only mathematical tool for studying such systems or such equations must be studied using numerical analysis (Lyashko 1991, 1995).

1.2. The Korteweg–de Vries equation and the soliton solutions

Most of the above-mentioned equations are models of fluid motion. The Korteweg–de Vries equation is used to mathematically describe the dynamics of solitary waves on a liquid surface. Later, in the middle of the 20th century, it was found that equation [1.2] and the KdV-like equations arise while studying many various phenomena and processes, in particular, in plasma, solid body theory, optic, biology, telecommunication systems, etc. Equation [1.2] is currently one of the fundamental equations of modern physics. If we consider this equation as an example, then we can find many different characteristic phenomena and properties that are inherent in hydrodynamic-type equations.

An in-depth study of the Korteweg–de Vries equation began in the late 1960s after the publication of the paper by Zabusky and Kruskal (1965). The authors pointed out the connection between the equation and the Fermi–Pasta–Ulam problem. Zabusky and Kruskal found that the Korteweg–de Vries equation

possesses solutions that have the property of retaining their waveform after collision with waves of the same nature. Such solutions are called *soliton solutions* or *solitons* (Zabusky and Kruskal 1965).

The amazing discovery drew great attention from both mathematicians and physicists to the Korteweg–de Vries equation. This attention increased significantly after the creation of a new technique called the inverse scattering transform, which has been successfully applied to the study of many integrable systems (Gardner et al. 1967). Subsequently, the inverse scattering transform became a powerful tool for studying numerous nonlinear systems of modern theoretical and mathematical physics.

Many monographs are devoted to different mathematical aspects of the theory of nonlinear integrable systems, including the Korteweg–de Vries equation. In particular, one-, two-, multi-soliton solutions, finite-gap solutions, etc. have been constructed for the KdV equation. For this equation and its generalizations, problems on existence, uniqueness and other properties (smoothness, exponential decay, etc.) of its solutions, and solutions to the Cauchy problem for various classes of initial functions were also studied (Sjöberg 1970; Hruslov 1976; Kato 1979; Faminskii 1990).

It has been found that properties of the solutions to the KdV equation essentially depend on the initial conditions (Pokhozhayev 2010; Turbal et al. 2015). In addition to soliton, periodic, finite-gap and shock wave solutions, the KdV equation has solutions with some singularities on the curves (Arkadyev et al. 1984), as well as solutions that are destroyed roughly or destroyed according to the gradient catastrophe scenario (Pokhozhayev 2010).

1.3. The Korteweg–de Vries equation with a small perturbation

The discovery of new classes of solutions to the KdV equation had a great influence on the development of the perturbation theory of nonlinear systems. Such research began with the study of the perturbed Korteweg–de Vries equation by asymptotic methods. In the paper by Miura and Kruskal (1974), the authors generalized the linear Wentzel–Kramers–Brillouin (WKB) method for the nonlinear equations and constructed the leading term of the asymptotic series for the quasi-periodic solution to the KdV equation with a small dispersion:

$$u_t + uu_x + \delta^2 u_{xxx} = 0, \quad [1.5]$$

where δ is a small parameter. Thus, the nonlinear WKB method was created. It has become a powerful mathematical tool for studying nonlinear systems of the hydrodynamic-type (Maslov 1982; Samoilenko and Samoilenko 2008, 2019; Lyashko et al. 2021) with a singular perturbation.

By means of the Bogoliubov M.M. averaging method (Bogoliubov and Mitropolsky 1961), Flaschka et al. (1980) studied the modulation of nonlinear waves, which are described by the Korteweg–de Vries equation. Although the equation under consideration does not contain a small parameter δ explicitly, the dependence of its finite-gap solutions on the small parameter manifests through a small deformation of the associated Riemann surfaces.

Furthermore, Lax and Levermore (1983a, 1983b, 1983c) considered the problem on the limit of the solution to the Cauchy problem for the singularly perturbed Korteweg–de Vries equation [1.5], as a small parameter tends to zero. While solving the problem they used the inverse scattering transform. In particular, the authors constructed eigenfunctions for the Lax operator with a small perturbation by the linear WKB method. Subsequently, this problem under other initial conditions was considered in many papers and studied mainly by numerical methods.

Asymptotic methods turned out to be an effective means of investigating other problems for the Korteweg–de Vries equation. So, for example, the equation with regular perturbation was considered by de Kerf (1988), who constructed asymptotic solutions to the Cauchy problem of the following form for the non-homogeneous Korteweg–de Vries equation

$$u_t - 6uu_x + u_{xxx} = \varepsilon f[u], \quad u(x, 0) = u_0(x), \quad [1.6]$$

where $f[u]$ is a functional and the initial function $u_0(x)$ is a soliton-like one.

