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Preface

A qualitatively new level of development of various branches of modern natural science is closely connected with theoretical and applied problems of the interaction of various media and fields. The problems of interaction are also fundamental in a new, rapidly progressing field of mechanics of a deformable solid body—in the theory of magnetoelasticity, the subject of which is the study of conjugate fields and processes in deformable bodies under the influence of external electromagnetic fields. Interest in research in this area is due to the importance of quantitative study and evaluation of the observed effects of the interaction of mechanical and electromagnetic processes and their practical application. The problems of interaction of various fields of physical origin are relevant and cover a wide class of general theoretical and applied problems that arise when considering a physical experiment, creating measuring equipment, solving problems of electromagnetic flaw detection, creating strong magnetic fields, electro-magneto-mechanical energy converters, signal processing, etc.

At present, a significant number of works are known in which various problems of magnetoelasticity are studied. In particular, the issues of oscillation and stability of thin elastic non-ferromagnetic conducting plates and shells in an electromagnetic field have been studied quite fully, and the interaction effects here turned out to be very significant. But similar questions for thin bodies made of ferromagnetic or superconducting material have received comparatively little attention in the literature. This is explained, firstly, by the fact that the solutions of the arising boundary value problems are associated with great mathematical difficulties, and secondly, by the fact that the interaction pattern becomes much more complicated when the material of the body has the property of magnetic polarizability. With this book, the authors intend to partially fill these gaps. It is devoted to the study of oscillations and stability of thin plates and shells (both magnetically active ferromagnetic and superconducting) in a magnetic field.

The purpose of this book is to introduce the reader to the methods of mathematical modeling and solving non-stationary (dynamic) problems of the theory of magnetoelasticity, as well as to give an idea of the richness of physical effects caused by the interaction of electromagnetic and mechanical phenomena in magnetoactive elastic thin bodies. Consideration is mainly limited to a model of isotropic body under the assumption of small deformations. The International System of Units (SI) is adopted.

Chapter 1 of the book is based on the basic connected nonlinear equations and relations of mechanics and quasi-static electrodynamics of continuous media; by linearization, a system of magnetoelasticity equations, surface conditions, and constitutive equations describing the behavior of perturbations in a magnetoactive medium (both magnetically soft or magnetostrictive ferromagnetic and superconducting), interacting with an external magnetic field, are obtained. On this basis, in the next two chapters (Chaps. 2 and 3), using the hypothesis of non-deformable normals, two-dimensional linearized equations and relations of magnetoelastic oscillations and the stability of magnetically soft thin plates and shells are derived. Mathematical modeling of dynamic processes occurring in the considered thin bodies is given. As a result, the problems of magnetoelastic oscillations and the stability of these thin bodies are reduced to solving a two-dimensional system of differential equations under the boundary conditions for fixing the edges of the body and the derived conditions for conjugation on its surface. On this basis, by solving specific applied problems, a number of qualitative and quantitative results were revealed, due to the interaction of mechanical and magnetic phenomena in ferromagnetic thin bodies, including loss of static stability under the action of both transverse and longitudinal constant magnetic fields; a significant increase in the damping effect of the magnetic field in the case of conducting ferromagnetic thin bodies; regulation of parametric oscillations (caused by external non-stationary forces) using a given constant magnetic field (e.g., a magnetic field with a magnetic induction of less than one tesla can significantly reduce the width of the main region of dynamic instability); excitation of controlled parametric oscillations by a harmonic magnetic field; by varying the magnitude of the induction of a given magnetic field, one can significantly increase or significantly decrease the amplitude of forced magnetoelastic oscillations. Moreover, it was found that with the help of a magnetic field, it is possible to regulate the location of resonance points and exclude the possibility of dangerous resonant oscillations, etc. An approximate formula is obtained for determining the magnetohydrodynamic pressure on the oscillating surfaces of a plate in a supersonic flow of an ideally conducting gas in the presence of a magnetic field. It is a generalization of the well-known formula, obtained on the basis of the piston theory of classical gas dynamics, for the case of magneto-gas-dynamic flow around thin bodies. On this basis, it became possible to solve complex problems of aero-magnetoelasticity and show the possibility of favorable control of the characteristics of supersonic linear and nonlinear flutter of thin bodies using a magnetic field.

In Chaps. 4 and 5, magnetoelastic processes in superconducting thin shells in stationary and non-stationary magnetic fields are studied. Here, based on the main provisions of the classical theory of thin shells, the theory of superconductivity and the results of previous chapters, on the basis of the hypothesis of non-deformable normals, two-dimensional equations and the corresponding conditions that characterize the oscillations and stability of superconducting deformable thin cylindrical and spherical shells under the action of a given magnetic field are obtained. On this basis, by solving specific problems, the possibility of losing both static and dynamic stability of thin superconducting bodies under the action of an external magnetic field has been established. The conditions are formulated under which in superconducting thin bodies, due to the presence of a constant magnetic field of relatively low strength, it is possible: to control resonant and flutter oscillations; to excite both transverse forced oscillations with the help of a magnetic field harmonic in time, and accompanying transverse oscillations under the action of a non-stationary longitudinal force; get rid of the appearance of bending stresses of magnetoelastic origin, exceeding the elastic limit of the shell material, etc.

Chapter 6 is devoted to mathematical modeling and research of the dynamics of magnetostrictive plates in magnetic fields (stationary and non-stationary) of various orientations. To achieve this goal (study of the processes of magnetoelastic interaction in the considered plate with complex physical properties of its material), the main provisions of the following theories and methods were used: the linearized theory of elastic stability of magnetostrictive solids; the classical theory of elastic plates; and the asymptotic method for solving boundary value problems in a rectangular region. As a result, the corresponding boundary value problems of mathematical physics are formulated, which describe dynamic processes in the considered magnetoelastic systems. By solving specific practically interesting problems, it has been established that: (a) there is a region of change in the geometric parameters of the plate and the magnetostrictive characteristics of its material, where the unperturbed state of the plate is stable at any value of the induction of the external magnetic field; (b) in the specified region, the magnetic field can lead to a significant increase in the frequency of magnetoelastic oscillations; (c) outside this region, the magnetostrictive effect has a destabilizing effect, leading to a significant decrease in the critical value of the magnetic induction (at which the plate loses stability) compared to the indicated critical value obtained in the absence of the magnetostrictive effect . It is also shown that due to the magnetostrictive effect, it is possible: to lose the static stability of the plate under the action of a constant transverse magnetic field; using a magnetic field to change significantly the value of the frequency of natural oscillations. The influence of the inhomogeneity of the plate on the processes under consideration was also investigated. Dynamic processes in layered plates are studied, and it is established that a magnetostrictive plate inhomogeneous in thickness (two-layer) in a harmonic magnetic field can become a source of disturbance propagation. In the case of a three-layer magnetostrictive plate, the material of the middle layer of which is a composite non-magnetic dielectric, it is shown that the inhomogeneity, magnetostrictive, and compositional properties of the layers are sufficient to optimally control dynamic processes, especially those that arise due to interaction. Here, an important role is played by the optimal choice of the angle of reinforcement of the composite material of the middle layer.

