Mauricio Porto Pato

Pseudo-Hermitian Random Matrices

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To my parents

Preface

This book comes as the result of a decade-long collaboration with the late Oriol Bohigas. After years dedicated to the study of Hermitian random matrices, he proposed the change of our interest to the non-Hermitian ones. We then started studying the structure of the trajectories, in the complex plane, of eigenvalues of a model constructed with the matrices of the non-Hermitian Unitary Ginibre ensemble. Once this investigation finished, the idea came of studying the removal of the Hermitian condition of the tridiagonal matrices of the so-called β-ensembles. That is, to do with the tridiagonal form of the Random Matrix Theory (RMT), the equivalent of what J. Ginibre, a long time before, did with the Gaussian matrices, just after E. Wigner had proposed them. The β-ensembles were a subject in which O. Bohigas was particularly interested. It is not well known but he was one of the first of the RMT community to ponder about what would happen if the Gaussian matrices were reduced to the tridiagonal form using Householder unitary transformations. Immediately engaged in the project, we were able to publish an article about non-Hermitian random matrices with real eigenvalues [1]. Chapter 3 is based on this article with the addition of the discussion of a Dyson 2 β effect recently discovered [2]. Unfortunately, just after the publication of the article Oriol died and I could not anymore count with his collaboration to develop the research field open by that article.

It is important to remark that at that time, we knew nothing about PT symmetry and pseudo-Hermitian operators. As a matter of fact, it was by chance that I came to learn about these matters and, moreover, to know that, concomitantly, there was an effort of working out random matrix models to deal with the subject of this new class of operators. Once the article was published, I found on the internet the announcement of a Conference to be held at the Koç University of Istanbul organized by Ali Mostafazadeh. My talk at the meeting with the discussions it ensued especially those with Naomichi Hatano and Eva-Maria Graefe were very important to me. After this meeting, it became fundamental to me to participate in the annual meetings of the PT community to talk as much as to learn.

By coincidence, just after my return from Istanbul, a student, Gabriel Marinello, came to me interested in doing his graduate studies having me as supervisor. Immediately, I accepted him, and we started to work in a project inspired by the Hatano article on the delocalization of the eigenvectors of complex eigenvalues. The results of our investigation were published [3] and turned out to become Gabriel's master's degree thesis and contribute in part to Chap. 4. To Gabriel's PhD project, the idea was to extend the concept of the pseudo-Hermitian condition to the RMT Gaussian matrices. This was done by turning complex the coupling constant of an RMT model used by me and the late nuclear physicist M. S. Hussein, another longtime collaborator of mine, to deal with the partial conservation of a quantum number. Imposing the pseudo-Hermitian condition, the model was obtained and a sequence of articles followed that constituted his doctorate work [4–8]. Chapters 10, 11, 12, 13, and 14 are based on these articles co-authored by me and Gabriel.

After Gabriel obtained his PhD, another student, Cleverson Andrade Goulart, started to work with me on the idea of applying the pseudo-Hermitian condition to the Laguerre ensembles. As is well known, Laguerre ensembles play a special role in the study of quantum random pure states. We were able to show that pseudo-Hermitian states can be used to investigate the behavior of the von Neumann entropy in the transition from real to complex eigenvalues. Chapter 15 is based on an article which Cleverson is co-author [9]. Another subject in which I could count with the collaboration with Cleverson was in the extension of the 2 β effect to the three β-ensembles whose results were published in an article with him [2] and parts of it appears in Chaps. 3, 6, and 9.

It is important to inform that my co-authors are aware of the use of our articles in this book.

São Paulo, Brazil Mauricio Porto Pato

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About the Author

Mauricio Porto Pato is a Senior Professor at the University of São Paulo with a large experience in the field of random matrix theory and applications. In the early 1990s, in collaboration with the nuclear physicist M. S. Hussein, he began a study of random matrices that resulted in the construction of an ensemble to be applied to a situation of partial conservation of a quantum number. The model was then, successfully, applied to the description of isospin data. In a collaboration with O. Bohigas, another important contribution of ours to be highlighted, was the formalism to deal with missing levels in correlated spectra, a study that evolved from his work with the experimentalist G. E. Mitchell. About 10 years ago, his interest moved from Hermitian to non-Hermitian operators, and this led to his involvement with the studies of the class of pseudo-Hermitian matrices associated to PT-symmetric systems, that is, systems invariant under parity and time-reversal transformations. This investigation started with the introduction of the pseudo-Hermiticity condition in the sparse tridiagonal matrices of the so-called beta-ensembles of the random matrix theory. Next, the pseudo-Hermiticity condition was extended to the standard Gaussian matrices with the creation of the pseudo-Hermitian Gaussian ensembles. All this effort, over the course of a decade, comprises about one dozen works among articles and theses.