Ilyin and Kalyakin studied the problem [1.6] when the initial condition described the perturbation of multisoliton solutions. The authors constructed perturbed soliton solutions to the Korteweg–de Vries equation (Kalyakin 1992) using the Krylov–Bogoliubov–Mitropolsky method (Bogoliubov and Mitropolsky 1961).

Later Glebov et al. (2005) and Glebov and Kiselev (2005) considered the Korteweg–de Vries equation with forced perturbation of the following form:

$$\varepsilon^2 f(\varepsilon x) \cos \left(\frac{S(\varepsilon^2 x, \varepsilon^2 t)}{\varepsilon^2} \right).$$

They found asymptotic solutions using the nonlinear WKB method. This made it possible to study the phenomenon of passage through resonance.

In the papers mentioned above, the Korteweg–de Vries equation with a small parameter was considered. It is important that under a regular perturbation, the unperturbed equation is the classical Korteweg–de Vries equation, while under a singular perturbation it is an equation with constant coefficients.

The presence of singular perturbations in such problems requires the study of challenging mathematical problems. The first papers on singularly perturbed nonlinear partial differential equations with variable coefficients are due to Maslov and his students, who considered equations that are integrable in particular cases. We should also mention the paper by Maslov and Omel'yanov (1981), in which an asymptotic solution of a differential equation with variable coefficients was constructed. They considered an equation describing the propagation of waves in shallow water with variable depth. It was the KdV-like equation of the following form:

$$u_t + (\rho_1 + 3\rho_2 u)u_x + \varepsilon^2 \rho_3 u_{xxx} + \rho_4 u = 0, \quad [1.7]$$

where the coefficients of the equation depend on the depth function $H(x) > 0$ of the non-perturbed liquid and have the form:

$$\rho_1 = \sqrt{gH(x)}, \quad \rho_2 = \sqrt{gH^{-1}(x)}/2, \quad \rho_3 = \sqrt{gH^5(x)}/6, \quad \rho_4 = \rho_{2x}/2,$$

where g is the acceleration of gravity, and ε is a small parameter that characterizes the value of the dispersion.

Using the nonlinear WKB method, the authors constructed asymptotic solutions to equation [1.7]. In structure, the solutions are close to solutions of the soliton type. These solutions are called *asymptotic soliton-like solutions* (Maslov and Omel'yanov 1981, 2001).

Furthermore, the problem of constructing asymptotic soliton-like solutions was considered in a more general statement when the coefficients of the KdV equation are variable and have a rather general form (Samoilenko and Samoilenko 2012a, 2012b).

Finally, we note that singularly perturbed partial differential equations with variable coefficients naturally arise in the study of mathematical models of wave processes and phenomena in media whose characteristics depend on time and spatial variables, and singular perturbation. These problems are generally non-integrable, although they have soliton solutions in the case of constant coefficients. Hence, the problem of constructing asymptotic soliton-like solutions for a singularly perturbed partial differential equation with variable coefficients is of great interest (Whitham 1974; Ablowitz 2011; Samoilenko and Samoilenko 2019).

1.4. The linear WKB technique and its generalization

To construct an asymptotic soliton-like solution to the Korteweg–de Vries equation with a singular perturbation, the nonlinear WKB method (Miura and Kruskal 1974) is used. This technique is generalization of the WKB method.

The WKB method is named after the German scientists Wentzel, Kramers and Brillouin. At the beginning of 1926, in fact, at the same time, they proposed this method (Brillouin 1926; Kramers 1926; Wentzel 1926) for the construction of approximate solutions to the one-dimensional stationary Schrödinger equation of quantum mechanics (Schrödinger 1926)

$$-\frac{\hbar^2}{2m} \frac{d^2 \Psi(x)}{dx^2} + V(x) \Psi(x) = E \Psi(x). \quad [1.8]$$

This was an efficient approach to studying the equation, since [1.8] is a non-integrable differential equation in general.

Here, the Planck constant \hbar is supposed to be a small parameter, and $\Psi = \Psi(x)$ is a wave (complex-valued) function which, according to its physical content, satisfies the conditions of regularity, i.e. the wave function is finite, single-valued and continuous with its first derivatives. These assumptions are satisfied, for example, under condition $E - V(x) > 0$.