The last chapter (Chap. 7) is devoted to the stability of dielectric thin plates in a supersonic flow of an ideally conducting gas in the presence of a magnetic field. The problems are studied in both linear and nonlinear formulations. Here, as well as in the classical flutter problems, in determining the forces acting on the body caused by the flow and the magnetic field, the asymptotic property of the supersonic flow is used, according to which gas particles move mainly in directions normal to the streamlined surface (the law of plane sections). Based on this, an approximate formula for determining the magnetodynamic pressure on the oscillating surface of a thin elastic plate is obtained. The formula for pressure is a certain generalization of the well-known formula, obtained on the basis of the “piston” theory of classical gas dynamics, for the case of magneto-hydrodynamic flow around thin bodies. On the basis of this formula, as well as the equations of the theory of magnetoelasticity of thin plates, specific stability problems are solved in order to identify the influence of a magnetic field and the effects of interaction on the critical flutter velocity and on the amplitude of steady flutter oscillations. It is shown that in the presence of a magnetic field: (a) the critical flutter velocity of a rectangular plate decreases significantly, (b) the critical velocity of divergence (static instability) in the case of a cylindrical shell with an internal flow increases significantly due to the conductivity of the gas inside the shell and is a monotonically increasing function of the parameter characterizing strength of a given magnetic field. The study of nonlinear magnetoelastic flutter oscillations of plates is given taking into account both types of nonlinearities: aerodynamic (square and cubic) and geometric (cubic). It is established that the presence of both a flow and a magnetic field can become a source of both quantitative and qualitative changes in the monotonically increasing nature of the amplitude-frequency dependence of steady-state flutter oscillations, which occurs in the absence of a magnetic field and a gas flow.

Based on the above, we can conclude that this book is useful not only for students, graduate students and researchers specializing in the fields of mechanics and electrodynamics of continuous media, but also of interest to specialists associated with applied physics and mathematics and their numerous applications.

The main part of the book consists of the results obtained by the authors and Ph.D. students of Gevorg Baghdasaryan: D. Hasanyan, P. Mkrtchyan, E. Danoyan, R. Saghoyan, I. Vardanyan. A lot of work on preparation of the manuscript was made by Iren Vardanyan. The authors express sincere gratitude to her, as well as to all the persons who assisted them in the work on the book.

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Contents

Chapter 1
Basic Equations and Relations of Magnetoelasticity of Magnetoactive Deformable Bodies

In this chapter, based on the basic coupled nonlinear equations and relations of mechanics and quasi-static electrodynamics of a continuous medium, by linearization, a system of magnetoelasticity equations, surface conditions and constitutive equations are obtained that describe the behavior of perturbations in a magnetically active (both soft ferromagnetic or superconducting, and magnetostrictive) medium interacting with an external magnetic field. The presentation is given in curvilinear coordinates with the use of tensor analysis and the Lagrange method for describing the motion of a continuous medium. We also note that in this chapter, when presenting the well-known basis of the theory of magnetoelasticity of magnetoactive bodies, the methods of presentation from the monograph *Bagdasaryan G.Y., Mikilyan M.A. Effects of Magnetoelastic Interactions in Conductive Plates and Shells. Springer, 2016* are widely used.

1.1 Deformed State of Magnetically Active Elastic Bodies Interacting with Magnetic Fields

In continuum mechanics, a deformable body is considered as a set of material points (particles) that continuously fill a part of the space occupied by the body. When moving (deforming), the body at different moments of time t occupies different regions Ω_t of the three-dimensional Euclidean space, called its configurations. Each material particle P of the body at a certain moment in time t occupies a certain point M in the area Ω_t. One of the body configurations Ω_0, at $t = t_0$, is considered the main (initial) and everything related to the body and its movement refers to this configuration.

To describe the motion (deformation) of a continuous medium, we introduce a curvilinear coordinate system $\alpha^n (n = 1, 2, 3)$. When formulating the basic equations and relations of the nonlinear theory of magnetoelasticity, as in the usual theory of

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elasticity, it is necessary to distinguish the coordinates of the initial and current configurations. We can individualize the material particles of the body in the initial configuration, by assigning to each material point from the region Ω_0 the triple of values $\left(\alpha^1, \alpha^2, \alpha^3\right)$, which specifies the position of an arbitrary point $M^{(0)}$ in the region Ω_0 before deformation. This set of numbers will characterize the same material particle occupying the point M in the region of the current configuration Ω_t and in the process of deformation. That is, we will use the Lagrangian approach, according to which the coordinates of the points of the deformable body are expressed in terms of the coordinates of the points before the deformation.

The position of an arbitrary point $M^{(0)}$ of the medium in the initial configuration is determined by the radius vector $\vec{r}^{(0)}$

$$\vec{r}^{(0)} = \vec{r}^{(0)}\left(\alpha^1, \alpha^2, \alpha^3\right). \tag{1.1.1}$$

Here and below, the index "0" means that the value refers to the undeformed state.

Let us briefly summarize some well-known facts from tensor analysis [22, 45], which are necessary for what follows.

Covariant basis vectors $\vec{g}_\lambda^{(0)}$ at the point $M^{(0)}$ are defined by the formulas

$$\vec{g}_\lambda^{(0)} = \frac{\partial \vec{r}^{(0)}}{\partial \alpha^\lambda}. \tag{1.1.2}$$

Using the covariant basis vectors (1.1.2), we define the covariant metric tensor $g_{ij}^{(0)}$:

$$g_{ij}^{(0)} = \vec{g}_i^{(0)} \cdot \vec{g}_j^{(0)} = g_{ji}^{(0)}, \tag{1.1.3}$$

contravariant metric tensor g_0^{ij}:

$$g_0^{ik} g_{kj}^0 = \delta_j^i \tag{1.1.4}$$

and the contravariant basis vector $\vec{g}_0^\lambda$ at the point $M^{(0)}$:

$$\vec{g}_0^\lambda = g_0^{\lambda k} \vec{g}_k^{(0)}. \tag{1.1.5}$$

In these relations, δ_j^i is the Kronecker symbol, and the notation $\vec{a} \cdot \vec{b}$ means the scalar product of the vectors $\vec{a}$ and $\vec{b}$.

