Synthesis Lectures on Electromagnetics

SYNTHESIS
COLLECTION OF TECHNOLOGY

Hamad M. Alkhoori

Concise Introduction to Electromagnetic Fields

Springer

Synthesis Lectures on Electromagnetics

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This series of short books focuses on a wide array of applications on electromagnetics, particularly in relation to design and interactions with advanced materials and devices. Topics include cutting-edge applications in bioengineering and biomaterials, optics, nanotechnology, and metamaterials.

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Preface

As the title suggests, this book serves as an introductory book to the subject of electromagnetics. Moreover, the word 'concise' from the book's title stresses that the book covers the fundamental topics a reader needs to acquire before consulting advanced-level references on electromagnetics. The book is intended to be used in an undergraduate-level course, whether in physics curriculum, or in electrical engineering curriculum. It can, as well, be used as a reference for researchers who wish to solidify their understanding of the subject.

Like any phenomenon being described by governing equations, electromagnetic phenomenon is described by Maxwell equations. In many undergraduate-level textbooks of electromagnetics, Maxwell equations are derived starting from Coulomb law, Biot-Savart law, Faraday law, and, lastly, Maxwell's correction to Ampere law. Consequently, the reader has to be exposed to the subjects of electrostatics and magnetostatics before discussing full-version electromagnetics in which time character emerges. While this approach is perfectly fine, I adopt a different approach in this book, whereby the full-version electromagnetics is introduced first. Then, electrostatics and magnetostatics can be regarded as special cases. It is to be mentioned that this approach is somewhat similar, in terms of the sequence, at least, to the approach adopted by Landau and Lifshitz, in which Maxwell equations are introduced from relativistic principles. However, due to the complexity of such an approach to be taught at an undergraduate-level course, Maxwell equations in this book are postulated in the beginning.

Electromagnetics can be divided into two regimes. These are (i) time-dependent regime and (ii) time-independent regime. This book is divided into four parts. Part I is about some required mathematical background, and an introduction to electromagnetics. Part II is about time-independent electromagnetics, namely, electrostatics and magnetostatics. Then, Part III discusses time-dependent electromagnetics in source-free regions. Finally, Part IV is about time-dependent electromagnetics in source regions. It should be emphasized that the division into source-free and source regions is made for the sake of facilitating the presentation. Once the concepts from source-free-region problems are grasped, transition to source-region problems becomes smoother.

In terms of chapters, this book comprises 14 chapters distributed in the four aforementioned parts. Chapter 1 presents a brief revision on vector algebra and vector calculus. Chapter 2 introduces Maxwell equations and divides the theory into two regimes. Chapter 3 is about electrostatics in which a static charge distribution gives rise to an electric field. Currents (i.e., moving charges) is discussed in Chap. 4, followed by magnetostatics, in which a current distribution gives rise to a magnetic field, in Chap. 5. Then, Chap. 6 is about the transition from time-independent regime to time-dependent regime. Chapter 7 discusses the propagation of electromagnetic fields in an unbounded, source-free region. This is followed by the propagation in the presence of an infinite-extent obstacle in Chap. 8. Chapters 9 and 10 treat the problem of propagation of electromagnetic fields in guided structures. Transition to source regions, namely, radiation problem, is discussed in Chap. 11. Chapter 12 discusses radiators, well known as antennas, and their properties. Chapter 13 discusses simple antenna structures. Finally, Chap. 14 briefly discusses the analysis of group of antennas, well known as antenna arrays.

Since the usage of computer programs to validate analytical procedures, to tackle problems not amendable to analytical solutions, or at least to gain a better visualization has significantly increased, some chapters are supplied with an appendix containing useful *Mathematica* computer programs. These can be used for the purpose of validating the solutions of end-of-chapter problems, validating the solutions of problems from other textbooks, or even validating the solutions of problems a reader can propose and solve. Furthermore, these computer programs can be used by researchers to produce various forms of plots (e.g., two-dimensional and three-dimensional plots for scalars, two-dimensional and three-dimensional streamline plots for vectors, etc.).

The reader is assumed to have some background in standard topics taught in junior undergraduate-level courses, or even in high school, such as differentiation and integration. Also, an exposure to elementary physics courses might be beneficial, though not necessary.

Al Ain, UAE Hamad M. Alkhoori

Contents

Part I

Introduction to Essential Mathematics and Electromagnetics

The first part of the book gives a brief review on mathematical topics needed in this book. Then, it gives an overview on electromagnetics from a system perspective. This part consists of two chapters. Chapter 1 discusses vector algebra and vector calculus in three coordinate systems (Cartesian, cylindrical, and spherical), as well as in a general curvilinear system. Chapter 2 presents Maxwell equations as governing equations of electromagnetics. Specialization to electrostatics and magnetostatics is discussed then as special cases from the general setting.

