COMBINATORIAL REASONING

An Introduction to the Art of Counting

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WILLIAM WEBB

WILEY

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Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication Data:

DeTemple, Duane W.
Combinatorial reasoning : an introduction to the art of counting / Duane DeTemple, William Webb.
pages cm
Includes bibliographical references and index.
ISBN 978-1-118-65218-3 (cloth)
1. Combinatorial analysis–Textbooks. 2. Mathematical analysis–Textbooks. I. Webb, William, 1944– II. Title.
QA164.D48 2013
511′.6–dc23

2013035522

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

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PREFACE

Counting problems, or more precisely enumerative combinatorics, are a source of some of the most intriguing problems in mathematics. Often these problems can be solved using ingenious and creative observations, what we call *combinatorial reasoning*. It is this kind of thinking that we stress throughout the descriptions, examples, and problems in this text.

Combinatorics has many important applications to areas as diverse as computer science, probability and statistics, and discrete optimization. But equally important, the subject offers many results of beautiful mathematics that are enjoyable to discover and problems that are simply fun to think about and solve in innovative ways.

Each of us has over 40 years of experience teaching combinatorics, as well as other mathematics courses, at both the undergraduate and graduate levels. We think that we have learned some effective ways to present this subject. Early versions of the notes for the book were used in both undergraduate and graduate courses, and our students found the approach both easy to understand and quite thorough.

FEATURES OF THIS TEXT

Chapter 1 introduces the reader to combinatorial thinking by considering topics of existence, construction, and enumeration that lead, by the end of the chapter, to general principles of combinatorics that are employed throughout the remainder of the text. The problems solved in this chapter, often involving dot patterns and tilings of rectangular boards, are easily described and visualized, and foreshadow much of what comes later.

More formal considerations of combinatorics are taken up in Chapter 2, where selections, arrangements, and distribution are treated in detail. Special attention is given to combinatorial models—block walking, tiling of rectangular boards, committee selection, and others. It is shown how general results can be derived by combinatorial reasoning based on an appropriate model. Most often, only the simplest of algebraic calculations are necessary.

An unusually complete discussion of generating functions—both ordinary and exponential—is given in Chapter 3, where the binomial series is shown to be a prototype of a much larger collection of generating functions. It is also shown how enumeration problems can often be solved with generating functions.

Chapter 4 begins with the DIE method (Description—Involution—Exceptions), which is shown to be a powerful combinatorial approach to the evaluation of alternating series. This leads naturally to the Principle of Inclusion/Exclusion. The chapter then turns to a section on rook polynomials that combines generating functions and inclusion–exclusion to solve an interesting class of restricted arrangement problems. The chapter concludes with an optional section on the Zeckendorf representation of integers and its application to creating a winning strategy for the game Fibonacci Nim.

Recurrence sequences are treated in detail in Chapters 5, using what we call the operator method. By employing the readily understood successor operator E, which simply replaces n by $n + 1$, the properties of recurrence sequences seem natural and easy to understand. This approach not only deepens comprehension but also simplifies many calculations.

Chapter 6 enlarges the library of special numbers that often answer combinatorial questions. Since the sections within this chapter are largely independent, there is freedom to pursue whatever topics seem of most interest—Stirling numbers, harmonic numbers, Bernoulli numbers, Eulerian numbers, partition numbers, or Catalan numbers.

Chapter 7 returns to the operator approach for solving linear recurrence relations that was introduced earlier in Chapter 5. Here, by viewing recurrence sequences as vector spaces, additional methods to solve recurrence relations become available. Moreover, we discover a powerful new approach to both discover and verify combinatorial identities.

Pólya-Redfield counting—the enumeration of arrangements that take symmetries into account—is the subject of Chapter 8, the final chapter. Here, abstraction is minimized by showing how general formulas can be derived from the consideration of carefully chosen simple figures and arrangements.

FLEXIBILITY FOR COURSES

A beginning course for undergraduates can be easily constructed using selected sections from Chapters 1–5. A course for advanced undergraduates and beginning graduate students might give quicker coverage to early chapters and include material chosen from Chapters 6, 7, and 8. For example, Sections 6.1–6.6, on special numbers, are largely independent, and any of these sections can be covered in any order. There are no special prerequisites for this material beyond a little exposure to power series. An elementary introduction to linear algebra is needed for Chapter 7. A little background in group theory is helpful for Chapter 8, but in our experience this chapter can provide a good introduction to this algebraic topic.