From relations (1.1.2)–(1.1.5) we obtain:

$$\begin{gathered} \vec{g}_0^\lambda \cdot \vec{g}_k^0 = \delta_k^\lambda, \\ \vec{g}_\lambda^0 = g_{\lambda k}^0 \vec{g}_0^k. \end{gathered} \tag{1.1.6}$$

Here and in what follows, summation over repeated indices is assumed.

The derivative of the covariant $\vec{g}_k^{(0)}$ and contravariant $\vec{g}_0^{\lambda}$ basis vectors with respect to the coordinate α^{ν} is defined as follows:

$$\frac{\partial \vec{g}_k^0}{\partial \alpha^{\nu}} = \dot{\Gamma}_{k\nu}^{\lambda} \vec{g}_{\lambda}^0,$$
$$\frac{\partial \vec{g}_0^{\lambda}}{\partial \alpha^{\nu}} = -\vec{g}_0^k \Gamma_{\nu k}^{\lambda} = -\Gamma_{\nu k}^{\lambda} g_o^{ik} \vec{g}_i^{(0)}, \qquad (1.1.7)$$

where $\Gamma_{\nu k}^{\lambda}$ are the Christoffel symbols, defined by the formula

$$2\Gamma_{k\nu}^{\lambda} = g_o^{\nu s} \left(\frac{\partial g_{\nu s}^{(0)}}{\partial \alpha^k} + \frac{\partial g_{ks}^{(0)}}{\partial \alpha^{\nu}} - \frac{\partial g_{k\nu}^{(0)}}{\partial \alpha^s} \right). \qquad (1.1.8)$$

Using (1.1.7), one can determine the partial derivatives of arbitrary vectors and tensors with respect to curvilinear coordinates. Bearing in mind that the vectors $\vec{g}_k^0$ and $\vec{g}_0^{\lambda}$ are basic vectors, any vector $\vec{a}$ can be represented as follows:

$$\vec{a} = a_{\lambda} \vec{g}_0^{\lambda} = a^k \vec{g}_k^0, \qquad (1.1.9)$$

where a_{λ} and a^k, respectively, are the covariant and contravariant components of this vector.

From (1.1.9), on the basis of (1.1.7), we obtain

$$\frac{\partial \vec{a}}{\partial \alpha^m} = \nabla_m a^k \vec{g}_k^0 = \nabla_m a_k \vec{g}_0^k, \qquad (1.1.10)$$

where the following notation is introduced for the covariant derivative of the covariant and contravariant vector components:

$$\nabla_m a^k = \frac{\partial a^k}{\partial \alpha^m} + \Gamma_{nm}^k a^n,$$
$$\nabla_m a_k = \frac{\partial a_k}{\partial \alpha^m} - \Gamma_{km}^n a_n \,. \qquad (1.1.11)$$

Similarly, the covariant derivative of the covariant and contravariant components of any tensor $\hat{A}$ of the second rank is calculated and the following expressions are obtained:

$$\nabla_m A^{ij} = \frac{\partial A^{ij}}{\partial \alpha^m} + \Gamma_{mk}^i A^{kj} + \Gamma_{mk}^j A^{ik},$$
$$\nabla_m A_{ij} = \frac{\partial A_{ij}}{\partial \alpha^m} - \Gamma_{im}^k A_{kj} - \Gamma_{jm}^k A_{ik} \,. \qquad (1.1.12)$$

Let us proceed to the description of the deformed state of the medium, mainly relying on the approaches described in [6, 7, 18, 22].

Let at an arbitrary moment of time t the considered material particle $M^{(0)}$ of a continuous medium occupy a point M in the region Ω_t, the position of which is determined by the radius vector $\vec{r}(\alpha^1, \alpha^2, \alpha^3, t)$

$$\vec{r} = \vec{r}^0(\alpha^1, \alpha^2, \alpha^3) + \vec{u}(\alpha^1, \alpha^2, \alpha^3, t) \tag{1.1.13}$$

where $\vec{u}$ is the displacement vector of the material particle $M^{(0)}$ corresponding to the transition of the point from the initial configuration Ω_0 to the current configuration Ω_t.

In a similar way, as was done above in the case of the undeformed state, the corresponding vector and tensor quantities of the deformed state are introduced and the actions on them are determined. In particular, the covariant and contravariant basis vectors and their corresponding covariant and contravariant metric tensors of the deformed state are introduced as follows:

$$\begin{aligned} &\vec{g}_\lambda = \frac{\partial \vec{r}}{\partial \alpha^\lambda}, \ \ g_{\lambda k} = \vec{g}_\lambda \cdot \vec{g}_k, \\ &\vec{g}^\lambda = g^{\lambda k}\vec{g}_k, \quad g^{\lambda k} g_{kl} = \delta_l^\lambda, \quad \vec{g}^n \vec{g}_m = \delta_m^n. \end{aligned} \tag{1.1.14}$$

All other values of the deformed state are determined according to formulas (1.1.7)–(1.1.12) with the replacement of $\vec{g}_\lambda^0$, $\vec{g}_0^\kappa$, g_{ij}^0 and g_0^{ij}, respectively, by $\vec{g}_\lambda$, $\vec{g}^k$, g_{ij} and g^{ij}.

Let us consider the changes in the distance between two infinitely close material particles after deformation. Let dl_0 be the distance between two infinitely close material particles in the initial configuration, and dl—the distance between the same material particles in the current configuration. Then

$$(dl)^2 - (dl_0)^2 = d\vec{r} \cdot d\vec{r} - d\vec{r}^0 \cdot d\vec{r}^0. \tag{1.1.15}$$

Taking into account (1.1.2), (1.1.10), (1.1.13) and (1.1.14), we calculate the differentials $d\vec{r}^0$ and $d\vec{r}$ in the form

$$\begin{aligned} &d\vec{r}^0 = \frac{\partial \vec{r}^0}{\partial \alpha^n} d\alpha^n = \vec{g}_n^0 d\alpha^n \\ &d\vec{r} = \frac{\partial \vec{r}}{\partial \alpha^n} d\alpha^n = \vec{g}_n d\alpha^n = \left(\vec{g}_n^0 + \vec{g}_m^0 \nabla_n u^m\right) d\alpha^n = \left(\vec{g}_n^0 + \vec{g}_0^m \nabla_n u_m\right) d\alpha^n, \end{aligned} \tag{1.1.16}$$

where u_m and u^m are covariant and contravariant components of the displacement vector $\vec{u}\left(\vec{u} = \vec{g}_m^0 u^m = \vec{g}_0^m u_m\right)$.