The main idea of the WKB method is that a solution of equation [1.8] is represented in the special form $\Psi(x) = e^{\Phi(x)}$, where the complex-valued function $\Phi(x)$ according to equation [1.8] satisfies the second-order nonlinear differential equation

$$\Phi''(x) + \Phi'^2(x) = \frac{2m}{\hbar^2}(V(x) - E). \quad [1.9]$$

It is natural to consider the real-valued functions $A(x) = \text{Re } \Phi'(x)$, $B(x) = \text{Im } \Phi'(x)$, characterized amplitude and phase of the oscillating solutions. From relation [1.8], it follows that these functions satisfy the first-order nonlinear differential equation

$$A' + A^2 - B^2 = \frac{2m}{\hbar^2}(V(x) - E), \quad B' + 2AB = 0, \quad [1.10]$$

the solutions of which can be constructed as asymptotic expansions in the value \hbar :

$$A(x) = \frac{1}{\hbar} \sum_{k=0}^{\infty} \hbar^k A_k(x), \quad B(x) = \frac{1}{\hbar} \sum_{k=0}^{\infty} \hbar^k B_k(x). \quad [1.11]$$

The coefficients of asymptotic series [1.11] are determined recursively from the equations found in the standard way from equation [1.10]. In particular, the main terms $A_0(x)$ and $B_0(x)$ in equation [1.11] have to satisfy the differential equations

$$A_0^2(x) - B_0^2(x) = 2m(V(x) - E), \quad A_0(x)B_0(x) = 0, \quad [1.12]$$

and the terms $A_1(x)$ and $B_1(x)$ satisfy the following differential equations:

$$A_0'(x) + 2A_0A_1 - 2B_0B_1 = 0, \quad [1.13]$$

$$B_0'(x) + 2A_0B_1 + 2A_1B_0 = 0. \quad [1.14]$$

Obviously, to find the coefficients $A_0(x)$, $B_0(x)$, $A_1(x)$ and $B_1(x)$ from relations [1.12]–[1.14], both cases $A_0(x) = 0$ and $B_0(x) = 0$ must be considered separately.

Taking into account the condition of regularization, we first consider the case $A_0(x) = 0$. This equality means that the amplitude of the oscillation changes more slowly than its phase. Therefore, from equations [1.12] to [1.14], we have

$$B_0(x) = \pm \sqrt{2m(E - V(x))},$$

$$A_1(x) = -\frac{1}{4} \frac{d}{dx} \ln(2m(E - V(x))), \quad B_1(x) = 0.$$

As a result, taking into account the first two terms of the asymptotic for the functions $A(x)$, $B(x)$, the asymptotic approximation for the solution to equation [1.8] is given in the form

$$\begin{aligned} \Psi(x) = & \frac{C_+}{\sqrt[4]{2m(E - V(x))}} \exp\left(\frac{i}{\hbar} \int \sqrt{2m(E - V(x))} dx\right) + \\ & + \frac{C_-}{\sqrt[4]{2m(E - V(x))}} \exp\left(-\frac{i}{\hbar} \int \sqrt{2m(E - V(x))} dx\right), \end{aligned} \quad [1.15]$$

where C_+ , C_- are arbitrary constants.

If $B_0(x) = 0$, i.e. the phase changes more slowly than the amplitude, then from equations [1.12]–[1.14], we have

$$A_0(x) = \pm \sqrt{2m(E - V(x))}, \quad B_0(x) = B_1(x) = 0,$$

and the asymptotic approximation for the solution to equation [1.8] is written as

$$\begin{aligned} \Psi_1(x) = & \frac{C_+}{\sqrt[4]{2m(E - V(x))}} \exp\left(\frac{1}{\hbar} \int \sqrt{2m(E - V(x))} dx\right) + \\ & + \frac{C_-}{\sqrt[4]{2m(E - V(x))}} \exp\left(-\frac{1}{\hbar} \int \sqrt{2m(E - V(x))} dx\right), \end{aligned} \quad [1.16]$$

where C_+ , C_- are arbitrary complex constants.

Note that while applying the WKB method, it is enough to construct only the first two terms of the asymptotics.

Formulas [1.15] and [1.16] asymptotically approximate the solutions to equation [1.8] for all real arguments x except for the neighborhoods of the turning points, where the equality $E - V(x) = 0$ holds.

In the neighborhoods of the turning points, the asymptotic solution to equation [1.8] is searched for in the other way, using the expansion of its right-hand side in a Taylor series in the neighborhood of the corresponding point. The main term of the asymptotic solution to equation [1.8] is defined from the Bessel equation and written as a linear combination of the Bessel functions.