We would like to thank Ken Davis (Hardin-Simmons University) for his very helpful suggestions and comments.

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PART I

THE BASICS OF ENUMERATIVE COMBINATORICS

1

INITIAL EnCOUNTers WITH COMBINATORIAL REASONING

1.1 INTRODUCTION

Although this text is devoted largely to enumerative combinatorics, Section 1.2 presents a brief encounter with a simple yet surprisingly versatile method to prove existence, the pigeonhole principle. Section 1.3 discusses some combinatorial construction problems associated with covering a chessboard with dominoes. In Section 1.4, we consider some number sequences that often arise in combinatorial problems such as *triangular* numbers 1,3,6,10, . . .; *square* numbers 1,4,9,16, . . .; and other *figurate numbers*, where the terminology alludes to the representation of these numbers by geometric patterns of dots. In Section 1.5, we count the number of ways a $1 \times n$ rectangle can be tiled with either unit squares of two contrasting colors or with a mixture of 1×1 squares and 1×2 dominoes. By counting the number of dots in a pattern or the number of tilings of a chessboard, we will discover several general principles of counting that are fundamental to enumerative combinatorics. In particular, we will encounter the addition and multiplication principles, which are explored in detail in Section 1.6, which concludes the chapter.

1.2 THE PIGEONHOLE PRINCIPLE

The pigeonhole principle was first applied in 1834 by Peter Dirichlet (1805–1859) to solve a problem in number theory. Soon, other mathematicians found his idea equally

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useful and referred to it as *Dirichlet's box principle* (*Schubfachprinzip* in German). Later, in the nineteenth century, the term *pigeonhole* was used in reference to the small boxes or drawers common in desks of that century. (It may be comforting to know that envelopes, and not pigeons, are placed in the pigeonholes.)

Dirichlet's idea is simply stated as follows.

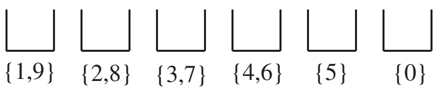
Theorem 1.1 (Pigeonhole Principle) If $n + 1$ or more objects are placed into n boxes, then at least one of the boxes contains two or more of the objects.

Proof (by contradiction). Suppose, to the contrary, that each of the n boxes contains no more than one object. Then the n boxes together contain no more than n objects, a contradiction. ■

The examples that follow show how the pigeonhole principle provides the basis for an existence proof. It is helpful to think of pigeons and pigeonholes as metaphorical terms for the objects and boxes of the theorem.

Example 1.2 In a family of seven, there must be two family members for which either the sum or difference in their ages can be given in decades, that is, as a multiple of 10.

Solution. A multiple of 10 is easy to identify by the digit 0 in the units position. If two family members have ages ending with the same digit, the difference in their ages is a multiple of 10. Also, if someone in the family has an age ending in the digit 1 and another family member's age ends with the digit 9, their sum of ages ends with the digit 0. Continuing with this type of reasoning suggests that we define the following pigeonholes:



The *pigeons* are the seven ages of the family member. When these are placed in the box labeled with the set containing the last digit of the age, the pigeonhole principle guarantees that at least one of the six boxes contains at least two people with one of the labeled ages. If these two ages happen to have the same unit's digit, then their difference is a multiple of 10. If the two ages have different last digits, these must be 1 and 9, or 2 and 8, or 3 and 7, or 4 and 6. In each case, the sum of the two ages is a multiple of 10. ■

The pigeonholes set up for Example 1.2 show why seven numbers were needed. There is no pair of numbers from the six members of the set $\{1, 2, 3, 4, 5, 10\}$ whose sum or difference is divisible by 10.

Example 1.3 There are five people in a 6 mile by 8 mile rectangular forest, each carrying a walkie-talkie with a range of 5 miles. Show that at least two of the five people can talk with one another on their walkie-talkies.

Solution. Divide the forest into four rectangular 3×4-mi plots; these are the *pigeonholes*. There are five people (the *pigeons*), so by the pigeonhole principle, at least two people are in one of the four plots. Since the maximum distance in a 3×4 rectangular plot is the 5-mi-long diagonal, these two people are within talking range. ■

The solution of a problem by means of the pigeonhole principle requires us to carry out these steps:

1. Recognize that the pigeonhole principle can be helpful.
2. Identify the pigeons and the pigeonholes.
3. Show that there are more pigeons than pigeonholes.
4. Show why the existence of two pigeons in the same pigeonhole solves the given problem.