Substituting (1.1.16) into (1.1.15), after a series of transformations, we obtain relations for determining the covariant components of the Green's strain tensor in the form [22, 27, 32]

$$dl^2 - dl_0^2 = 2\varepsilon_{ij} d\alpha^i d\alpha^j,$$
$$2\varepsilon_{ij} = \nabla_i u_j + \nabla_j u_i + \nabla_i u^k \nabla_j u_k,$$
$$\varepsilon_{ij} = \varepsilon_{ji}, \quad g_{ij} = g_{ij}^0 + 2\varepsilon_{ij}, \tag{1.1.17}$$

where ε_{ij} are covariant components of the Green's strain tensor $\hat{\varepsilon}$.

Based on the above relationships, it is possible to calculate the change in other physical and geometric quantities during the deformation process. In particular, the velocity $\vec{\mathrm{v}}$ and acceleration $\vec{w}$ vectors of the particles of the material medium for a given law of motion (1.1.13) are determined by the formulas [22, 32]

$$\vec{\mathrm{v}} = \frac{\partial \vec{r}}{\partial t} = \frac{\partial u^n}{\partial t}\vec{g}_n^0 = \frac{\partial u_m}{\partial t}\vec{g}_0^m; \qquad \vec{\mathrm{v}} = \mathrm{v}^n \vec{g}_n^0 = \mathrm{v}_m \vec{g}_0^m,$$
$$\vec{w} = \frac{\partial^2 \vec{r}}{\partial t^2} = \frac{\partial^2 u^n}{\partial t^2}\vec{g}_n^0 = \frac{\partial^2 u_m}{\partial t^2}\vec{g}_0^m; \quad \vec{w} = w^n \vec{g}_n^0 = w_m \vec{g}_0^m, \tag{1.1.18}$$

The values v_n and v^n; w_n and w^n are called covariant and contravariant components of the velocity and acceleration vectors in the Lagrangian description of the motion of a continuous medium. When obtaining (1.1.18), it was taken into account that the basis vectors and Lagrangian coordinates do not depend on time.

In conclusion of this section, we present the change in the components of the normal vector to the material surface in the process of motion. Let us denote by $\vec{N}^0$ the unit vector of the normal to the surface of the initial configuration, and by $\vec{N}$—the unit vector of the normal to the same material surface in the current configuration. Then, the contravariant components of the vector $\vec{N}$ are expressed in terms of the covariant components of the vector $\vec{N}^0$ as follows [22, 27]:

$$N^j = \frac{g^{kn}\left(\delta_n^j + \nabla_n u^j\right)}{\sqrt{g^{\alpha\beta} N_\alpha^0 N_\beta^0}} N_k^0, \quad \vec{N} = N^i \vec{g}_i^0, \tag{1.1.19}$$

where $g^{\alpha\beta}$, taking into account (1.1.7) and (1.1.3), are determined according to (1.1.14).

1.2 Description of the Stress State of a Magnetically Active Deformable Body

To describe the stress state of a continuous medium, we will use the stress vector related to elementary areas before deformation. In the initial configuration, let us consider an infinitesimal tetrahedron dQ_0, three faces $dS_{n^0}^0$ of which are formed by the coordinate surfaces $\alpha = \mathrm{const}$, and the fourth face $dS_{n^0}^0$ is determined by the normal vector $\vec{N}^0$. In the process of deformation, the indicated material tetrahedron transforms into an infinitely small tetrahedron dQ in the current configuration. As applied to the tetrahedron dQ, we introduce the following notation [22]: $\vec{t}^{(n)}$ is the stress vector on the elementary area dS_n referred to the area $dS^0_{n^0}$; $\vec{t}^{(N)}$ is the stress vector on the elementary area dS_N referred to the area $dS^0_{N^0}$.

From the condition of dynamic equilibrium of an infinitely small material tetrahedron dQ, the following vector relation is obtained [22, 27]:

$$\vec{t}^{(N)} = \sum_{k=1}^{3} N^0_k \sqrt{g_0^{kk}} \vec{t}^{(k)};$$
$$\vec{N}^0 = N^0_k \vec{g}^k_0, \tag{1.2.1}$$

which relates the stress vector on the inclined area dS_N with ort $\vec{N}$ with the stress vectors on the coordinate areas dS_n in the current configuration.

In accordance with (1.2.1), the Kirchhoff stress tensor $\hat{t}$ and the Lagrange tensor $\hat{s}$ are introduced as follows [22, 27, 32]:

$$\vec{t}^{(n)} \sqrt{g_0^{nn}} = t^{nm} \vec{g}^0_m = s^{nm} \vec{g}_m,$$
$$\vec{t}^{(N)} = t^{nm} \vec{g}^0_m N^0_n = s^{nm} \vec{g}_m N^0_n. \tag{1.2.2}$$

Taking into account, that

$$\vec{g}_m = \vec{g}^0_m + \vec{g}^0_k \nabla_m u^k, \tag{1.2.3}$$

which follows from (1.1.13), from (1.2.2) we obtain the following relation between the components of the tensors $\hat{t}$ and $\hat{s}$

$$t^{nl} = s^{nm} \left(\delta^l_m + \nabla_m u^l\right). \tag{1.2.4}$$

To compose the equations of motion and reveal the symmetry properties of stress tensors, let us consider in the current configuration an infinitely small material parallelepiped formed by the coordinate surfaces $\alpha^i = \text{const}$ and $\alpha^i + d\alpha^i = \text{const}$. From the conditions of equality to zero of the main vector and the main moment, applied to the specified infinitely small material parallelepiped of all forces (surface and volume) and volume moments, respectively, we obtain [12, 22, 39]

$$\nabla_i t^{ik} + \rho_0 f^k = \rho_0 \frac{\partial^2 u^k}{\partial t^2},$$
$$\vec{g}_i \times t^{ij} \vec{g}^0_j + \rho_0 \vec{c} = 0,$$
$$\nabla_i t^{ik} \equiv \frac{\partial t^{ij}}{\partial \alpha^i} + t^{in} \Gamma^j_{in} + t^{nj} \Gamma^i_{in}, \quad \vec{f} = f^k \vec{g}^0_k, \tag{1.2.5}$$

where $\rho_0\vec{f}$ and $\rho_0\vec{c}$ are volume forces and volume moments of the current configuration, respectively; ρ_0 is the density of the medium in the initial configuration.