Abbreviations and Operations

$(\cdot)^*$	Complex Conjugation
$(\cdot, \eta\, \cdot)$	η-Internal Product
$\langle \cdot, \cdot \rangle$	Ensemble Average
GOE	Gaussian Orthogonal Ensemble
GSE	Gaussian Symplectic Ensemble
GUE	Gaussian Unitary Ensemble
i.i.d.	Independent Identically Distributed
IPR	Inverse Participation Ratio
MEP	Maximum Entropy Principle
N	Size of Full Matrices
n	Size of the Tridiagonal Matrices
NNSD	Nearest Neighbor Spacing Distribution
pHGOE	pseudo-Hermitian Gaussian Orthogonal Ensemble
pHGUE	pseudo-Hermitian Gaussian Unitary Ensemble
pHGSE	pseudo-Hermitian Gaussian Symplectic Ensemble
$\leftrightarrow P$	Permutation Operator
PT	Parity/Time Reversal Symmetry
RMT	Random Matrix Theory
SVD	Singular Value Decomposition
Σ^2	Number Variance
σ^2	Variances

Chapter 1
Introduction

By the end of last century, evidence has been gathered that the complex Hamiltonian may have a real spectrum [1, 2]. The breakthrough was the study of the deformed harmonic oscillator Hamiltonian

$$H = \frac{p^2}{2m} - (ix)^N \tag{1.1}$$

that revealed that the eigenvalues are real for $N \geq 2$ and move in conjugate pairs into the complex plane for $N < 2$. It also has been identified that this property is a consequence of the invariance of the complex Hamiltonian under the combined parity and time-reversal transformations, the so-called *PT*-invariance [3]. This result led to the question if an extension of quantum mechanics can be constructed to include this special class of non-Hermitian Hamiltonians [4–6]. For instance, if the scalar product of two wave functions satisfying the Schrödinger equation with the non-Hermitian Hamiltonian is preserved in time. It was then realized that for this to happen, the scalar product must be modified by introducing an appropriate metric.

The connection of the *PT*-invariance with the class of pseudo-Hermitian operators was set by *A*. Mostafazadeh in a series of papers [7–9]. He took a clue from two main features of the *PT* systems, namely, that they have eigenvalues which are real or complex conjugate and that they require an internal product with a metric that makes it invariant under a time evolution obeying the Schrödinger equation. To see the consequence of these two demands, let us consider the η-internal product $(\Phi, \eta\Psi)$ of two wave functions that are supposed to satisfy the Schrödinger equation with a Hamiltonian H. Under this circumstance, we deduce that

$$i\partial_t(\Phi, \eta\Psi) = \left(\Phi, \left(\eta H - H^\dagger \eta\right)\Psi\right) = 0$$

such that the preservation in time of the internal product implies in the relation $\eta H = H^\dagger \eta$, that is, in the condition

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$$H^{\dagger} = \eta H \eta^{-1} \tag{1.2}$$

that shows that the operators of this non-Hermitian class must be connected to their respective adjoints by a similarity transformation. This condition defines the class of pseudo-Hermitian operators whose main property is to have a spectrum of real or complex conjugate eigenvalues. We observe that the metric η is not unique, for instance, the product $C\eta$, where C is a real number, satisfies the above equation. More general, if μ is an invertible Hermitian matrix that commutes with H, then the matrix $\mu\eta\mu^{-1}$ is also a valid metric. Along the book, we will see cases of metric constructed with the elements of the matrix itself but, also, metrics fixed independently from the matrix.

It is important to observe that, independently of the Schrödinger equation, the above pseudo-Hermitian condition (1.2) also can be deduced by considering that, given an internal product with a metric η, to an operator A to be Hermitian, it should satisfy

$$(\Phi, \eta A \Psi) = (A\Phi, \eta\Psi). \tag{1.3}$$

At the same time, however, using the adjoint $A^{\dagger}$ of the operator we also have the relation $(A\Phi, \eta\Psi) = (\Phi, A^{\dagger}\eta\Psi)$ such that by comparing with the left-hand side of (2.3), the pseudo-Hermitian condition follows without requiring time independence of the metric [10, 11]. This fact poses the question if the Schrödinger equation can be extended to the case of time-dependent metric, this will be treated in the last chapter of the book.