These steps are carried out to solve the following problem.

Example 1.4 The *lattice plane* is the set of points in the Cartesian plane with integer coordinates. Given any five points of the lattice plane, show that the midpoint of some pair of points is a point in the lattice plane.

Solution. If (a, b) and (c, d) are two lattice points, their midpoint is $((a + c)/2, (b + d)/2)$, the average of the x and y coordinates. This will be a point in the lattice plane if, and only if, both $a + c$ and $b + d$ are even; that is, a and c must have the same parity, and b and d must have the same parity. This observation suggests that we use parity to define these four pigeonholes:

Box 1: x even, y even

Box 2: x even, y odd

Box 3: x odd, y even

Box 4: x odd, y odd

Since five points are placed in the four boxes, there is some box with at least two members. Both of these points have x and y coordinates with the same parity, so the midpoint of these two points has integer coordinates; that is, their midpoint is a point of the lattice plane. ■

1.2.1 Applications to Ranges and Domains of Functions

There are a number of useful variations and interpretations of the pigeonhole principle. For example, suppose that A is a set of objects and B is a set of distinct boxes. Then any placement of the objects into the boxes describes a function $f : A \to B$. If no two objects are assigned to the same box, then the function is *one-to-one*, or *injective*. The pigeonhole principle requires there to be at least as many pigeonholes as pigeons, so if $|A|$ and $|B|$ denote the respective number of elements in set A and B, we have the following theorem.

Theorem 1.5 If the function $f : A \to B$ is one-to-one, then $|A| \leq |B|$.

In the opposite direction, suppose that we have a placement of objects from set A into the boxes of set B that leaves no box empty. In other words, we have a function $f : A \to B$ that is *onto* or *surjective*, meaning that its range is all of set B. Since there must be at least as many objects as boxes, we have this theorem.

Theorem 1.6 If $f : A \to B$ is a surjective function from A onto B, then $|A| \geq |B|$.

A function that is both one-to-one and onto (i.e., is both injective and surjective), is said to be *bijective*. When the two theorems above are combined, we get the following result.

Theorem 1.7 If $f : A \to B$ is a bijective function of A onto B, then $|A| = |B|$.

Finally, we have the following result.

Theorem 1.8 If $f : A \to B$ and $|A| = |B|$, then f is one-to-one if and only if f is onto.

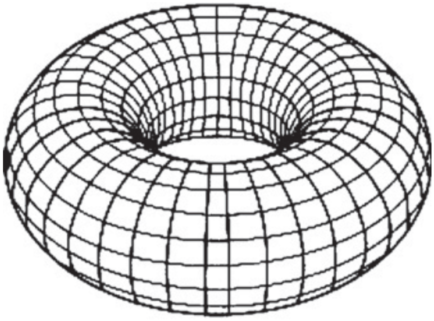
Proof. Let $|A| = |B|$ and first suppose that f is one-to-one. Then $|A| = |\text{range}(f)|$ and therefore $|\text{range}(f)| = |B|$. This shows us that $\text{range}(f) = B$, and we see that f is onto. Similarly, if f is not one-to-one, then $|A| > |\text{range}(f)|$, so $\text{range}(f)$ is a proper subset of B and f is not onto. ■

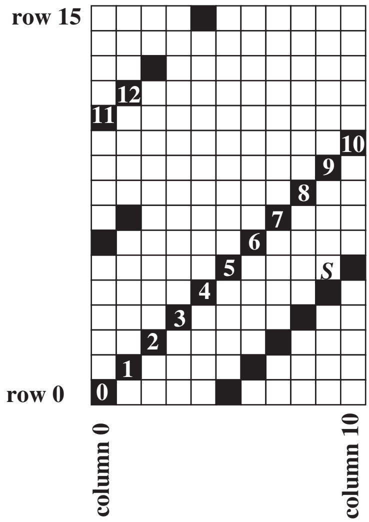
Bijections are immensely useful in combinatorial reasoning. Suppose that we have a difficult problem counting the number of elements in set A, but we can find a bijection of set A onto a set B, and B is more easily counted. Since $|A| = |B|$, our difficulties are over! This strategy is known as a *bijective proof*, and we'll see several examples of this type of combinatorial reasoning later.

1.2.2 An Application to the Chinese Remainder Theorem

Theorem 1.8 will be useful for our next application. Readers with a background or interest in number theory may find the following approach to this topic interesting. However, it is optional and can be skipped since it will not be used later.