Substituting (1.2.4) into (1.2.5) we obtain the following equations of motion of a continuous medium with respect to the Lagrange stress tensor $\hat{s}$ [12, 22]:

$$\nabla_i\left[s^{im}\left(\delta_m^k + \nabla_m u^k\right)\right] + \rho_0 f^k = \rho_0 \frac{\partial^2 u^k}{\partial t^2},$$
$$\sqrt{g}\,\varepsilon_{imk}s^{im} + \rho_0 c_k = 0; \quad \vec{c} = c_k \vec{g}^k, \tag{1.2.6}$$

where $g = \det\left\|g_{pq}^0\right\|$, and ε_{imk} the Levi–Civita tensor, whose components have the form

$$\varepsilon_{imk} = \varepsilon^{imk} = \begin{cases} +1 & \text{if}(i, m, k) \text{ is}(1, 2, 3) \text{ or } (2, 3, 1) \text{ or } (3, 1, 2) \\ -1 & \text{if}(i, m, k) \text{ is } (3, 2, 1) \text{ or } (1, 3, 2) \text{ or } (2, 1, 3) \\ 0 & \text{if} i = m \text{ or } m = k \text{ or } i = k. \end{cases}$$

It follows from (1.2.5) and (1.2.6) that the stress tensors $\hat{t}$ and $\hat{s}$ are asymmetric. If $\vec{c} = 0$, then the Lagrange stress tensor $\hat{s}$, due to the second equation of system (1.2.6), will be symmetric.

1.3 Determination of Mass Forces and Mass Moments of Magnetic Origin. Equations of a Quasi-Stationary Magnetic Field

If a dielectric ferromagnetic medium is in a magnetic field, then body forces $\rho_0\vec{f}$ and mass moments $\rho_0\vec{c}$ arise in it, and are determined by the following formulas [26, 37, 43]:

$$\rho_0\vec{f} = \mu_0\vec{M}\,\nabla\vec{H},$$
$$\rho_0\vec{c} = \mu_0\vec{M} \times \vec{H}, \tag{1.3.1}$$

where $\vec{H}$ is the magnetic field strength, $\vec{M}$ is the medium specific magnetization vector (the magnetic moment per volume unit), ∇ is the nabla operator, μ_0 is the magnetic constant $\left(\mu_0 = 4\pi \times 10^{-7}\text{N/A}^2\right)$.

The vectors $\vec{H}$ and $\vec{M}$ are related to the magnetic induction vector $\vec{B}$ by the relation $\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right)$ and satisfy (in the quasistatic approximation) the following Maxwell equations [26, 43]:

$$\text{rot}\vec{H} = 0,$$
$$\text{div}\vec{B} = 0. \tag{1.3.2}$$

In component form, Eq. (1.3.2) can be written as

$$\begin{aligned} &\varepsilon^{ijk}\nabla_i H_j = 0, \\ &\nabla_m B^m = 0. \end{aligned} \tag{1.3.3}$$

Similarly, from (1.3.1) we have

$$\begin{aligned} \rho_0 f^i &= \mu_0 M^n \nabla_n H^i, \\ \rho_0 c_i &= \mu_0 \sqrt{g}\, \varepsilon_{nmi} \bar{M}^n \bar{H}^m, \\ \vec{M} &= M^n \vec{g}_n^{\,0} = \bar{M}^n \vec{g}_n, \\ \vec{H} &= H^i \vec{g}_n^{\,0} = \bar{H}^i \vec{g}_i. \end{aligned} \tag{1.3.4}$$

Substituting (1.3.4) into (1.2.6), we obtain the following symmetry conditions

$$\varepsilon_{nmi}\left(s^{im} + \mu_0 \bar{M}^i \bar{H}^m\right) = 0, \tag{1.3.5}$$

from which it follows that the tensor s^{im} can be represented in the form

$$s^{im} = s_c^{im} + \frac{1}{2}\mu_0\left[\bar{M}^m \bar{H}^i - \bar{M}^i \bar{H}^m\right], \tag{1.3.6}$$

where s_c^{im} is the symmetric part of tensor s^{im}.

In accordance with (1.3.3) and (1.3.4), the expressions for $\rho_0 f^i$ can be represented as [6, 34, 37]

$$\rho_0 f^i = \nabla_m T^{mi}, \tag{1.3.7}$$

where the Maxwell stress tensor T^{mi} is [6, 17, 28, 30, 37]

$$T^{mi} = H^m B^i - \frac{1}{2}\mu_0 g_0^{mi} \vec{H}^2. \tag{1.3.8}$$

Considering (1.2.5) [or (1.2.6)] and (1.3.1), we notice that the problem of mechanics of a continuous deformable medium turns out to be related to the problem of magnetostatics of a deformable non-conducting ferromagnetic body. In the future, when writing the equations of state and boundary conditions, we will see that there is also a reverse bond, so that both problems turn out to be interconnected.

1.4 State Equations of Magnetically Active Elastic Media

Let us consider the formulation of relations for the characteristics of the deformed state (elastic stresses, deformations, magnetic field strength and magnetization) for an elastic ferromagnetic medium, expanding one of the thermodynamic potentials into a power series in powers of a small parameter. In this case, we will consider quasi-static processes, neglecting the thermoelastic effects. Let us choose the specific free energy F as the initial thermodynamic potential and expand it into a series in powers of the strain tensor components and the specific magnetization vector.

Changes in the free energy of a unit mass, as shown in [6, 26, 34, 47], can be represented as

$$\delta F = \frac{1}{\rho} s^{ij} \alpha_{ki} \delta\beta_{jk} + \mu_0 H^k \delta\left(\frac{1}{\rho} M_k\right),$$
$$\alpha_{ki} = \delta_{ki} - \nabla_k u^i, \quad \beta_{jk} = \delta_{jk} + \nabla_j u^k, \tag{1.4.1}$$

where α_{ki} and β_{jk} are Euler and Lagrange strain gradient components respectively, δ_{ki} is Kronecker symbol.

