Synthesis Lectures on
Wave Phenomena in the Physical Sciences

SYNTHESIS
COLLECTION OF TECHNOLOGY

Sanichiro Yoshida

Physics and Mathematics Behind Wave Dynamics

Springer

Synthesis Lectures on Wave Phenomena in the Physical Sciences

Series Editor

Sanichiro Yoshida, Department of Chemistry and Physics, Southeastern Louisiana University, Hammond, LA, USA

The aim of this series is to discuss the science of various waves. An emphasis is laid on grasping the big picture of each subject without dealing formalism, and yet understanding the practical aspects of the subject. To this end, mathematical formulations are simplified as much as possible and applications to cutting edge research are included.

Sanichiro Yoshida

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Sanichiro Yoshida
Department of Chemistry and Physics
Southeastern Louisiana University
Hammond, LA, USA

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This book is dedicated to the memory of my late father, Akira Yoshida, who introduced me to the joy of science and engineering; my mother, Sonoko Yoshida, who taught me how to learn; and my wife, Yuko Yoshida, for filling my life with love, joy, and happiness.

Preface

This book grew from my laboratory notebook. I conducted various experimental and theoretical research projects using a wave to characterize the properties of physical systems, e.g., ultrasonic nondestructive testing to probe anomalies in metal materials, an optical interferometric research project to diagnose the deformation status of solid materials, a development of signal processing algorithms to improve the performance of cochlear implant devices, and gravitational wave detection with an optical interferometer.

Through these activities, waves and wave dynamics have fascinated me. At the same time, I have encountered difficulties in treating the wave signals. Often, the signal-to-noise ratio is beyond my control. Another time, the signal is too complicated to process. These experiences motivated me to write this book.

The reader may notice that half of this book discusses frequency domain analysis, such as Fourier Transform, Laplace Transform, and Transfer Functions. I have a reason for this. During these times, I have learned frequency domain analysis can ease these difficulties. By dividing the signal into different frequency ranges, we can interpret the observed signal better and have better insight into the underlying phenomenon.

The organization of this book is as follows. Chapter 1 introduces waves as a moving oscillatory pattern. Chapter 2 focuses on the concept of harmonic oscillation. In Chap. 3, we discuss the mathematical aspect of wave dynamics. We derive wave equations and solve them. Chapter 4 discusses some basic properties of waves. In Chaps. 5 and 6, we view waves in the frequency domain. We discuss various mathematical concepts related to frequency domain analysis.

I hope this book helps students to deepen their understanding of wave dynamics and apply the knowledge to their research activities.

Hammond, LA, USA Sanichiro Yoshida
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1 Description of Oscillation and Wave Motions

We can view a wave as a snapshot of a periodic pattern moving in a certain direction at a constant velocity. We can also view a wave as a pattern in which different locations experience the same temporal oscillation with a certain time lag. Take a wave formed by the spectators in a soccer stadium as an example. If you look at the wave from a helicopter hovering above the stadium, you will see a periodic pattern of people's hands going around the stadium. This is the first view. If you look at several spectators next to each other, you will see each of them move their hands up and down at the same rate with a certain delay. When the first person in the group raises his or her hands at the highest position, the hands of the person on the left are about to reach the highest position, reaching the highest position a few seconds later. This is the second view of the wave. In this case, the wave travels toward the right of this group of spectators.

In this chapter, we discuss waves according to these two views. We will find that the spatial and temporal periodic patterns constitute a wave motion. Several basic quantities associated with these views of a wave will be introduced. Some mathematical tools that facilitate the description of waves are also discussed.

1.1 Wave as a Moving Oscillatory Pattern

We can view a wave as an oscillation moving in space. The spectator of a soccer stadium can make a wave because one person's up-and-down hand motion moves around the stadium. If all the spectators move their hands simultaneously, no wave is formed. It is simply an oscillation.

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Mathematically, we can express the above fact as follows. Suppose $f(t, x)$ is a periodic function of time t and space x where the period is θ.

$$f(\phi) = f(\phi \pm n\theta) \tag{1.1}$$

Here ϕ is the phase and n is an integer. The left-hand side of (1.1) represents the value of function $f(t, x)$ when the combination of time and position makes the phase equal to ϕ. The entire Eq. (1.1) expresses that the value of the function is the same when the phase varies by an integer multiple of period θ. Here θ is a constant.

Consider that ϕ is a linear function of the time and spatial coordinates as

$$\phi = \omega t \pm kx \tag{1.2}$$

where ω and k are constants whose meanings will be clarified shortly. At a fixed location x, the phase varies as a function of t. Selecting this location to be $x = 0$ does not lose the generality. Under this condition, from (1.1) and (1.2) we find it as follows.

$$f(\omega t) = f(\omega t + \theta) = f(\omega(t + \tau)) \tag{1.3}$$

Expression (1.3) indicates that every τ s (seconds) the function $f(t, 0)$ takes the same value. Thus, we can interpret τ to be the period in time. (We can say that θ is the period in phase.) The same argument holds at an arbitrary location x_0,

$$f(\omega t + kx_0) = f(\omega t + kx_0 + \theta) = f(\omega(t + \tau) + kx_0) \tag{1.4}$$

At the fixed location x_0, the phase ϕ varies only depending on t. So, the function $f(\phi)$ takes the same value every τ s.