While using the linear WKB method, the solution of equation [1.8] is sought as an exponential function in some asymptotic expansion. Miura and Kruskal (1974) proposed to construct a solution to the nonlinear differential equation with a singular perturbation as an asymptotic series in a small parameter with quickly oscillating coefficients. They called this technique the nonlinear WKB method. This approach has proven to be quite effective (Samoilenko and Samoilenko 2005, 2008, 2019, 2012a, 2012b; Lyashko et al. 2021) for constructing the asymptotic soliton-like solutions to the following equation:

$$\varepsilon^n u_{xxx} = a(x, t, \varepsilon) u_t + b(x, t, \varepsilon) u u_x, \quad n \in N, \quad [1.17]$$

where $\varepsilon > 0$ is a small parameter, the coefficients $a(x, t, \varepsilon)$, $b(x, t, \varepsilon)$, $(x, t) \in R \times [0; T]$ are infinitely differentiable and are given with asymptotic (according to Poincaré) series

$$a(x, t, \varepsilon) = \sum_{k=0}^N \varepsilon^k a_k(x, t) + O(\varepsilon^{N+1}),$$

$$b(x, t, \varepsilon) = \sum_{k=0}^N \varepsilon^k b_k(x, t) + O(\varepsilon^{N+1}),$$

and the condition $a_0(x, t) b_0(x, t) \neq 0$, $(x, t) \in R \times [0; T]$, takes place.

Note that due to the presence of the variable coefficients in equation [1.17], the problem of finding its asymptotic solutions becomes much more complicated, and at

the same time, a number of new interesting mathematical problems appear. In addition, the degree of the singularity in the equation [1.17] also has a significant impact on the structure of the constructed asymptotic soliton-like solution, the algorithm for finding it and the properties of the solutions.

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1.6. References

- Ablowitz, M.J. (2011). *Nonlinear Dispersive Waves. Asymptotic Analysis and Solitons*. Cambridge University Press, Cambridge.
- Arkadyev, V.A., Pogrebkov, A.K., Polivanov, M.K. (1984). Singular solutions to the Korteweg–de Vries equation and inverse scattering transform. *Zapiski nauchnykh seminarov Leningradskogo otdeleniya Matematicheskogo instituta*, 133, 17–37 [in Russian].
- Benjamin, T.B., Bona, J.L., Mahony, J.J. (1972). Model equations for long waves in nonlinear dispersive systems. *Philosoph. Trans. Royal Society of London. Series A, Mathematical and Physical Sciences*, 272(1220), 47–78.
- Blackmore, D., Prykarpatsky, A.K., Samoilenko, V.H. (2011). *Nonlinear Dynamical Systems of Mathematical Physics. Spectral and Integrability Analysis*. World Scientific, Singapore.
- Bogoliubov, N.N. and Mitropolsky, Y.A. (1961). *Asymptotic Methods in the Theory of Nonlinear Oscillations*. Gordon and Breach, New York.
- Brillouin, L. (1926). Remarques sur la mécanique ondulatoire. *Journ. Phys. Radium.*, 7, 353–368.
- Bullough, R.K. and Caudrey, P.J. (1980). Further remarks on John Scott Russel and on the early history of his solitary wave. In *Solitons*, Bullough, R.K. and Caudrey, P.J. (eds). Springer, Berlin, Heidelberg.
- Burgers, J.M. (1939). Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion. *Verhandelingen der Koninklijke Nederlandsche Akademie van Wetenschappen, Afdeling natuurkunde, Eerste sectie, Deel XVII* (2), 1–53.
- Dodd, R.K., Eilbeck, J.G., Gibbon, J.D., Morris, H.C. (1982). *Solitons and Nonlinear Wave Equations*. Academic Press, London.
- Faddeev, L.D. and Takhtajan, L. (1987). *Hamiltonian Methods in the Theory of Solitons*. Springer, Berlin, Heidelberg.