In what follows, the specific free energy F will be considered as a function of the quantities β_{jk} and magnetization $\vec{I} = \rho^{-1}\vec{M}$. In this case, for the variation of the chosen thermodynamic potential, we obtain the following relation:

$$\delta F = \frac{\partial F}{\partial \beta_{ij}} \delta\beta_{ij} + \frac{\partial F}{\partial I_k} \delta I_k. \tag{1.4.2}$$

From (1.4.1) and (1.4.2) we obtain relations for determining the contravariant components of the tensor $\hat{s}$ and the magnetic field strength vector in the following form:

$$s_{ij} = \rho\beta_{ik} \left(\frac{\partial F}{\partial \beta_{jk}}\right)_{M_k},$$
$$H^k = \frac{1}{\mu_0} \left(\frac{\partial F}{\partial I_k}\right)_{\varepsilon_{ij}}. \tag{1.4.3}$$

In (1.4.3), subscripts indicate which variables are considered constant during differentiation.

Thus, having the expression for the specific free energy, we obtain the dependence of the components of the stress tensor $\hat{s}$ and the magnetic field strength vector on the components of the Lagrange deformation gradient and the specific magnetization vector.

1.5 The Boundary and Initial Conditions

The system of Eqs. (1.2.6) and (1.3.3), which determine the behavior of a magnetic field and an elastic ferromagnetic body moving in it, must be accompanied by conditions on the surface that bounds the body, the initial conditions, and the conditions at infinity.

Let us confine ourselves to considering the case when a dielectric ferromagnetic body is in contact with an external non-polarizable non-conductive medium. Then the surface of the body serves as the interface between two media with different magnetic properties and is the surface of a strong discontinuity.

Let us formulate the conditions that must be satisfied on the discontinuity surface. Let us first of all consider the boundary conditions for the magnetic field, assuming that there are no external electric charges on the surface of the body. From the equations of magnetostatics written in an integral form, based on the accepted assumptions, the continuity of the normal component of the magnetic induction vector $\vec{B}$ and the continuity of the tangential components of the magnetic field strength vector $\vec{H}$ on the surface of the ferromagnet S of the current configuration [9, 26, 28, 43]

$$\left(\vec{B}-\vec{B}^{(e)}\right)\times\vec{N}=0,$$
$$\left(\vec{H}-\vec{H}^{(e)}\right)\times\vec{N}=0. \tag{1.5.1}$$

In (1.5.1), $\vec{N}$ is the vector of the outer normal to the surface S of a ferromagnetic medium, determined according to (1.1.19), and the index “e” here and in what follows means that the quantity under consideration belongs to the external medium. According to the accepted assumptions that the external medium is non-conductive and non-polarizable, and the process is quasi-static, the equations of the magnetic field in this medium relative to $\vec{B}^{(e)}$ and $\vec{H}^{(e)}$ will have the form

$$\mathrm{div}\vec{H}^{(e)}=0,$$
$$\mathrm{rot}\vec{H}^{(e)}=0,$$
$$\vec{B}^{(e)}=\mu_0\vec{H}^{(e)}. \tag{1.5.2}$$

The boundary conditions for the functions characterizing the mechanical part of the problem, if they are formulated in displacements, have the form

$$\vec{u}|_{S_0}=\vec{u}_*,$$

where $\vec{u}_*$ is the given vector of the surface points moving, S_0 is the body surface in the initial configuration.

If surface forces $\vec{F}$ are given on the surface of the body, then the boundary conditions must be formulated in stresses. To obtain such conditions, we single out the volume V containing part of the discontinuity surface S, and with respect to this volume we apply the momentum equation. Using (1.3.8) and the Gauss-Ostrogradsky theorem, we transform the volume integrals in the momentum equation into surface integrals over the S_1 surface of volume V. Then, constricting the surface S_1 from both sides to the discontinuity surface S and proceeding similarly as in [39], taking into account (1.2.2)–(1.2.4) and (1.4.1), we obtain the following conditions at the points of the surface S_0 [6, 36, 39]:

$$\beta_{mi}\left[s^{km} - s^{km(e)}\right]N_k^0 = R^i. \tag{1.5.3}$$

Here s^{km} and $s^{km(e)}$ are the components of tensor (1.2.2), respectively, for the body and the medium; R is the surface plane of distribution on S of the external forces acting on the body;

$$\begin{aligned} &R^i = F^i + \left[T^{km(e)} - T^{km}\right]\beta_{mi}N_k^0,\\ &\vec{R} = R^i\vec{g}_i^{\,0}, \quad \vec{F} = F^i\vec{g}_i^{\,0},\\ &T^{ki(e)} = \mu_0 H^{k(e)}H^{i(e)} - \frac{1}{2}\mu_0 g_0^{ki}\left[\vec{H}^{(e)}\right]^2; \end{aligned} \tag{1.5.4}$$

T^{ki} and $T^{ki(e)}$ are the Maxwell stress tensors for the body and the medium, respectively, $\vec{H}^{(e)}$ is the magnetic field outside the ferromagnet, N_k^0 are the covariant components of the unit vector of the normal to the body surface S_0 in the undeformed state.

In particular, when the external medium is a vacuum, conditions (1.5.3) take the form [6, 9, 28, 37, 38]

$$\left[s^{km}\left(\delta_m^i + \nabla_m u^i\right)\right]N_k^0 = F^i + \left[T^{km(e)} - T^{km}\right]\left(\delta_m^i + \nabla_m u^i\right)N_k^0. \tag{1.5.5}$$

In specific problems of magnetoelasticity, the initial conditions should also be appropriately specified and limits on the behavior of the solution at infinity should be set. The initial conditions, in the study of unsteady processes, in most cases, are reduced to setting at a fixed point in time the main sought-for functions and their time derivatives. The conditions at infinity must be such as to ensure the uniqueness of the solution of the considered problems [10, 11, 13, 20, 21, 27, 31].

1.6 Equations and Boundary Conditions of the Perturbed State and Their Linearization

In the future, under the concept of a magnetoelastic system, we mean either an elastic magnetically active ferromagnetic or elastic superconducting medium and a magnetic field. Moreover, in the general case, the magnetic field will exist not only in the area occupied by the body (the inner area), but also in the outer area, where it will satisfy Eq. (1.5.2). For definiteness, it is further assumed that the body under consideration is in a medium whose electrodynamic properties are identified with the properties of vacuum.

Let us consider two states of a magnetoelastic system. The first state will be called unperturbed, and all quantities related to this state will be marked with the index “non” from above or below, depending on convenience. The second state will be called perturbed. All quantities related to the second state will be marked with the sign “~ ” and presented as the sum of quantities related to the unperturbed state and the perturbations of the corresponding quantities $\left(\tilde{Q} = Q_{non} + q\right)$ [33]. The perturbations will be considered small compared with the corresponding values of the unperturbed state and we will not give them with any additional indices. The derivation of the basic equations and the corresponding conditions is given for two types of considered media: elastic dielectric magnetoactive and elastic superconducting.