Example 1.9 Suppose that a *torus* (i.e., the surface of a doughnut) is divided into quadrangular regions by 11 circles in one direction that are crossed by 16 circles in the orthogonal direction. If the surface is cut along one circle of each type, the surface can be unrolled and stretched to form an 11×16 rectangle partitioned into $11 \cdot 16 = 176$ squares. As shown in the diagram below, a path of squares numbered 0,1,2, . . . has been initiated that spirals around the torus, starting with square 0 in row 0 and column 0. Prove that a continuation of the spiral path covers the entire torus, so that each square is assigned a unique number 0,1,2, . . .,175.





Solution. It suffices to show that the spiral path includes all 16 of the squares located in column 0, the far left column. These are the squares numbered by the entries in the set

$$A = \{0 \cdot 11, 1 \cdot 11, 2 \cdot 11, 3 \cdot 11, \dots, 15 \cdot 11\} \tag{1.1}$$

We still need to know that no number in the left column is repeated, where the rows are numbered by the set

$$B = \{0,1,2, \dots ,15\} \tag{1.2}$$

The row of square $k \cdot 11 \in A$ is given by its remainder when divided by 16. For example, square $2 \cdot 11 = 22$ has the remainder of 6 when divided by 16, and we see from the figure that square 22 is in row 6. This suggests we consider the function $f : A \to B$ defined by mapping each element of set A to the corresponding remainder r in set B. To see why f is one-to-one, suppose that two values, say, $j \cdot 11$ and $k \cdot 11$, $j, k \in \{0,1,2, \dots .15\}$, have the same remainder r; that is, suppose that

$$j \cdot 11 = p \cdot 16 + r \text{ and } k \cdot 11 = q \cdot 16 + r \tag{1.3}$$

for the quotients p and q. When these equations are subtracted from one another, we see that

$$(j - k) \cdot 11 = (p - q) \cdot 16 \tag{1.4}$$

Since 16 divides the right side of (1.4) it must also divide the left side $(j-k)\cdot 11$. Since 16 is relatively prime to 11 (i.e., 11 and 16 have no common prime divisors), we see that 16 divides $j-k$. However, $0 \le |j-k| < 15$, and only $j-k=0$ is divisible by 16. Therefore, f is a one-to-one function that maps set A to the set B, and $|A| = |B| = 16$. We now see that each of the 16 squares in column 0 is along the spiral path, and so every square of the entire rectangle is assigned a unique number along the spiral path of squares. ■

Except for its geometric interpretation as a spiral path on a torus, Example 1.8 gives a proof of a special case of the *Chinese remainder theorem*, which first appeared in a third-century (A.D.) book *Sun Zi Suanjing* written by the Chinese mathematician Sun Tzu. It was important that $m = 11$ and $n = 16$ have no positive common divisor other than 1. More generally, we say that two integers m and n are *relatively prime* when their largest common integer divisor is 1. The following theorem can be proved similarly to the approach followed in Example 1.9.

Theorem 1.10 (Chinese Remainder Theorem) Let m and n be relatively prime positive integers, and let a and b be integers with $0 \le a < m$ and $0 \le b < n$. Then there is a unique k, $0 \le k < mn$, for which $k = mj + a$ and $k = nj' + b$ for some j and j'.

1.2.3 Generalizations of the Pigeon Principle

For some problems, we want to know that there are not just two but some larger number of pigeons in some box. In these cases, we can turn to a generalized version of the pigeonhole principle.

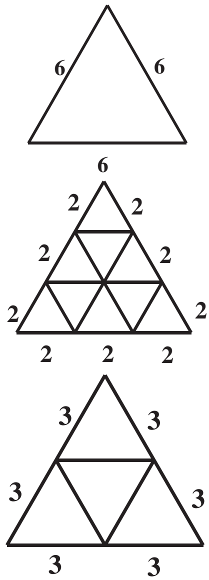
Theorem 1.11 (Generalized Pigeonhole Principles)

(a) If $nk + 1$ or more pigeons are put into n pigeonholes, then at least one pigeonhole has $k + 1$ or more pigeons.

(b) If $p_1 + p_2 + \cdots + p_n + 1$ or more pigeons are put into n pigeonholes numbered 1 through n, then, for some r, pigeonhole r has $p_r + 1$ or more pigeons.