1 Vector Algebra and Vector Calculus

This chapter is devoted to vector algebra and vector calculus. In Sect. 1.1, we give an overview on vector algebra, including definition, Cartesian bases and vector expansion, vector arithmetic operators, and position and distance vectors. Then, Sect. 1.2 discusses the various coordinates systems encountered in this book (e.g., Cartesian, cylindrical, and spherical), as well as transformation among them. We then discuss vector calculus in Sect. 1.3, including vector integral calculus, and vector differential calculus. These are discussed first in Cartesian, cylindrical, and spherical coordinate systems, and then are extended to a general curvilinear coordinate system in Sect. 1.4. Finally, time-harmonic vectors is discussed in Sect. 1.5. Useful computer programs are given in the appendix at the end of the chapter.

1.1 Vector Algebra

1.1.1 Definition and Expansion

A scalar is a quantity that has a magnitude only (e.g., mass, charge, temperature, etc.), whereas a vector is a quantity that has a magnitude and a direction (e.g., velocity, acceleration, force, momentum, etc.). The magnitude of a vector $\mathbf{A}$ is written as $|\mathbf{A}|$, or A, and its direction is written as $\hat{\mathbf{A}}$ given by

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|}. \tag{1.1}$$

The vector $\hat{\mathbf{A}}$ is called a unit vector because its magnitude is unity. A vector $\mathbf{A}$ can be expanded into Cartesian unit vectors as

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}, \tag{1.2}$$

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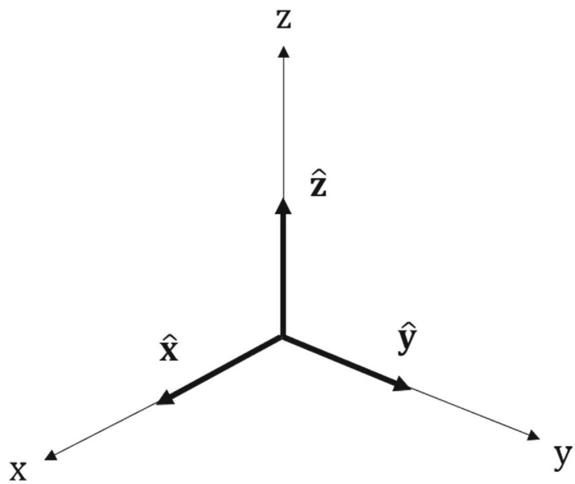


Fig. 1.1 Cartesian bases vectors

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors in the direction of the x axis, the y axis, and the z axis, respectively; see Fig. 1.1. These unit vectors can be called Cartesian bases. The scalars A_x, A_y, and A_z are components of the vector $\mathbf{A}$ in the direction of the x axis, the y axis, and the z axis, respectively.

1.1.2 Vector Addition and Subtraction

In addition to $\mathbf{A}$, let us define the vectors $\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$ and $\mathbf{C} = C_x\hat{\mathbf{x}} + C_y\hat{\mathbf{y}} + C_z\hat{\mathbf{z}}$. Addition between two vectors $\mathbf{A}$ and $\mathbf{B}$ can be done using

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}. \tag{1.3}$$

Subtraction between two vectors $\mathbf{A}$ and $\mathbf{B}$ can be done as $\mathbf{A} + (-\mathbf{B})$. Addition is commutative (i.e., $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$), associative [i.e., $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$], and distributive (i.e., $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$), where α is a scalar..

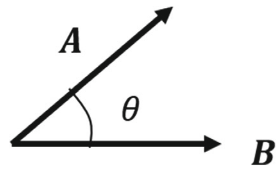
1.1.3 The Dot Product

The dot product between two vectors $\mathbf{A}$ and $\mathbf{B}$ can be performed as

$$\mathbf{A} \bullet \mathbf{B} \equiv |\mathbf{A}||\mathbf{B}| \cos\theta, \tag{1.4}$$

where θ (in rad) is the angle between $\mathbf{A}$ and $\mathbf{B}$; see Fig. 1.2.

Fig. 1.2 Dot product



When (i) $\theta = 0°$, **A** and **B** are coparallel to each other, (ii) when $\theta = 90°$, **A** and **B** are perpendicular (normal or orthogonal) to each other, and (iii) when $\theta = 180°$, **A** and **B** are antiparallel to each other. Note that $\hat{\mathbf{x}} \bullet \hat{\mathbf{x}} = \hat{\mathbf{y}} \bullet \hat{\mathbf{y}} = \hat{\mathbf{z}} \bullet \hat{\mathbf{z}} = 1$, whereas $\hat{\mathbf{x}} \bullet \hat{\mathbf{y}} = \hat{\mathbf{x}} \bullet \hat{\mathbf{z}} = \hat{\mathbf{y}} \bullet \hat{\mathbf{z}} = 0$. In component form, the dot product can be written as

$$\mathbf{A} \bullet \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \tag{1.5}$$

From Eqs. (1.4) and (1.5), we see that

$$\mathbf{A} \bullet \mathbf{A} = |\mathbf{A}|^2 = A_x^2 + A_y^2 + A_z^2. \tag{1.6}$$

Hence, the magnitude of the vector **A** is

$$|\mathbf{A}| = \sqrt{\mathbf{A} \bullet \mathbf{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{1.7}$$

The dot product is commutative (i.e., $\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$), and associative (i.e., $\mathbf{A} \bullet (\mathbf{B} + \mathbf{C}) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$).