1.6.1 The Case of Elastic Magnetoactive Ferromagnetic Materials

So, according to the above, we will represent the quantities characterizing the perturbed state of the considered magnetoelastic system in the form

$$\begin{aligned}
&\tilde{s}^{ik} = s^{ik}_{non} + s^{ik}, \quad \tilde{u}^i = u^i_{non} + u^i,\\
&\tilde{f}^k = f^k_{non} + f^k, \quad \tilde{N}_k = N^{non}_k + N_k,\\
&\tilde{B}^n = B^n_{non} + b^n, \quad \tilde{H}^n = H^n_{non} + h^n,\\
&\tilde{M}^n = M^h_{non} + m^n, \quad \tilde{B}^{(e)n} = B^{n(e)}_{non} + b^{(e)n},\\
&\tilde{H}^{(e)n} = H^{n(e)}_{non} + h^{n(e)}_{non},\\
&\tilde{T}^{km} = T^{km}_{non} + T^{km}, \quad \tilde{T}^{km(e)} = T^{km(e)}_{non} + T^{km(e)},\\
&\tilde{\rho} = \rho_{non} + \rho, \quad \tilde{\rho} = \rho_0\left(1 - \nabla_k \tilde{u}^k\right),\\
&\rho_{non} = \rho_0\left(1 - \nabla_k \tilde{u}^k_{non}\right).
\end{aligned} \tag{1.6.1}$$

moreover, the formulas for $\tilde{\rho}$ and ρ_{non} follow from the law of conservation of mass.

Here, all quantities with index “non” characterize the unperturbed state and, according to (1.2.6), (1.3.3), (1.3.4), (1.5.1)–(1.5.5), satisfy the following equations and boundary conditions:

The unperturbed state equations

$$\begin{aligned}
&\nabla_i\left[s^{im}_{non}\left(\delta^k_m + \nabla_m u^k_{non}\right)\right] + \rho_0 f^k_{non} = \rho_0 \frac{\partial^2 u^k_{non}}{\partial t^2},\\
&\rho_0 f^k_{non} = \mu_0 M^n_{non} \nabla_n H^k_{non},\\
&\varepsilon^{ijk} \nabla_i H^{non}_j = 0, \quad \nabla_m B^m_{non} = 0,\\
&\text{rot} \vec{H}^{(e)}_{non} = 0, \quad \text{div} \vec{H}^{(e)}_{non} = 0\,, \quad \vec{B}^{(e)}_{non} = \mu_0 \vec{H}^{(e)}_{non};
\end{aligned} \tag{1.6.2}$$

the boundary conditions of the unperturbed state:

$$\begin{aligned}&\left(\vec{B}_{non}-\vec{B}_{non}^{(e)}\right)\cdot\vec{N}_{non}=0,\\&\left(\vec{H}_{non}-\vec{H}_{non}^{(e)}\right)\times\vec{N}_{non}=0\,,\\&\left[s_{non}^{km}\left(\delta_m^i+\nabla_m u_{non}^i\right)\right]N_k^0=F^i+\left[T_{non}^{km(e)}-T_H^{km}\right]\left(\delta_m^i+\nabla_m u_{non}^i\right)N_k^0.\end{aligned}\tag{1.6.3}$$

The characteristics of the perturbed state $\tilde{Q}$ must also satisfy equations and boundary conditions of the type (1.6.2) and (1.6.3) with the subscript "non" replaced by the sign "~" in (1.6.2) and (1.6.3). Substituting into the equations and boundary conditions of the perturbed state obtained in this way instead of $\tilde{Q}$ their expression according to (1.6.1) and taking into account that the quantities Q_{non} obey Eqs. (1.6.2) and boundary conditions (1.6.3), we obtain the following equations for perturbations q:

$$\begin{aligned}&\nabla_i\left[s^{ik}+s_{non}^{im}\nabla_m u^k+s^{im}\nabla_m\left(u_{non}^k+u^k\right)\right]+\rho_0 f^k=\rho_0\frac{\partial^2 u^k}{\partial t^2},\\&\rho_0 f^k=\mu_0 M_{non}^i\nabla_i h^k+\mu_0 m^i\nabla_i\left(H_{non}^k+h^k\right),\\&\mathrm{rot}\vec{h}^{(e)}=0,\quad \mathrm{div}\vec{h}^{(e)}=0,\quad \vec{b}^{(e)}=\mu_0\vec{h}^{(e)};\end{aligned}\tag{1.6.4}$$

and boundary conditions

$$\begin{aligned}&\left(\vec{B}_{non}-\vec{B}_{non}^{(e)}\right)\cdot\vec{N}+\left(\vec{b}-\vec{b}^{(e)}\right)\cdot\vec{N}_{non}+\left(\vec{b}-\vec{b}^{(e)}\right)\cdot\vec{N}=0,\\&\left(\vec{H}_{non}-\vec{H}_{non}^{(e)}\right)\times\vec{N}+\left(\vec{h}-\vec{h}^{(e)}\right)\times\vec{N}_{non}+\left(\vec{h}-\vec{h}^{(e)}\right)\times\vec{N}=0,\\&s_{non}^{km}\nabla_m u^i N_k^0+s^{km}\left[\delta_m^i+\nabla_m\left(u_{non}^i+u^i\right)\right]N_k^0\\&\quad=\left[T_{non}^{km(e)}-T_{non}^{km}\right]\nabla_m u^i N_k^0+\left[T^{km(e)}-T^{km}\right]\left[\delta_m^i+\nabla_m\left(u_{non}^i+u^i\right)\right]N_k^0.\end{aligned}\tag{1.6.5}$$

We note once again that all the quantities included in (1.6.4), (1.6.5) and not marked with the index "non" are the perturbations of the corresponding quantities.