Proof. Since part (a) follows from (b) by setting $p_r = k$, $1 \le r \le n$, it suffices to prove part (b). As before, it is easy to give a proof by contradiction. If each hole r has at most p_r pigeons, then all n holes together contain at most $p_1 + p_2 + \cdots + p_n$ pigeons. This contradicts the hypothesis that more than $p_1 + p_2 + \cdots + p_n$ pigeons were placed in the holes. ■

Example 1.12 An equilateral triangle has sides of length 6. Show that (a) if there are 10 points inside the triangle, then at least 2 of them are within 2 units of each other; and (b) if there are 9 points inside the triangle, there are at least 3 of them for which each is at most 3 units from each of the others.



Solution

(a) The triangle can be partitioned into nine equilateral triangles with sides of length 2. Given any 10 points within or on the large triangle, there must be 2 in the same small triangle, and these are at a distance of at most 2 units.

(b) Again using the partitioning of the large triangle into four congruent equilateral triangles of side length 3, not each of the 4 small triangles can contain just two of the nine points. Thus, some small triangle has three or more of the nine points, and each of these is at a distance of no more than 3 units from the other two points. ■

For the next example, it will be helpful to introduce some notations and terminologies that are often used in combinatorics and elsewhere in mathematics:

- The set of the first n natural numbers will be denoted by $[n] = \{1,2,\dots,n\}$.
- A *sequence* of length n is an ordered list $(a_1, a_2, \dots, a_n)$. Equivalently, a sequence of length n is any function f: $[n] \to A$, where $f(j) = a_j$ is the jth *term* of the sequence.
- A *permutation* of $[n]$ is an ordered arrangement of the n elements of set $[n]$. Equivalently, a permutation is a bijection $\pi : [n] \to [n]$.
- A sequence $(a_1, a_2, \dots, a_n)$ of real numbers in *monotone increasing* if $a_1 < a_2 < \cdots < a_n$ and *monotone decreasing* if $a_1 > a_2 > \cdots > a_n$.

- A *subsequence* of $(a_1, a_2, \dots, a_n)$ is a sequence formed by deleting some of the terms of the given sequence but preserving the order in which the remaining terms are listed.

For example, (8,3,5,2,6,1,4,10,9,7) is a permutation of [10]. The subsequence (3,5,6,10) is monotone increasing. According to the following result, we can always find a monotone sequence of length 4 for any permutation of [10].

Example 1.13 Ten students, all of different heights, are standing shoulder to shoulder in a line. Prove that four of the students can take a step forward so that they form a line of students whose heights either decrease or increase from left to right.

Solution. Suppose that the students' heights are $h_1, h_2, \dots, h_{10}$ from left to right. Assume that there is no subsequence of four of the students (in the same left-to-right order, of course) with increasing heights. We will then show that a subsequence of four students with decreasing heights can be found. Starting with student 1 at the left, let s_1 be the largest number of students that can step forward to form a row with increasing height with student 1 at the left of the row. Similarly, let s_2 be the largest number of students that can take a step forward to form a row with increasing height with student 2 at the far left. More generally, let s_k be the largest number of students that can step forward to form a row of increasing height and with student k at the left. Since we cannot find a subsequence of four students with increasing height, we know that $1 \le s_k \le 3$, $k = 1, 2, \dots, 10$; that is, we have 10 numbers that have one of the values 1, 2, or 3. By Theorem 1.11 part (a), at least 4 of the 10 numbers are the same, say, $s_a = s_b = s_c = s_d$, $a < b < c < d$. We see that student a must be taller than student b, since otherwise student a could be added to the left of the longest increasing subsequence starting with student b, and then s_a would be larger than s_b. Thus, we have $h_a > h_b$. By the same reasoning, $h_b > h_c$ and $h_c > h_d$, so that altogether $h_a > h_b > h_c > h_d$. We see that if students *a, b, c,* and *d* take a step forward, they have decreasing heights from left to right. ■

The reasoning used in Example 1.13 can be extended to prove this theorem of Erdös and Szekeres [1].

Theorem 1.14 Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of $n + 1$ numbers that is either increasing or decreasing.

PROBLEMS

1.2.1. A bag contains seven blue, four red, and nine green marbles. How many marbles must be drawn from the bag without looking to be sure that we have drawn

(a) a pair of red marbles?

(b) a pair of marbles of the same color?

(c) a pair of marbles with different colors?

(d) three marbles of the same color?