- Faminskii, A.V. (1990). Cauchy problem for the Korteweg–de Vries equation and its generalizations. *J. Sov. Math.*, 50(1), 1381–1420.
- Flaschka, H., Forest, M.G., McLaughlin, D.W. (1980). Multiphase averaging and the inverse spectral solution of the Korteweg–de Vries equation. *Comm. Pure Appl. Math.*, 33(6), 739–784.
- Gardner, C.S., Green, J.M., Kruskal, M.D., Miura, R.M. (1967). Method for solving the Korteweg–de Vries equation. *Phys. Rev. Lett.*, 19, 1095–1097.
- Glebov, S.G. and Kiselev, O.M. (2005). The stimulated scattering of solitons on a resonance. *J. Nonlin. Math. Phys.*, 12(3), 330–341.
- Glebov, S.G., Kiselev, O.M., Lazarev, V.A. (2003). Birth of solitons during passage through local resonance. *Proc. Steklov Inst. Math. Suppl.*, 1, 84–90.
- Glebov, S.G., Kiselev, O.M., Lazarev, V.A. (2005). Slow passage through resonance for a weakly nonlinear dispersive waves. *SIAM J. Appl. Math.*, 65(6), 2158–2177.
- Hruslov, E.J. (1976). Asymptotics of the solutions of the Cauchy problem for the Korteweg–de Vries equation with initial data of step type. *Math. USSR-Sbornik*, 28(2), 229–248.
- Kalyakin, L.A. (1992). Perturbation of the Korteweg–de Vries soliton. *Theor. Math. Phys.*, 92(1), 736–747.
- Kato, T. (1979). On the Korteweg–de Vries equation. *Manuscripta Math.*, 28, 89–99.
- de Kerf, F. (1988). Asymptotic analysis of a class of perturbed Korteweg–de Vries initial value problems. *Centrum voor Wiskunde en Informatica*, Amsterdam.
- Kramers, H.A. (1926). Wellenmechanik und halbzahlige Quantisierung. *Z. Phys.*, 39, 828–840.
- Lamb Jr., G.R. (1980). *Elements of Soliton Theory*. John Wiley & Sons, New York.
- Lax, P.D. and Levermore, C.D. (1983a). The small dispersion limit of the Korteweg–de Vries equation I. *Comm. Pure Appl. Math.*, 36(3), 253–290.
- Lax, P.D. and Levermore, C.D. (1983b). The small dispersion limit of the Korteweg–de Vries equation II. *Comm. Pure Appl. Math.*, 36(5), 571–593.
- Lax, P.D. and Levermore, C.D. (1983c). The small dispersion limit of the Korteweg–de Vries equation III. *Comm. Pure Appl. Math.*, 36(6), 809–829.
- Lyashko, S.I. (1991). The approximate solution of a pseudoparabolic equation. *Comput. Math. Math. Phys.*, 31(12), 107–111.
- Lyashko, S.I. (1995). Numerical solution of pseudoparabolic equations. *Cybernet. Systems. Anal.*, 31(5), 718–722.
- Lyashko, S.I., Samoilenko, V.H., Samoilenko, Y.I., Lyashko, N.I. (2021). Asymptotic analysis of the Korteweg–de Vries equation by the nonlinear WKB method. *Math. Model. and Comput.*, 8(3), 368–378.

- Maslov, V.P. and Omel'yanov, G.A. (1981). Asymptotic soliton-like solutions of equations with small dispersion. *Russ. Math. Surveys*, 36(3), 73–149.
- Maslov, V.P. and Omel'yanov, G.A. (2001). *Geometric Asymptotics for PDE. I*. American Mathematical Society, Providence.
- Miura, R.M. and Kruskal, M.D. (1974). Application of nonlinear WKB-method to the Korteweg–de Vries equation. *SIAM Appl. Math.*, 26(2), 376–395.
- Newell, A.C. (1985). *Solitons in Mathematics and Physics*. SIAM, Philadelphia.
- Peregrin, D.H. (1966). Calculations of the development of an undular bore. *J. Fluid Mech.*, 25(2), 321–330.
- Pokhozhayev, S.I. (2010). On the singular solutions of the Korteweg–de Vries equation. *Math. Notes*, 88(5), 741–747.
- Russell, J.S. (1844). Report on waves. Report, Fourteenth Meeting of the British Association, John Murray, London.
- Samoilenko, V.H. and Samoilenko, Y.I. (2005). Asymptotic expansions for one-phase soliton-type solutions of the Korteweg–de Vries equation with variable coefficients. *Ukr. Math. J.*, 57(1), 132–148.
- Samoilenko, V.H. and Samoilenko, Y.I. (2007). Asymptotic solutions of the Cauchy problem for the singularly perturbed Korteweg–de Vries equation with variable coefficients. *Ukr. Math. J.*, 59(1), 126–139.
- Samoilenko, V.H. and Samoilenko, Y.I. (2008). Asymptotic two-phase soliton-like solutions of the singularly perturbed Korteweg–de Vries equation with variable coefficients. *Ukr. Math. J.*, 60(3), 449–461.
- Samoilenko, V.H. and Samoilenko, Y.I. (2012a). Asymptotic m -phase soliton-type solutions of a singularly perturbed Korteweg–de Vries equation with variable coefficients. *Ukr. Math. J.*, 64(7), 1109–1127.
- Samoilenko, V.H. and Samoilenko, Y.I. (2012b). Asymptotic m -phase soliton-type solutions of a singularly perturbed Korteweg–de Vries equation with variable coefficients II. *Ukr. Math. J.*, 64(8), 1241–1259.
- Samoilenko, V.H. and Samoilenko, Y.I. (2019). Asymptotic soliton-like solutions to the singularly perturbed Benjamin–Bona–Mahony equation with variable coefficients. *J. Math. Phys.*, 60(1), 011501-1–011501-13.
- Schrödinger, E. (1926). Quantisierung als Eigenwertproblem. *Ann. Phys.*, 384(4), 361–376.
- Sjoberg, A. (1970). On the Korteweg–de Vries equation: Existence and uniqueness. *J. Math. Anal. Appl.*, 29(3), 569–579.
- Turbal, Y., Turbal, M., Bomba, A., Sokh, A. (2015). T -transformation method for studying the multi-soliton solutions of the Korteweg–de Vries type equations. *J. Math. System Sci.*, 4, 164–169.