Example 1.1 Let $\mathbf{A} = \hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} + 5\,\hat{\mathbf{z}}$ and $\mathbf{B} = -\,\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} + 3\,\hat{\mathbf{z}}$. Find the angle θ between **A** and **B**.

Solution. We have $\mathbf{A} \bullet \mathbf{B} = 18$, $|\mathbf{A}| = 5.47$, and $|\mathbf{B}| = 3.74$. Therefore, $\theta = \cos^{-1} \left(\frac{18}{5.47 \times 3.74} \right) = 0.498\,(28.56°)$. ◁

Position and Distance Vectors

The position vector of a point represented by coordinates (x, y, z) is given in Cartesian coordinates by

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}. \tag{1.8}$$

Suppose that another point is represented by (x', y', z'), with a corresponding position vector

$$\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}. \tag{1.9}$$

Then, the distance vector between the two points is given by

$$\mathbf{R} = \mathbf{r} - \mathbf{r}'. \tag{1.10}$$

The magnitude of the distance vector

$$R = |\mathbf{r} - \mathbf{r}'| = \sqrt{|\mathbf{r}|^2 + |\mathbf{r}'|^2 - 2\mathbf{r} \cdot \mathbf{r}'} \tag{1.11}$$

gives the distance between two points.

Example 1.2 Find the distance between the two points (1, −1, 3) and (5, 0, 3).

Solution. Let $\mathbf{r} = \hat{\mathbf{x}} - \hat{\mathbf{y}} + 3\,\hat{\mathbf{z}}$, and $\mathbf{r}' = 5\,\hat{\mathbf{x}} + 3\,\hat{\mathbf{z}}$. Then, $\mathbf{R} = \mathbf{r} - \mathbf{r}' = -4\,\hat{\mathbf{x}} - \hat{\mathbf{y}}$. So, $R = |\mathbf{R}| = 4.12$. ◁

Component of a Vector

The scalar component (projection) of a vector **A** in the direction of a vector **B** is written as A_B, which is given by

$$A_B = \mathbf{A} \cdot \hat{\mathbf{B}}. \tag{1.12}$$

The vector component of a vector **A** in the direction of a vector **B** is written as $\mathbf{A}_B$, which is given by

$$\mathbf{A}_B = (\mathbf{A} \cdot \hat{\mathbf{B}})\hat{\mathbf{B}}. \tag{1.13}$$

This can be used in defining the normal and tangential components of a vector with respect to a surface characterized by a unit normal $\hat{\mathbf{n}}$. Given a surface with a unit normal $\hat{\mathbf{n}}$, the vector component normal to the surface, denoted by $\mathbf{A}_\perp$, is

$$\mathbf{A}_\perp = (\mathbf{A} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}, \tag{1.14}$$

whereas the the vector component tangential to the surface, denoted by $\mathbf{A}_\parallel$, is

$$\mathbf{A}_\parallel = \mathbf{A} - (\mathbf{A} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}. \tag{1.15}$$

1.1.4 The Cross Product

The cross product between two vectors **A** and **B** is performed as

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta\, \hat{\mathbf{n}}, \tag{1.16}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both **A** and **B**. In component form, the cross product can be written as

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{\mathbf{x}}(A_y B_z - A_z B_y) + \hat{\mathbf{y}}(A_z B_x - A_x B_z) + \hat{\mathbf{z}}(A_x B_y - A_y B_x). \tag{1.17}$$

If we set $\mathbf{A} = \hat{\mathbf{x}}$ and $\mathbf{B} = \hat{\mathbf{y}}$ in Eq. (1.17), we find that $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$. By a similar approach, it can be seen that $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ and $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$. The cross product is distributive (i.e., $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$), but it is not commutative (i.e., $\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$).

A combination of dot and cross products is also encountered in electromagnetics. These are scalar triple product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}), \tag{1.18}$$

and vector triple product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{C} \cdot \mathbf{A}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \tag{1.19}$$

1.2 Coordinate Systems and Transformations

A coordinate system in three dimensions is comprised of spatial variables (i.e., coordinates) $\{v_1, v_2, v_3\}$, as well as bases $\{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3\}$. We discuss in this section the most common three coordinate systems. These are Cartesian coordinates, circular cylindrical (or simply cylindrical) coordinates, and spherical coordinates. A general curvilinear coordinate system is discussed in Sect. 1.4 after learning vector calculus.