(e) a red marble, a blue marble, and a green marble?

1.2.2. Given 10 French books, 20 Spanish books, 8 German books, 15 Russian books, and 25 Italian books, how many books must be chosen to guarantee there are

(a) twelve books of the same language?

(b) a book of each language?

1.2.3. There are 10 people at a dinner party. Show that at least two people have the same number of acquaintances at the party.

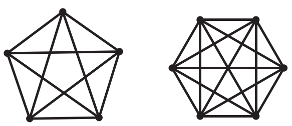
1.2.4. Given 10 distinct numbers chosen from the arithmetic sequence 1,4,7, . . . , $1+3k$, . . . , 40,43,46, prove there is at least one pair of the 10 chosen integers that has the sum 50.

1.2.5. Given any five points in the plane, with no three on the same line, show that there exists a subset of four of the points that form a convex quadrilateral.[1] [*Hint:* Consider the *convex hull* of the points; that is, consider the convex polygon with vertices at some or all of the given points that encloses all five points. This scenario can be imagined as the figure obtained by bundling the points within a taut rubber band that has been snapped around all five points. There are then three cases to consider, depending on whether the convex hull is a pentagon, a quadrilateral containing the fifth point, or a triangle containing the other two given points.]

1.2.6. Given four points on a circle, show that some three of the points lie in some closed semicircle (a closed semicircle includes its two endpoints).

1.2.7. Given five points on a sphere, show that some four of the points lie in a closed hemisphere [2].
(*Note:* A closed hemisphere includes the points on the bounding great circle.)

1.2.8. A *graph* is a set of points known as *vertices* together with a set of line segments called *edges* that connect some of the pairs of vertices. If every pair of vertices are joined by an edge, the graph is said to be *complete.* The complete graphs on five and six vertices, K_5 and K_6, are shown below:



K_5 K_6

[1]This well-known problem is called the “happy ending” problem, since two of its first investigators, Esther Klein and George Szekeres, would later be married to one another.

(a) Show that all of the edges of K_5 can be colored blue or red so that no triangle exists in K_5 with its three edges having the same color.

(b) Show that every red and blue coloring of the edges of K_6 contains a triangle with all of its edges of the same color.

1.2.9. Suppose that 51 numbers are chosen randomly from $[100] = \{1,2,\ldots,100\}$. Show that two of the numbers have the sum 101.

1.2.10. Assuming that there are 48 different pairs of people who know each other at a party of 20 people, show that some person has four or fewer acquaintances.

1.2.11. Choose any 51 numbers from $[100] = \{1,2,\ldots,100\}$. Show that two of the chosen numbers are relatively prime (i.e., have no common divisor other than 1).

1.2.12. Show that any subset of eight distinct integers between 1 and 14 contains a pair of integers m and n such that m divides n.

1.2.13. Choose any 51 numbers from $[100] = \{1,2,\ldots,100\}$. Show that there are two of the chosen numbers for which one divides the other.

1.2.14. State and prove a theorem that generalizes the results of Problems 1.2.12 and 1.2.13.

1.2.15. Consider a string of $3n$ consecutive natural numbers. Show that any subset of $n+1$ of the numbers has two members that differ by at most 2.

1.2.16. Let $(a_1, a_2, \ldots, a_n)$ be any sequence of n natural numbers. Show that there is a subsequence of consecutive members of the sequence that is divisible by n. [*Hint:* Consider the sums $s_k = a_1 + a_2 + \cdots + a_k$.]

1.2.17. Suppose that the numbering of the squares along the spiral path shown in Example 1.9 is continued. What number k is assigned to the square S whose lower left corner is at the point (9, 5)?

1.2.18. Suppose that a torus is divided into mn quadrangular regions by m circles crossed orthogonally by n circles, as in Example 1.9. By the Chinese remainder theorem 1.10, if m and n are relatively prime, then each of the regions is reached by the unique spiral path on the torus.

(a) Using Example 1.9 as a model, draw and number the squares along the spiral path in the case that $m = 4$ and $n = 5$.

(b) How many distinct spiral paths can be found when $m = 4$ and $n = 6$?

(c) Repeat part (b) for $m = 3$ and $n = 6$.

1.2.19. Generalize the results of Problem 1.2.18.

(a) How many spiral paths exist on the torus if $m = n$?

(b) Suppose that $d \geq 2$ is the largest common divisor of m and n. How many distinct spiral paths exist on the torus?