Now we use the conditions for the smallness of the deformations of the unperturbed state and the conditions for the smallness of the perturbations to linearize the equations and boundary conditions for both the initial and the perturbed states. The following simplifications are based on the version of the theory of small deformations, according to which the relative elongations, shifts and covariant derivatives of the components of the displacement vector are small compared to unity and can be neglected compared to unity [6, 9, 22, 33]. To linearize the equations and boundary conditions of the unperturbed state, the characteristics of the unperturbed magnetic field, according to the above, can be represented as

$$\begin{aligned}\vec{M}_{non}&=\vec{M}_*+\vec{m}_0,\\\vec{H}_{non}&=\vec{H}_*+\vec{h}_0,\end{aligned}$$

$$\vec{B}_{non} = \vec{B}_* + \vec{b}_0, \tag{1.6.6}$$

where $\vec{M}_*$, $\vec{B}_*$ and $\vec{H}_*$ are the magnetization, magnetic induction, and magnetic field strength of the non-deformed body, and $\vec{m}_0$, $\vec{b}_0$ and $\vec{h}_0$ are the additions to the indicated quantities due to the deformation of the unperturbed state. The quantities marked with "$*$" are the solution to the following magnetostatics problem:

Equations of magnetostatics in the inner region

$$\begin{aligned} &\text{rot}\vec{H}_* = 0\,, \\ &\text{div}\vec{B}_* = 0\,, \\ &\vec{B}_* = \mu_0\left(\vec{H}_* + \vec{M}_*\right), \end{aligned} \tag{1.6.7}$$

equations in the outer region

$$\begin{aligned} &\text{rot}\vec{H}_*^{(e)} = 0\,, \\ &\text{div}\vec{B}_*^{(e)} = 0\,, \\ &\vec{B}_*^{(e)} = \mu_0\vec{H}_*^{(e)}\,, \\ &\vec{M}_* = 0; \end{aligned} \tag{1.6.8}$$

conjugation conditions on the surface S_0 of a non-deformed body and conditions at infinity

$$\begin{aligned} &\left(\vec{B}_* - B_*^{(e)}\right)\cdot\vec{N}^0 = 0\,, \\ &\left(\vec{H}_* - \vec{H}_*^{(e)}\right)\times\vec{N}_0 = 0, \\ &\vec{H}_*^{(e)} \to \vec{H}_0 \quad \text{for} \quad |r| \to \infty, \end{aligned} \tag{1.6.9}$$

where $\vec{H}_0$ is the given external magnetic field in which the considered ferromagnetic body is placed.

In addition to the accepted assumptions of the geometrically linear theory, it is considered that the additions $\vec{m}_0$, $\vec{b}_0$ and $\vec{h}_0$ are small values in comparison with the corresponding values characterizing the magnetic field of an undeformed body.

By virtue of the accepted assumptions, it follows from (1.6.2) that the characteristics of the unperturbed state must satisfy the equations

$$\begin{aligned} &\nabla_i s_{non}^{ik} + \rho_0 f_{non}^k = \rho_0\frac{\partial^2 u_{non}^k}{\partial t^2}, \\ &f_{non}^k = \mu_0 M_{*i}\nabla_i\left(H_*^k + h_0^k\right) + \mu_0 m_0^i\nabla_i H_*^k, \\ &\varepsilon_{ijk}\nabla_i h_j^0 = 0, \quad \nabla_m b_0^m = 0, \quad b_0^m = \mu_0\left(m_0^m + h_0^m\right) \end{aligned} \tag{1.6.10}$$

in the inner region and equations

$$\begin{aligned}&\mathrm{rot}\vec{h}_0^{(e)} = 0\,,\\&\mathrm{div}\vec{b}_0^{(e)} = 0\,,\\&\vec{b}_0^{(e)} = \mu_0\vec{h}_0^{(e)}\end{aligned} \tag{1.6.11}$$

in the outer region.

The solutions of Eqs. (1.6.10) and (1.6.11), according to (1.6.3) and the accepted assumptions, satisfy the following conjugation conditions on the surface of the body in the initial configuration:

$$\begin{aligned}&s_{non}^{ki}N_k^0 = F^i + \left[T_{non}^{ki(e)} - T_{non}^{ki}\right]N_k^0,\\&\left[b_0^k - b_0^{k(e)}\right]N_k^0 - \left[B_m^* - B_m^{*(e)}\right]\nabla_m u_{non}^i N_0^i = 0,\\&\varepsilon_{nmk}\left\{\left[h_0^n - h_0^{n(e)}\right]N_0^m - \left[H_*^n - H_*^{n(e)}\right]\nabla_m u_{non}^i N_0^i\right\} = 0;\end{aligned} \tag{1.6.12}$$

where, due to (1.3.8) and (1.5.4), we have

$$\begin{aligned}T_{non}^{ki} &= H_*^k B_*^i - \frac{1}{2}\mu_0 g_0^{ik}\vec{H}_*^2 + H_*^k b_0^i + h_0^k B_*^i - \mu_0 g_0^{ik}\vec{H}_*\cdot\vec{h}_0,\\T_{non}^{ki(e)} &= \mu_0 H_*^{k(e)}H_*^{i(e)} - \frac{1}{2}\mu_0 g_0^{ik}\left[\vec{H}_*^{(e)}\right]^2 + \mu_0 H_*^{k(e)}h_0^{i(e)}\\&\quad + \mu_0 h_0^{k(e)}H_*^{i(e)} - \mu_0 g_0^{ik}\vec{H}_*^{(e)}\cdot\vec{h}_0^{(e)}.\end{aligned} \tag{1.6.13}$$

The systems of Eqs. (1.6.10), (1.6.11) must also be supplemented with the initial conditions and the conditions at infinity.

Let us proceed to the linearization of the equations and boundary conditions of the perturbed state, accepting the main assumptions of the geometrically linear theory and the smallness of the perturbations. Then, with the accuracy of the accepted assumptions, taking into account (1.1.19), we obtain the following simplified relations:

$$\begin{aligned}&\left[T^{km(e)} - T^{km}\right]\left[\delta_m^i + \nabla_m\left(u_H^i + u^i\right)\right] \approx T^{ki(e)} - T^{ki},\\&s^{im}\left[\delta_m^k + \nabla_m\left(u_{non}^k + u^k\right)\right] \approx s^{ik}.\end{aligned} \tag{1.6.14}$$

Taking into account (1.6.14), from (1.6.4) and (1.6.5), within the accepted accuracy, with respect to the perturbations of the corresponding magnetoelastic values of the unperturbed state, we obtain the following linearized equations:

$$\begin{aligned}&\nabla_i\left[s^{ik} + s_{non}^{im}\nabla_m u^k\right] + \mu_0 M_{non}^i\nabla_i h^k + \mu_0 m^i\nabla_i H_{non}^k = \rho_0\frac{\partial^2 u^k}{\partial t^2},\\&\varepsilon^{ijk}\nabla_i h_j = 0,\quad \nabla_k b^k = 0\end{aligned} \tag{1.6.15}$$

in the inner region and the equations