1.2.1 Cartesian Coordinates

Cartesian coordinates $\{x, y, z\}$, where $x \in (-\infty, \infty)$, $y \in (-\infty, \infty)$, and $z \in (-\infty, \infty)$, constitute the simplest coordinate system. The following Cartesian surfaces arise when one coordinate is fixed.

- The equation $x = x_0$ defines an infinite plane on the yz plane with a coordinate $x = x_0$.
- The equation $y = y_0$ defines an infinite plane on the xz plane with a coordinate $y = y_0$.
- The equation $z = z_0$ defines an infinite plane on the xy plane with a coordinate $z = z_0$.

Notice that all of the aforementioned surfaces are infinite because only one coordinate is specified, while the range of the other two are not. If the range of each one of the other two coordinates is specified and is finite, then, the resulting plane will no more be infinite. Figure 1.3 shows Cartesian surfaces.

Cartesian bases $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ can be used for expanding a vector $\mathbf{A}$ as

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}. \tag{1.20}$$

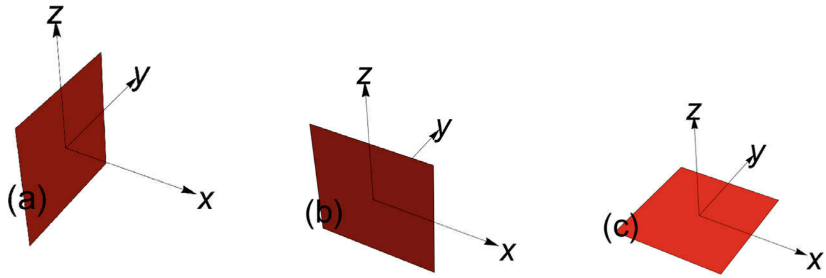


Fig. 1.3 **a** $x = 0$, **b** $y = 0$, and **c** $z = 0$

Here, we can regard A_x as the projection of $\mathbf{A}$ into $\hat{\mathbf{x}}$, A_y is the projection of $\mathbf{A}$ into $\hat{\mathbf{y}}$, and A_z is the projection of $\mathbf{A}$ into $\hat{\mathbf{z}}$. It is to be noted that, each of A_x, A_y, and A_z is a scalar that can be a function of the variables x, y, and z. That is,

$$\mathbf{A}(x, y, z) = A_x(x, y, z)\,\hat{\mathbf{x}} + A_y(x, y, z)\,\hat{\mathbf{y}} + A_z(x, y, z)\,\hat{\mathbf{z}}. \tag{1.21}$$

For shorthand notation, we can let $(x, y, z) \to \mathbf{r}$. Then, Eq. (1.21) can be written as

$$\mathbf{A}(\mathbf{r}) = A_x(\mathbf{r})\,\hat{\mathbf{x}} + A_y(\mathbf{r})\,\hat{\mathbf{y}} + A_z(\mathbf{r})\,\hat{\mathbf{z}}.^1 \tag{1.22}$$

We already saw that Cartesian bases satisfy

$$\left.\hat{\mathbf{x}} \bullet \hat{\mathbf{x}} = \hat{\mathbf{y}} \bullet \hat{\mathbf{y}} = \hat{\mathbf{z}} \bullet \hat{\mathbf{z}} = 1 \qquad \hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \qquad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}} \qquad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}\right\}. \tag{1.23}$$

Figure 1.4 shows streamline plots of Cartesian bases. These bases can be represented more simply as in Fig. 1.1. Notice that Cartesian bases are constant vectors (i.e., do not depend on either x, y, or z). Also, we see that $\hat{\mathbf{x}}$ points in the direction of increase of x, $\hat{\mathbf{y}}$ points in the direction of increase of y, and $\hat{\mathbf{z}}$ points in the direction of increase of z. As to been seen in Sect. 1.3.3, this is not a mere coincidence.

1.2.2 Cylindrical Coordinates

Cylindrical coordinates $\{\rho, \phi, z\}$, as well as cylindrical bases $\{\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}\}$ are shown in Fig. 1.5. Cylindrical coordinates are related to the Cartesian coordinates through

$$\left.x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z\right\}. \tag{1.24}$$

Given a point P in space, we see that $\rho \in [0, \infty)$ is the distance between the z axis and the point P, and $\phi \in [0, 2\pi)$ (called the azimuthal angle) is measured from the x axis to the