1.2.20. **(a)** Find a permutation of $[9] = \{1, 2, \ldots, 9\}$ for which no subsequence of length 4 is either monotone increasing or monotone decreasing (see Example 1.13).

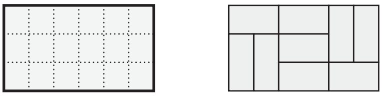
(b) Place 10 at the right end of your sequence from part (a) and underline the four terms of an increasing subsequence.

(c) Place a 10 at the left end of your sequence from part (a) and underline the four terms of a decreasing subsequence.

1.3 TILING CHESSBOARDS WITH DOMINOES

In this section, our attention turns to combinatorial *construction*. A construction settles the question of existence in a very satisfying way, since the constructed object provides an explicit example with the required properties. Sometimes it can be shown that an object cannot possibly be constructed, so that the existence question is answered in the negative.

In each of the examples that follow, we consider how a shape formed with unit squares can be completely covered with nonoverlapping dominoes. A *domino* is a 1×2 rectangle that is viewed simply as a tile with no attention given to the dots, known as *pips,* that are imprinted on actual dominoes. For example, the 3×6 rectangle in Figure 1.1(a) can be tiled with nine dominoes in many ways, such as the tiling constructed in Figure 1.1(b). Note that a tiling must cover the entire figure and the dominoes can touch along their edges but can never overlap one another.



(a) (b)

FIGURE 1.1 A 3×6 rectangle (a) and one way to tile it with dominoes (b).

In enumerative combinatorics, we would ask "in how many ways" can a given shape be tiled with dominoes, but here we will be content to consider only the existence question:

> Given a chessboard, possibly with some of its squares deleted, can we construct a tiling of the board with dominoes? If no construction is found, can it be explained why no tiling exists?

It will soon become apparent why we consider chessboards, since we will be able to take advantage of the alternating pattern of the colors of the unit squares.

Insights into the general case are often given by the examination of special cases. In the examples that follow we will consider the five chessboards shown in Figure 1.2.

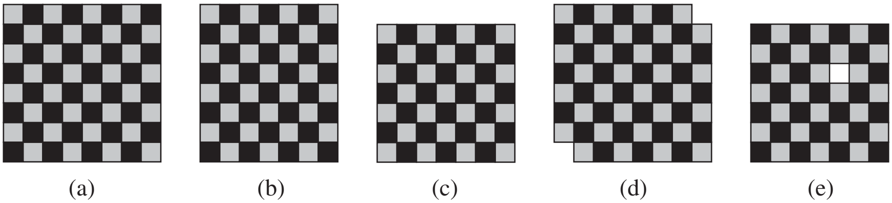


FIGURE 1.2 Five examples of chessboards.

Example 1.15 Chessboard (a) is standard chessboard, with 8 unit squares in each row that are colored with an alternating pattern of black and gray colors. The bottom row can be tiled easily with 4 horizontal dominoes laid end to end. Indeed, each row can be tiled in this way, so the entire 8×8 chessboard can be tiled with horizontally aligned dominoes. Of course, there are many other ways to construct a tiling as well.

Example 1.16 If the last column of an 8×8 standard chessboard is removed, this leaves board (b) with 7 unit squares in each row. Horizontally aligned dominoes no longer can be used to tile the rows as for board (a). However, each column is 8 units high, and therefore the 8×7 chessboard can be tiled with vertically oriented dominoes.

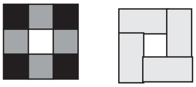
Example 1.17 Board (c) is a 7×7 board, so it has 49 unit squares, an odd number. But each domino in a tiling covers 2 unit squares. This means that any tiling by dominoes covers an even number of squares of the chessboard, so there is no possible way to tile board (c) with dominoes.

Our analysis of the first three chessboards can be generalized to rectangular chessboards of any size.

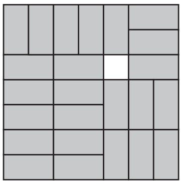
Theorem 1.18 A rectangular $m \times n$ chessboard can be tiled with dominoes if and only if at least one of its dimensions m or n is an even number.

Example 1.19 Chessboard (d) is obtained by removing two opposite corner squares from the 8×8 standard chessboard, leaving a trimmed board with 62 unit squares. It might seem, since 62 is even, that a tiling with dominoes exists. However, a closer look reveals that the 2 unit squares that were removed were both black, leaving a board with 32 gray and 30 black unit squares. But a domino, whether vertical or horizontal, simultaneously covers both a gray unit square and an adjacent black one. Thus, any trimmed chessboard cannot be tiled if it has an unequal number of gray and black unit squares. In particular, chessboard (d) cannot be tiled with dominoes.