Since ϕ is a linear function of t, $\omega(t + \tau) = \omega t + \omega\tau$. So, from (1.3) we find the following equation.

$$\tau = \frac{\theta}{\omega} \tag{1.5}$$

The reciprocal of τ indicates how many times the oscillation occurs in 1 s, i.e., how frequent the oscillation is.

$$\nu \equiv \frac{1}{\tau} \tag{1.6}$$

This quantity ν is referred to as the frequency of oscillation and its SI unit (Système International d'unités, or the International System of Units) is Hz (= 1/s). The quantity ω is called the angular frequency (sometimes called the frequency as well) and its unit is rad/s.

Repeating the same argument as above, we can define the spatial periodicity λ as follows.

$$\lambda = \frac{\theta}{k} \tag{1.7}$$

The quantity λ is referred to as the wavelength. It is the spatial period of the wave in the SI unit of m. The quantity k is called the wave number. It is the spatial frequency of the wave in the unit of rad (radian).

1.2 Phase Velocity

Multiplication of (1.6) and (1.7) yields the following equation.

$$\nu\lambda = \frac{\omega}{k} \tag{1.8}$$

The left-hand side of (1.8) indicates how many times the wavelength repeats in 1 s. If the observer stays at a fixed place, he will see the pattern of the spatial periodicity passing him $\nu\lambda$ times a second. Changing the perspective, we can also say that (1.8) represents the velocity of an observer that sits on the wave at a constant phase. For this observer, the phase does not change. Therefore, $d\phi/dt = 0$. By differentiating ϕ with respect to time, we obtain the following expression.

$$\frac{d\phi}{dt} = \omega \pm k\frac{dx}{dt} \tag{1.9}$$

Setting (1.9) to zero, we obtain the following equation.

$$\frac{dx}{dt} = \pm\frac{\omega}{k} \tag{1.10}$$

We can interpret that the left-hand side of (1.10) represents the change in the position of a constant phase over time, and call it the phase velocity of the wave. (1.9) indicates that the phase velocity is the product of the frequency and wavelength of a wave.

$$v_p = \nu\lambda = \pm\frac{\omega}{k} \tag{1.11}$$

The sign in front of ω/k on the right-hand side of (1.11) represents the direction of the wave's motion.

By substituting (1.2) into the expression on the left-hand side of (1.1) and using (1.11), we can express the wave function f in the following way.

$$f(\phi) = f(\omega t \pm kx) = f\left(\omega\left(t - \frac{x}{\pm v_p}\right)\right) = f\left(k\left(\pm v_p t - x\right)\right) \tag{1.12}$$

Consider the constant phase condition, $\phi = \phi_0$, in (1.12).

$$\omega t \pm kx = \phi_0 \tag{1.13}$$

If the sign in front of kx is positive, x must decrease to keep the phase constant at ϕ_0 over time (when t increases). This means that the wave moves in the negative direction along the

x-axis. If it is negative, the wave moves in the positive x direction. Note that the former case where the wave moves in the negative x direction corresponds to $v_p < 0$, and the latter case where the wave moves in the positive x direction corresponds to $v_p > 0$ on the right-hand side of (1.12). Thus, allowing for the phase velocity to take a positive or negative value, we can express the wave function in the following form.

$$f(\phi) = f\left(\omega\left(t - \frac{x}{v_p}\right)\right) = f\left(k\left(v_p t - x\right)\right) \tag{1.14}$$

In the form of (1.14), a positive v_p represents a wave moving in the positive x direction and a negative v_p represents a wave in the negative x direction.

1.3 Amplitude and Phase

The concepts of the amplitude and phase of an oscillatory system are most naturally and easily understood in association with a circular motion. Consider in Fig. 1.1 that point P moves along the circumference of the circle of radius A with a constant angular velocity of ω. The angle subtended by the arc between points P and $(x, y) = (A, 0)$ varies linearly with time as

$$\theta(t) = \omega t + \phi_0 \tag{1.15}$$

Here ϕ_0 is the initial angle at $t = 0$. The projection of point P onto the x-axis varies as

$$P_x(t) = A\cos\theta(t) = A\cos(\omega t + \phi_0) \tag{1.16}$$

In the next chapter, we will discuss the harmonic oscillation of a point mass connected to a spring. There we describe the dynamics using the displacement of the mass from its equilibrium position, $\xi(t)$. By viewing the origin $(x, y) = (0, 0)$ as the equilibrium point and $P_x(t)$ as the position of the mass connected to the spring as discussed above, we can

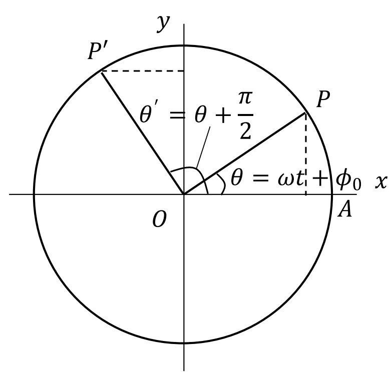

Fig. 1.1 Point P moves along the circumference at a constant circular speed