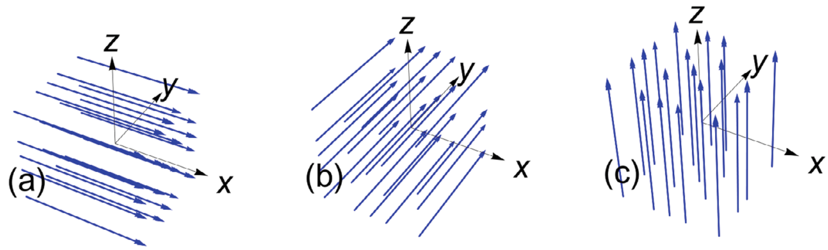
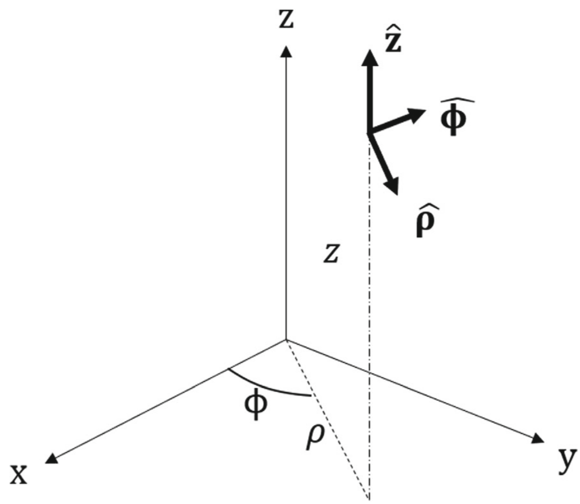


Fig. 1.4 **a** $\hat{\mathbf{x}}$, **b** $\hat{\mathbf{y}}$, and **c** $\hat{\mathbf{z}}$

Fig. 1.5 Cylindrical coordinates and bases



projection of the point P on the xy plane. Equations (1.24) can be used when converting from Cartesian coordinates to cylindrical coordinates. Conversion from cylindrical coordinates to Cartesian coordinates can be done upon inverting Eqs. (1.24), which gives

$$\left.\rho = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x} \qquad z = z\right\}. \tag{1.25}$$

The following cylindrical surfaces arise when one coordinate is fixed.

- The equation $\rho = \rho_0$ is the equation of an infinite cylinder with a radius ρ_0. Notice that if the range of z is specified and is finite, then the cylinder will no more be infinite.
- The equation $\phi = \phi_0$ is the equation of a semi-infinite plane making an angle ϕ_0 with respect to the positive x axis. Notice that if the ranges of ρ and z are specified and are finite, then the plane will no more be semi infinite.
- The equation $z = z_0$ is the equation of an infinite plane on the xy plane with a coordinate $z = z_0$.

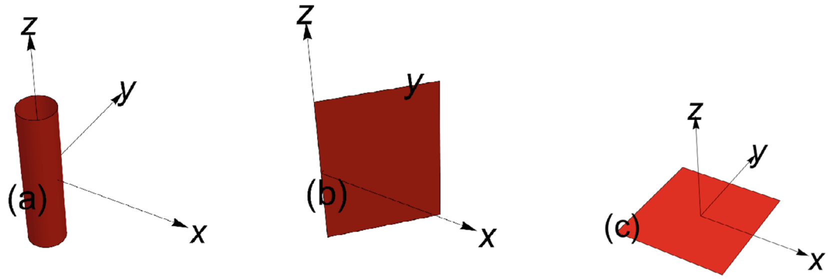


Fig. 1.6 **a** $\rho = 1$, **b** $\phi = \pi/4$, and **c** $z = 0$

Figure 1.6 shows examples of cylindrical surfaces.

Cylindrical bases $\{\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}\}$ can be used for expanding a vector $\mathbf{A}$ as

$$\mathbf{A}(\mathbf{r}) = A_\rho(\mathbf{r})\,\hat{\boldsymbol{\rho}} + A_\phi(\mathbf{r})\,\hat{\boldsymbol{\phi}} + A_z(\mathbf{r})\,\hat{\mathbf{z}}, \tag{1.26}$$

where A_ρ is the projection of $\mathbf{A}$ into $\hat{\boldsymbol{\rho}}$, A_ϕ is the projection of $\mathbf{A}$ into $\hat{\boldsymbol{\phi}}$, and A_z is the projection of $\mathbf{A}$ into $\hat{\mathbf{z}}$. Those cylindrical bases satisfy

$$\left.\hat{\boldsymbol{\rho}} \bullet \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}} \bullet \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \bullet \hat{\mathbf{z}} = 1 \qquad \hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \qquad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\boldsymbol{\rho}} \qquad \hat{\mathbf{z}} \times \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}\right\}. \tag{1.27}$$

Consequently, like Cartesian coordinates, the dot and the cross products in cylindrical coordinates can be done, respectively, as

$$\mathbf{A} \bullet \mathbf{B} = A_\rho B_\rho + A_\phi B_\phi + A_z B_z, \tag{1.28}$$

and

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\boldsymbol{\rho}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_\rho & A_\phi & A_z \\ B_\rho & B_\phi & B_z \end{vmatrix}. \tag{1.29}$$

Figure 1.7 shows streamline plots of cylindrical bases. These bases can be represented more simply as in Fig. 1.5. Notice that the cylindrical bases $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ are not constant vectors. Also, we see that $\hat{\boldsymbol{\rho}}$ points in the direction of increase of ρ, and $\hat{\boldsymbol{\phi}}$ points in the direction of increase of ϕ.