Wentzel, G. (1926). Eine Verallgemeinerung der Quantenbedingung für die Zwecke der Wellenmechanik. *Z. Phys.*, 38, 518–529.

Whitham, G.B. (1974). *Linear and Nonlinear Waves*. John Wiley & Sons, New York.

Zabusky, N.J. and Kruskal, M.D. (1965). Interaction of solitons in a collisionless plasma and recurrence of initial states. *Phys. Rev. Lett.*, 15, 240–243.

2

The Nonlinear WKB Technique and Asymptotic Soliton-like Solutions to the Korteweg–de Vries Equation with Variable Coefficients and Singular Perturbation

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Here we consider the problem of the mathematical description of soliton-like solutions of hydrodynamic models, such as the Korteweg–de Vries (KdV) equation and its generalizations, describing wave processes in inhomogeneous media with variable characteristics and a small dispersion. Mainly, attention is paid to the development of an algorithm for constructing approximate (asymptotic) solutions of soliton type. These solutions contain a regular part, which is a background function, and a singular part, which reflects the soliton properties of the solutions. They are

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constructed using the nonlinear WKB method. The construction of the regular and singular parts of the searched asymptotic solution is described in detail.

Problems that arise while using the algorithm as well as while its justification are discussed. Statements about the accuracy with which the constructed asymptotic solution satisfies the original equation are proved.

This approach provides us with effective tools for studying the influence of model parameters on the properties of the dynamical system under consideration. In particular, the obtained results can be used for the subsequent analysis of these solutions in different ways, for example, by using computer simulation.

2.1. Introduction

In the papers by Dobrokhotov and Maslov (1981), and Maslov and Omel'yanov (1981), the notion of an asymptotic soliton-like solution to equations of integrable type with a small singular perturbation is proposed. In accordance with the kind of small parameter, the nonlinear WKB method (Miura and Kruskal 1974) was used. Later, these results were detailed in the monograph (Maslov and Omel'yanov 2001). Much attention is also paid to other issues related to the properties of solutions to equations with a singular perturbation. In particular, the problem regarding discontinuous solutions to the unperturbed problems is considered. These solutions can be obtained from the formulas for the exact or asymptotic solutions in the limiting case as a small parameter tends to zero (Lax and Levermore 1983a, 1983b, 1983c; Samoilenko and Samoilenko 2010).

In the work on this subject (Maslov and Omel'yanov 1981), the equation of hydrodynamic KdV-like equation

$$u_t + (\rho_1 + 3\rho_2 u)u_x + \varepsilon^2 \rho_3 u_{xxx} + \rho_4 u = 0, \quad [2.1]$$

was considered. Here, the coefficients of the equation depend on the depth function $H(x) > 0$ of the non-perturbed liquid and have the form

$$\rho_1 = \sqrt{gH(x)}, \quad \rho_2 = \sqrt{gH^{-1}(x)}/2, \quad \rho_3 = \sqrt{gH^5(x)}/6, \quad \rho_4 = \rho_{2x}/2,$$

where g is the acceleration of gravity, and ε is a small parameter characterized value of dispersion.