Example 1.20 Chessboard (e) is a 7×7 board has one of its black squares removed, leaving a board with 48 unit squares, 24 gray and 24 black. The reasoning that we have used to examine boards (c) and (d) is not applicable, so it may yet be possible to tile the board with dominoes. As with boards (a) and (b), we can see if a particular tiling can be constructed. Rather than try brute force on the entire board, it may be best to consider first a simpler related problem. For example, what if the center black unit square is removed from a 3×3 chessboard? The following diagram shows that the trimmed 3×3 board can tiled with four dominoes:



This example suggests that we look for a tiling that combines both horizontally and vertically aligned dominoes. Nicely enough, we quickly find a tiling of board (e):

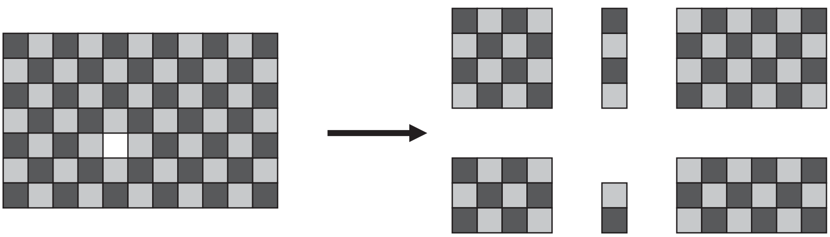


More generally, we can prove the next theorem.

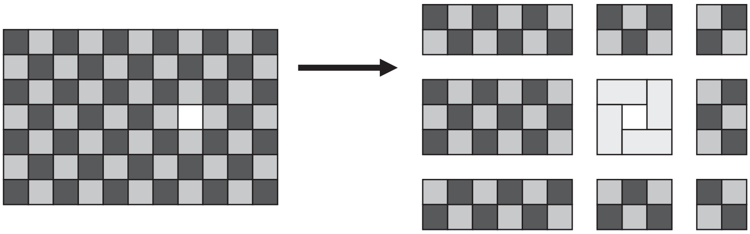
Theorem 1.21 Suppose that an $m \times n$ chessboard, m and n odd, has black corners. If any black square is removed from the board, the trimmed board can be tiled with dominoes.

Proof. Suppose that the black square that has been removed was in column i and row j. Then i and j are either both odd or both even. We consider the two cases separately.

Case 1: i and j are odd. This means the remainder of the board is a collection of rectangular boards each with at least one even dimension, so it can be tiled with dominoes by Theorem 1.18. Here is a typical example



Case 2: i and j even. This means that the 3×3 square with a deleted black center can be tiled, and that the remainder of the board is a collection of rectangular boards each with at least one even dimension, so it can be tiled with dominoes by Theorem 1.18. Here is a diagram illustrating this case:



■

It may seem curious that we have avoided a consideration of the enumerative question: How many ways can an $m \times n$ board be tiled with dominoes?

This question, although easy to ask, is not easy to answer! In 1961, the following result was derived independently by Temperley and Fisher [3] and Kasteleyn [4].

Theorem 1.22 An $m \times n$ board can be tiled by $mn/2$ dominoes in

$$\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\frac{j\pi}{m+1}+4\cos^2\frac{k\pi}{n+1}\right)^{1/4} \tag{1.5}$$

ways.

Fortunately, the problem is much easier when m is small. Later we will introduce enumerative methods to determine the number of ways to tile the $2 \times n$ and $3 \times n$ boards with dominoes, and in Section 1.5 we will discuss the number of ways to tile $1 \times n$ boards with either colored squares or a mixture of squares and dominoes.

PROBLEMS

1.3.1. Consider an $m \times n$ chessboard, where m is even and n is odd. Prove that if two opposite corners of the board are removed, the trimmed board can be tiled with dominoes.

1.3.2. Consider an $m \times n$ chessboard, where both m and n are even. Prove that if any two unit squares of opposite color are removed, then the trimmed board can be tiled with dominoes.

1.3.3. Suppose that the lower left $j \times k$ rectangle is removed from an $m \times n$ chessboard, leaving an angle-shaped chessboard. Prove that that angular board can be tiled with dominoes if it contains an even number of squares.