Cylindrical bases can be transformed into Cartesian bases as follows. The basis $\hat{\boldsymbol{\rho}}$, like any vector, can be expanded into Cartesian bases as

$$\hat{\boldsymbol{\rho}} = \alpha\,\hat{\mathbf{x}} + \beta\,\hat{\mathbf{y}} + \gamma\,\hat{\mathbf{z}}, \tag{1.30}$$

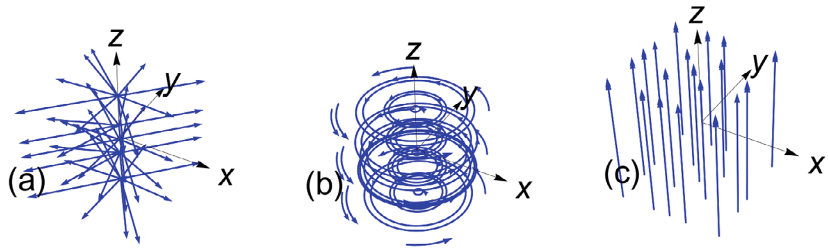


Fig. 1.7 **a** $\hat{\rho}$, **b** $\hat{\boldsymbol{\phi}}$, and **c** $\hat{\mathbf{z}}$

where α, β, and γ are unknown that have to be determined. Using the fact that Cartesian bases are orthogonal, one can find that $\alpha = \hat{\boldsymbol{\rho}} \bullet \hat{\mathbf{x}} = \cos\phi$, $\beta = \hat{\boldsymbol{\rho}} \bullet \hat{\mathbf{y}} = \sin\phi$, and $\gamma = \hat{\boldsymbol{\rho}} \bullet \hat{\mathbf{z}} = 0$. Therefore,

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}. \tag{1.31}$$

Equation (1.31) transforms the cylindrical basis $\hat{\boldsymbol{\rho}}$ into the Cartesian bases $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. Transforming $\hat{\boldsymbol{\phi}}$ can be done similarly. Hence, transformation between cylindrical bases to Cartesian bases can be written in matrix form as

$$\begin{pmatrix} \hat{\boldsymbol{\rho}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}. \tag{1.32}$$

Notice that, unlike Cartesian bases, cylindrical bases $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ depend on the coordinate ϕ. Conversion relation from Cartesian bases to cylindrical bases can be established upon inverting the square matrix appearing in Eq. (1.32). Since this matrix is orthogonal, its inverse is simply its transpose. Therefore,

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\rho}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{pmatrix}, \tag{1.33}$$

1.2.3 Spherical coordinates

Spherical coordinates $\{r, \theta, \phi\}$, as well as spherical bases $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ are shown in Fig. 1.8. Spherical coordinates are related to the Cartesian coordinates through

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta\Big\}. \tag{1.34}$$

Given a point P in space, we see that $r \in [0, \infty)$ is the distance between the origin and the point P, $\theta \in [0, \pi]$, called the colatitude (or polar) angle, is an angle drawn from the z axis

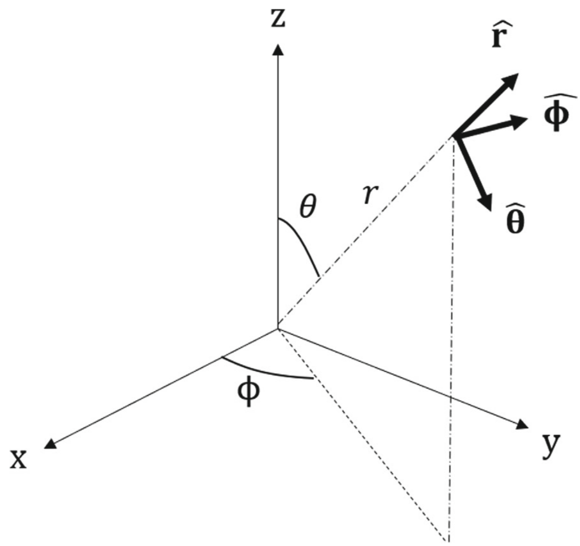


Fig. 1.8 Spherical coordinates and bases

to the line formed by the origin and the point P, and ϕ is defined same as before. Equations (1.34) can be used when converting from Cartesian coordinates to spherical coordinates. Conversion from spherical coordinates to Cartesian coordinates can be done upon inverting Eqs. (1.34), which gives

$$\left. r = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} \qquad \phi = \tan^{-1} \frac{y}{x} \right\}. \tag{1.35}$$

The following spherical surfaces arise when one coordinate is fixed.