Turning now to random matrices, everything started when, in the late 1950s of the last century, E. Wigner borrowed from J. Wishart the Gaussian matrices then used by this statistician to construct his ensemble of invariant matrices and create the random matrix theory (RMT) (see Ref. [12] with a collection of articles). Spectral properties of these ensembles, the so-called Wigner-Dyson statistics [13–15], are characterized by the correlations generated by the repulsion among the levels and properties which are directly connected to the classical polynomials, namely, the Hermite ones, in the Wigner case, and the Laguerre ones, in the Wishart case. From the symmetry point of view, these ensembles are invariant under unitarity transformations and define three classes: the Orthogonal, the Unitary, and the Symplectic labeled by the Dyson index β with values 1, 2, 4, respectively. These values correspond to matrices with real, complex and quaternion elements. These three values have been named by F. Dyson the threefold way and physically correspond to systems invariant under time reversal with integer spin ($\beta = 1$) and half integer spin ($\beta = 4$), while ($\beta = 2$) refers to system without the time-reversal invariance [16]. A huge boost in the RMT applications came with the already proved Bohigas-Giannoni-Schmit conjecture that states the connection between Wigner-Dyson statistics and the manifestations of chaos in the quantum systems [17].

More recently, this picture had undergone a generalization with the introduction of the families of tridiagonal matrices parametrized by a real positive value of β

whose spectral statistical properties are the same of the full matrices for the values 1, 2, 4 of the parameter β [18–21]. The tridiagonal ensembles are three: the β-Hermite, the β-Laguerre, and the β-Jacobi ensembles and as their designations denote, their properties are related to the classical orthogonal polynomials as it occurs with the Gaussian matrices. This does not exclude the possibility of other β-ensembles.

The effort to construct random matrix ensembles as tools to simulate *PT*-symmetric systems started right from the early days of the subject, initially, considering small matrices of size 2×2 [22–24]. In our case, the breakthrough was the realization that the tridiagonal form of the β-ensembles constitutes a very convenient framework to impose the pseudo-Hermitian condition, Eq. (1.2), to non-Hermitian matrices. This led to the development of pseudo-Hermitian models based on the β-ensembles. On the other hand, for the full Gaussian matrices, inspired by Eq. (1.1), the idea was to use complex coupling constants in the RMT ensembles that have been built to describe the partial conservation of a quantum number. By imposing the pseudo-Hermitian condition, the appropriate form of the constants was derived, and the pseudo-Hermitian Gaussian models were then introduced.

It is important to mention that non-Hermiticity has already a notable history in RMT. In fact, few years after Wigner had proposed the Hermitian Gaussian ensemble, J. Ginibre undertook the task of investigating Gaussian matrices with no Hermitian condition imposed [25]. As a matter of fact, the present book is the result of a research project that started with the idea of removing the Hermitian condition from the tridiagonal matrices of the β-ensembles as J. Ginibre successfully had done in the Gaussian case. Our investigation, naturally, leads to the pseudo-Hermiticity concept.

It is important to remark that the pseudo-Hermitian condition, Eq. 1.2, has been considered in the context of the Cartan classification of non-Hermitian random matrices [26]. It is one of the discrete symmetries used to classify the classes of non-Hermitian ensembles. These symmetries entail a characterization of the forms the matrices and the metrics that they can have. The results obtained in our investigation of the tridiagonal and full pseudo-Hermitian matrices agree with the prediction of the Cartan classification [27].

The book starts by showing in the present chapter how the concept of pseudo-Hermiticity historically emerged from the studies of *PT*-symmetric systems, studies that also aroused the interest of the RMT community. Chapter 2 discusses the consequences of the pseudo-Hermitian condition to the eigen-decomposition of the non-Hermitian matrices. The following chapters can be divided in two parts: one constituted by Chaps. 3 to 9 is dedicated to the sparse random matrices in the tridiagonal form, while Chaps. 10 to 15 discuss full matrices. The six chapters dedicated to pseudo-Hermitian random matrices in tridiagonal form, the cases with real eigenvalues, and the appearance of complex eigenvalues generated by unbound and/or nonpositive metrics are discussed for the β-Hermite, Chaps. 3, 4, and 5; β-Laguerre, Chaps. 6, 7, and 8; and, finally, β-Jacobi, Chap. 9. Chapter 10 introduces the pseudo-Hermitian Gaussian matrices, and some of its properties are described in the following four chapters. Finally, in the last Chap. 15, the time invariance of the metric is suspended and a pseudo-Hermitian model with a time dependent metric is constructed to discuss the time evolution of the von Neumann entropy of entangled states.

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