- The equation $r = r_0$ is the equation of a sphere with a radius r_0.
- The equation $\theta = \theta_0$ is the equation of an infinite cone of an angle θ_0. Notice that if the range of r is specified and is finite, then the cone will no more be infinite.
- The equation $\phi = \phi_0$ is the equation of a semi-infinite plane making an angle ϕ_0 with respect to the positive x axis. Notice that if the ranges of r and θ are specified, and the range of r is finite, then the plane will no more be semi infinite.

Spherical bases $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ can be used for expanding a vector $\mathbf{A}$ as

$$\mathbf{A}(\mathbf{r}) = A_r(\mathbf{r})\,\hat{\mathbf{r}} + A_\theta(\mathbf{r})\,\hat{\boldsymbol{\theta}} + A_\phi(\mathbf{r})\,\hat{\boldsymbol{\phi}}, \tag{1.36}$$

where A_r is the projection of $\mathbf{A}$ into $\hat{\mathbf{r}}$, A_θ is the projection of $\mathbf{A}$ into θ, and A_ϕ is the projection of $\mathbf{A}$ into $\hat{\boldsymbol{\phi}}$. Those spherical bases satisfy

$$\left. \hat{\mathbf{r}} \bullet \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} \bullet \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \bullet \hat{\boldsymbol{\phi}} = 1 \qquad \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \qquad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{r}} \qquad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} \right\}. \tag{1.37}$$

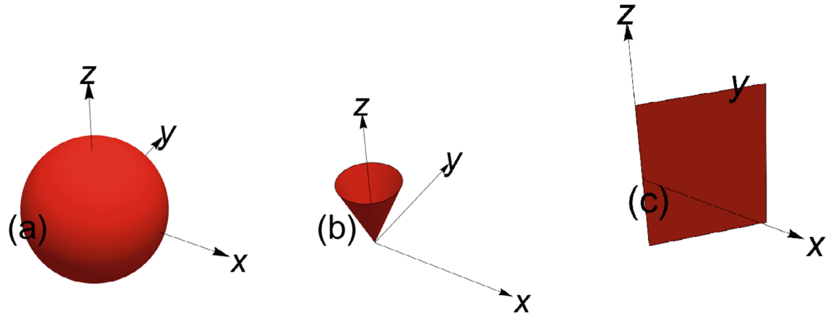


Fig. 1.9 **a** $r = 4$, **b** $\theta = \pi/8$, and **c** $\phi = \pi/4$

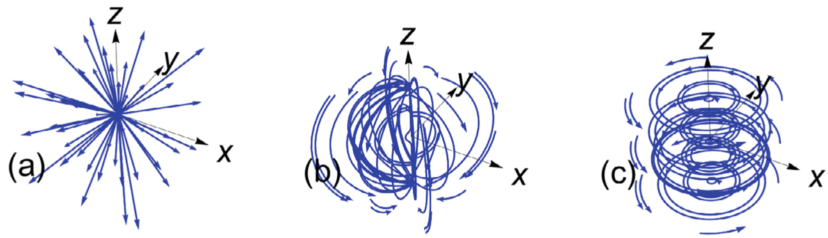


Fig. 1.10 **a** $\hat{\mathbf{r}}$, **b** $\hat{\boldsymbol{\theta}}$, and **c** $\hat{\boldsymbol{\phi}}$

Consequently, like Cartesian coordinates, the dot and the cross products in spherical coordinates can be done, respectively, as

$$\mathbf{A} \bullet \mathbf{B} = A_r B_r + A_\theta B_\theta + A_\phi B_\phi, \tag{1.38}$$

and

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix}. \tag{1.39}$$

Figure 1.10 shows streamline plots of spherical bases. These bases can be represented more simply as in Fig. 1.8. Notice that spherical bases are not constant vectors. Also, we see that $\hat{\mathbf{r}}$ points in the direction of increase of r, and $\hat{\boldsymbol{\theta}}$ points in the direction of increase of θ.

Using the procedure used in cylindrical coordinates, spherical bases can related to the Cartesian bases using Fig. 1.8 as

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}. \tag{1.40}$$

Conversion from Cartesian bases to spherical bases can then be done by

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix}. \tag{1.41}$$

It should be mentioned that transformation between bases formulas can be derived more systematically using the gradient operator (see Sect. 1.3.3).

Example 1.3 Given $f(x, y, z) = x^2 y + \frac{z}{x}$, express $f(x, y, z)$ in spherical coordinates [i.e., obtain $f(r, \theta, \phi)$].

Solution.

$$f(r, \theta, \phi) = r^3 \sin^3\theta \sin\phi \cos^2\phi + \cot\theta \sec\phi.$$

◁

Example 1.4 Given $\mathbf{A} = y\hat{\mathbf{x}}$, express $\mathbf{A}$ in (a) cylindrical coordinates and in (b) spherical coordinates.

Solution.
(a)

$$\mathbf{A} = y\hat{\mathbf{x}} = \rho\sin\phi(\cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\phi}}) = \rho\sin\phi\cos\phi\,\hat{\boldsymbol{\rho}} - \rho\sin^2\phi\,\hat{\boldsymbol{\phi}}.$$

(b)

$$\begin{aligned} \mathbf{A} = y\hat{\mathbf{x}} &= r\sin\theta\sin\phi(\sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}) \\ &= r\sin^2\theta\sin\phi\cos\phi\,\hat{\mathbf{r}} + r\sin\theta\cos\theta\sin\phi\cos\phi\,\hat{\boldsymbol{\theta}} - r\sin\theta\sin^2\phi\,\hat{\boldsymbol{\phi}}. \end{aligned}$$

◁

Example 1.5 Given $\mathbf{A} = A_0\hat{\mathbf{r}}$, where A_0 is a constant, determine $\mathbf{A} \bullet \hat{\mathbf{x}}$.

Solution. We have $\mathbf{A} \bullet \hat{\mathbf{x}} = A_0\hat{\mathbf{r}} \bullet \hat{\mathbf{x}}$. From

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix},$$

we see that $\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$. Thus,

$$\hat{\mathbf{r}} \bullet \hat{\mathbf{x}} = (\sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}) \bullet \hat{\mathbf{r}} = \sin\theta\cos\phi.$$

Therefore, $\mathbf{A} \bullet \hat{\mathbf{x}} = A_0 \sin\theta\cos\phi$. ◁

Example 1.6 Given $\mathbf{A} = \hat{\boldsymbol{\rho}} + 3\hat{\boldsymbol{\phi}} + 6\hat{\mathbf{z}}$, determine the scalar component of **A** parallel to the y axis.

Solution. We have

$$\mathbf{A} \bullet \hat{\mathbf{y}} = (\hat{\boldsymbol{\rho}} + 3\hat{\boldsymbol{\phi}} + 6\hat{\mathbf{z}}) \bullet \hat{\mathbf{y}}.$$

Using

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\rho}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{pmatrix},$$

we write $\hat{\mathbf{y}} = \sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\hat{\boldsymbol{\phi}}$. Then,

$$\mathbf{A} \bullet \hat{\mathbf{y}} = (\hat{\boldsymbol{\rho}} + 3\hat{\boldsymbol{\phi}} + 6\hat{\mathbf{z}}) \bullet (\sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\hat{\boldsymbol{\phi}}) = \sin\phi + 3\cos\phi.$$

Notice that $\phi = \pi/2$ along the y axis. Hence, $\mathbf{A} \bullet \hat{\mathbf{y}}\Big|_{\phi=\pi/2} = 1.$ ◁

Example 1.7 Given $\mathbf{A} = \rho z \cos^2\phi\,\hat{\boldsymbol{\rho}} + \sin\phi\,\hat{\boldsymbol{\phi}} + \rho\,\hat{\mathbf{z}}$, determine (a) the scalar component of **A** parallel to the x axis, (b) the vector component of **A** normal to the surface $\rho = 1$ [i.e., $(\mathbf{A} \bullet \hat{\boldsymbol{\rho}})\hat{\boldsymbol{\rho}}$ when $\rho = 1$], and (c) the vector component of **A** tangential to the plane $z = 0$.

Solution.

(a) We have

$$\mathbf{A} \bullet \hat{\mathbf{x}} = (\rho z \cos^2\phi\,\hat{\boldsymbol{\rho}} + \sin\phi\,\hat{\boldsymbol{\phi}} + \rho\,\hat{\mathbf{z}}) \bullet \hat{\mathbf{x}}.$$

Using

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\rho}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{pmatrix},$$

we write $\hat{\mathbf{x}} = \cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}$. Thus,

$$\mathbf{A} \bullet \hat{\mathbf{x}} = \rho z \cos^3\phi - \sin^2\phi.$$

But since $\phi = 0$ along the x axis, we get $\mathbf{A} \bullet \hat{\mathbf{x}}\Big|_{\phi=0} = \rho z.$

(b) We have

$$\mathbf{A}_\perp = (\mathbf{A} \bullet \hat{\boldsymbol{\rho}})\hat{\boldsymbol{\rho}} = \rho z \cos^2\phi\,\hat{\boldsymbol{\rho}}.$$

But this has to be evaluated at $\rho = 1$. Hence, $\mathbf{A}_\perp\Big|_{\rho=1} = z\cos^2\phi\,\hat{\boldsymbol{\rho}}.$

(c) We have

$$\mathbf{A}_\parallel = \mathbf{A} - \mathbf{A}_\perp = \mathbf{A} - (\mathbf{A} \bullet \hat{\mathbf{z}})\hat{\mathbf{z}} = \rho z \cos^2\phi\,\hat{\boldsymbol{\rho}} + \sin\phi\,\hat{\boldsymbol{\phi}}.$$

But this has to be evaluated at $z = 0$. Hence, $\mathbf{A}_\parallel\Big|_{z=0} = \sin\phi\,\hat{\boldsymbol{\phi}}.$ ◁